Portugal. Math. (N.S.)
Vol. 69, Fasc. 2, 2012, 141–158
DOI 10.4171/PM/1910

Portugaliae Mathematica

Uniqueness of entropy solution for general anisotropic convection-diffusion problems*

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Abstract. This work is an attempt to develop the uniqueness theory of entropy solution for the Cauchy problem associated to a general non-isotropic nonlinear strongly degenerate parabolic-hyperbolic equation. Our aim is to extend, at the same time, results of [1] and [11]. The novelty in this paper is the fact that we are dealing with general anisotropic diffusion problems, not necessarily with Lipschitz convection-diffusion flux functions in the whole space. Moreover, the source term depends on the unknown function of the problem. Under an abstract lemma and an additional assumption, we ensure the comparison principle which leads us to the uniqueness. In unbounded domains without Lipschitz condition on the convection and diffusion flux functions, this assumption seems to be optimal to establish uniqueness (cf. [1], [3], [14]).

Mathematics Subject Classification (2010). 35K65, 35L65.

Keywords. Degenerate parabolic-hyperbolic equations, entropy solution, Kato's Inequality, anisotropic diffusion, non-Lipschitz flux, conservation law.

1. Introduction

Let $Q =]0, T[\times \mathbb{R}^N$ with $T > 0$, $N \geq 1$ and $a = (a_{ij})_{1 \leq i,j \leq N}$.

We consider the class of Cauchy problems $(\mathrm{CP}) = (\mathrm{CP})(a, F, f, u_0)$:

$$(\mathrm{CP}) \quad \begin{cases} \partial_t u + \nabla.F(u) - \sum_{i,j=1}^N \partial_{x_i}\big(a_{ij}(u)\partial_{x_j} u\big) = f(u) & \text{in } Q, \\ u(0,.) = u_0 & \text{on } \mathbb{R}^N, \end{cases}$$

where

$$F \in C(\mathbb{R}, \mathbb{R}^N), \qquad F(0) = 0; \tag{1.1}$$

here a is a $N \times N$ symmetric nonnegative matrix with locally integrable coefficients.

*The referee's careful reading and remarks improved the paper.

So we can always write

$$a_{ij}(u) = \sum_{k=1}^{N} \sigma_{ik}(u)\sigma_{jk}(u), \qquad \sigma_{ik} \in L^2_{\mathrm{loc}}(\mathbb{R}), \tag{1.2}$$

and $(\sigma_{ik}(u))$ is its square root matrix.

It is well known that the problem (CP) possesses discontinuous solutions and that weak solutions are not uniquely determined by their initial data: if $a \equiv 0$, (CP) degenerates to a scalar hyperbolic conservation law where there is no uniqueness of weak solution in general. So we must interpret it in the sense of entropy solution. Note that this problem is a model of degenerate anisotropic diffusion-convection motions of fluid and has important applications as in two phase flows in porous media (cf. [9] and the references cited therein) and sedimentation-consolidation processes (cf. [6], [7]). It has received much attention in the last years. Its solutions are computed by many numerical methods (cf. [12], [13]) and well-posedness theory must be given. Under various conditions, many authors have proved uniqueness and existence results of entropy solutions:

For $N = 1$, well-posedness results in the sense of entropy solution are given by Wu and Yin [18] and Bénilan and Touré [4].

The multidimensional isotropic problem was treated in [15] using Kruzhkov's doubling-of-variables device and a generalization has recently been given by Andreianov and Maliki in [1]. Both works are based on the approach developped by Carrillo in [8]. Other important results have been given by Blanchard and Porretta in [5] and Andreu, Igbida, Mazón and Toledo in [2].

The general anisotropic diffusion context that we consider is more delicate and was brilliantly solved by Chen and Perthame in [11]: they suppose that

$$F' \in L^\infty_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^N), \qquad \sigma_{ij} \in L^\infty_{\mathrm{loc}}(\mathbb{R}),$$

and introduce a notion of kinetic solution to handle this problem.

In the present investigation, we wish to extend the uniqueness results of [11] to a more general set of convection functions and diffusion matrix: the derivative of F is not necessarily bounded, and we do not impose the coefficients of A to be locally Lipschitz continuous functions. Instead of kinetic formulation as done in [11], we use Kruzhkov's doubling-of-variables method. Another contribution is that we upgrade the uniqueness results given in [1] to an anisotropic case. Our main ingredient is to make some hypothesis on the modulus of continuity of F and A. This kind of assumption on the modulus of continuity appeared for the first time in [3], where Benilan and Kruzhkov show its optimality for the uniqueness in scalar conservation laws. The outline of this work is as follows. The Section 1 is the introduction. In Section 2, we present some notations and recall the notion of entropy solution as done in [11]. Section 3 is devoted to the statement of an abstract lemma and our basic well-posedness result.

2. Notion of entropy solution of (CP)

In this section we suppose that the initial data and source term satisfy the hypothesis

$$\begin{cases} u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \\ f \in \mathrm{Lip}(\mathbb{R}) \text{ and } f(0) = 0. \end{cases} \tag{2.1}$$

Let $i, j \in I_N = \{1, 2, \ldots, N\}$ and

$$A_{ij}(u) = \int_0^u a_{ij}(s)\, ds, \qquad 1 \le i, j \le N. \tag{2.2}$$

In what follows, we make the following assumptions:

- we assume that $F_i \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$;
- $\omega_{A_{ij}}$ and α_{F_i} denote respectively the modulus of continuity of A_{ij} and F_j, $i, j = 1, \ldots, N$;
- we define the operators H, H_0, H^1_ε, H^2_ε respectively by

$$H(s) = \begin{cases} 1 & \text{if } s > 0, \\ [0,1] & \text{if } s = 0, \\ 0 & \text{if } s < 0, \end{cases} \qquad H_0(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \le 0. \end{cases} \tag{2.3}$$

$$H^1_\varepsilon(s) = \begin{cases} 1 & \text{if } s > \varepsilon, \\ \sin\left(\frac{\pi}{2\varepsilon} s\right) & \text{if } 0 \le s \le \varepsilon, \\ 0 & \text{if } s < 0, \end{cases} \qquad H^2_\varepsilon(s) = \begin{cases} -1 & \text{if } s < -\varepsilon, \\ \sin\left(\frac{\pi}{2\varepsilon} s\right) & \text{if } -\varepsilon \le s \le 0, \\ 0 & \text{if } s > 0, \end{cases} \tag{2.4}$$

and for $k \in \mathbb{R}$, the corresponding entropy functions by

$$\begin{aligned} \gamma^1_\varepsilon(r,k) &= \int_k^r H^1_\varepsilon(s-k)\, ds, & \gamma^2_\varepsilon(r,k) &= \int_k^r H^2_\varepsilon(s-k)\, ds, \\ \theta^{i,1}_\varepsilon(r,k) &= \int_k^r H^1_\varepsilon(s-k) F_i'(s)\, ds, & \theta^{i,2}_\varepsilon(r,k) &= \int_k^r H^2_\varepsilon(s-k) F_i'(s)\, ds, \\ v^{i,j,1}_\varepsilon(r,k) &= \int_k^r H^1_\varepsilon(s-k) a_{ij}(s)\, ds, & v^{i,j,2}_\varepsilon(r,k) &= \int_k^r H^2_\varepsilon(s-k) a_{ij}(s)\, ds \end{aligned} \tag{2.5}$$

for $l \in I_N$ and $\eta \in C(\mathbb{R})$, we set $\rho_l = (\rho_{1l}, \ldots, \rho_{Nl})$ and $\rho_l^\eta = (\rho_{1l}^\eta, \ldots, \rho_{Nl}^\eta)$ with

$$\rho_{il}(r) = \int_0^r \sigma_{il}(\tau)\,d\tau, \qquad \rho_{il}^\eta(r) = \int_0^r \eta(\tau)\sigma_{il}(\tau)\,d\tau \qquad \text{for all } i \in I_N;$$

- for any convex C^2 entropy function $\gamma : \mathbb{R} \to \mathbb{R}$, we define the entropy flux

$$\theta = (\theta_i) : \mathbb{R} \to \mathbb{R}^N, \qquad \theta'(r) = \gamma'(r)F'(r), \tag{2.6}$$

$$v = (v_{ij}) : \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^N, \qquad v'(r) = \gamma'(r)a(r); \tag{2.7}$$

- we introduce the set

$$L^2\big(0, T; L^2(\mathrm{div}; \mathbb{R}^N)\big) = \big\{v \in \big(L^2\big((0,T) \times \mathbb{R}^N\big)\big)^N : \mathrm{div}(v) \in L^2\big((0,T) \times \mathbb{R}^N\big)\big\}.$$

We now consider the Cauchy problem (CP) = (CP)(A, F, f, u_0).

Definition 2.1 (Entropy solution of (CP)(A, F, f, u_0)). Let u_0 and f be such that (2.1) is fulfilled. An entropy solution of (CP)(A, F, f, u_0) is a measurable function $u : Q \to \mathbb{R}$ such that:

(i)

$$u \in L^\infty\big(0, T; L^1(\mathbb{R}^N)\big) \cap L^\infty(Q), \tag{2.8}$$

(ii)

$$\rho_l(u) \in L^2\big(0, T; L^2(\mathrm{div}; \mathbb{R}^N)\big) \qquad \text{for } l = 1, \ldots, N, \tag{2.9}$$

(iii)

$$\mathrm{div}\,\rho_l^\eta(u) = \eta(u)\,\mathrm{div}\,\rho_l(u) \qquad \text{for } l = 1, \ldots, N \text{ and } \eta \in C(\mathbb{R}), \tag{2.10}$$

(iv) For any entropy flux triple (γ, θ, v),

$$\partial_t\gamma(u) + \nabla.\theta(u) - \sum_{i,j=1}^N \partial_{x_i x_j}^2 v_{ij}(u) - \gamma'(u)f(u) \leq -m^{u,\gamma''} \qquad \text{in } D'(Q). \tag{2.11}$$

where

$$m^{u,\eta}(t,x) := \eta\big(u(t,x)\big) \sum_{l=1}^N \big(\mathrm{div}\,\rho_l\big(u(t,x)\big)\big)^2 \qquad \text{for } \eta \in C(\mathbb{R}).$$

(v) The initial condition is assumed in the following strong L^1 sense:

$$\lim_{t \downarrow 0} \|u(t,.) - u_0\|_{L^1(\mathbb{R}^N)} = 0.$$

One of the important contributions of Chen and Perthame is to show in [11] that the chain rule (2.10) must be included in the definition of an entropy solution for the anisotropic case. They note also that (2.10) is automatically fulfilled when $a(u)$ is a diagonal matrix. In this situation, this point can be deleted in the Definition 2.1. For a deeper discussion on this subject, we refer the reader to [11] and [17].

3. Existence and uniqueness of entropy solution of (CP)

Here we give an abstract lemma which will be a main ingredient in the proof of our uniqueness result.

Lemma 3.1. *For $(i,j) \in I_N \times I_N$, take nonnegative functions ω_{ij}, α_i with the following properties:*

$$\lim_{\varepsilon\to 0} \frac{\alpha_j(\varepsilon) + \left(\alpha_j(\varepsilon)^2 + 2\varepsilon\omega_{ij}(\varepsilon)\right)^{1/2}}{\varepsilon} = +\infty; \tag{3.1}$$

$$\liminf_{\varepsilon\to 0} \varepsilon^{1-N} \prod_{1\le j\le N} \left[\sum_{i=1}^{N} \left(\alpha_j(\varepsilon) + \left(\alpha_j(\varepsilon)^2 + 2\varepsilon\omega_{ij}(\varepsilon)\right)^{1/2}\right)\right] < +\infty \tag{3.2}$$

in the case $N > 2$, or

$$\liminf_{\varepsilon\to 0} \varepsilon^{1-N} \prod_{1\le j\le N} \left[\sum_{i=1}^{N} \left(\alpha_j(\varepsilon) + \left(\alpha_j(\varepsilon)^2 + 2\varepsilon\omega_{ij}(\varepsilon)\right)^{1/2}\right)\right] = 0 \tag{3.3}$$

in the case $N = 2$. Let $h \in L^1_{\mathrm{loc}}(Q)$ such that $h^+ = \max(h,0) \in L^1(Q)$ and consider $W_0 \in L^1(\mathbb{R}^N)$, $W \in L^\infty(Q) \cap L^\infty\left(0,T;L^1(\mathbb{R}^N)\right)$, $W \ge 0$ satisfying

$$\iint_Q \Big(W\partial_t\xi + \sum_{(i,j)\in I_N\times I_N} (W+\varepsilon)\frac{\omega_{ij}(\varepsilon)}{\varepsilon}|\partial^2_{x_i x_j}\xi| + \sum_{j=1}^{N}(W+\varepsilon)\frac{\alpha_j(\varepsilon)}{\varepsilon}|\partial_{x_j}\xi|\Big)\,dx\,dt$$
$$+\iint_Q h\xi\,dx\,dt \ge 0 \qquad \textit{for any } \varepsilon > 0 \textit{ and } \xi \in D(Q),\ \xi \ge 0. \tag{3.4}$$

Suppose $\left(W(t,.) - W_0\right)^+ \to 0$ in $L^1(\mathbb{R}^N)$ for some $W_0 \in L^1(\mathbb{R}^N)$, when $t \to 0$ essentially. Then

$$h \in L^1(Q) \qquad \textit{and} \qquad \int_{\mathbb{R}^N} W(r,x)\,dx \le \int_{\mathbb{R}^N} W_0(x)\,dx + \int_{Q_r} h\,dx\,dt \tag{3.5}$$

for $r \in (0,T)$ a.e. with $Q_r = (0,r)\times\mathbb{R}^N$.

Remark 3.2. One gets (3.1) if for i and j at least one of the functions $\frac{\omega_{ij}(\varepsilon)}{\varepsilon}$ and $\frac{\alpha_i(\varepsilon)}{\varepsilon}$ goes to infinity as $\varepsilon \to 0$.

If $N = 1$, we enter an isotropic problem (cf. [1]) which is of no interest to us in this paper. So we do not consider it in the above lemma.

Proof of Lemma 3.1. Let $s \le r$. For a.e. $s, r \in (0, T)$, inequality (3.4) is equivalent to

$$\begin{aligned}\int_{\mathbb{R}^N} & W(r,.)\xi\,dx \\ & \le \int_{\mathbb{R}^N} W(s,.)\xi\,dx + \int_s^r \Big(\int_{\mathbb{R}^N} \sum_{(i,j)\in I_N\times I_N} (W+\varepsilon)\frac{\omega_{ij}(\varepsilon)}{\varepsilon}|\partial^2_{x_ix_j}\xi|\,dx\Big)\,dt \\ & \quad + \int_s^r \Big(\int_{\mathbb{R}^N} \sum_{j=1}^N (W+\varepsilon)\frac{\alpha_j(\varepsilon)}{\varepsilon}|\partial_{x_j}\xi|\,dx\Big)\,dt + \int_s^r\int_{\mathbb{R}^N} h\xi\,dx\,dt, \qquad (3.6)\end{aligned}$$

for any $\varepsilon > 0$ and $\xi \in D(\mathbb{R}^N)$, $\xi \ge 0$. Sending s to 0^+, this inequality implies that for any $\varepsilon > 0$, $0 \le \xi \in D(\mathbb{R}^N)$ and $r \in (0, T)$ a.e.,

$$\begin{aligned}\int_{\mathbb{R}^N} & \Big(W(r,.) + \int_0^r h^-(t,.)\,dt\Big)\xi\,dx \\ & \le \int_{\mathbb{R}^N} W_0\xi\,dx + \int_{Q_r} h^+\xi\,dx\,dt \\ & \quad + \int_{\mathbb{R}^N} \sum_{(i,j)\in I_N\times I_N} \Big(\int_0^r W(t,.)\,dt + \varepsilon r\Big)\Big(\frac{\omega_{ij}(\varepsilon)}{\varepsilon}|\partial^2_{x_ix_j}\xi| + \frac{\alpha_j(\varepsilon)}{\varepsilon}|\partial_{x_j}\xi|\Big)\,dx, \quad (3.7)\end{aligned}$$

where $h^+ = \max(h, 0)$ and $h = h^+ - h^-$.

Note that in a bounded domain, we can end this proof here by taking in (3.7), a test function ξ which is identically equal to a positive constant. Of course, that does not work in the whole space and one of the main difficulties lies in the fact that the constants are not integrable anymore. So, in the sequel of the proof, our goal is to construct a test function ξ which goes to 1 in $\mathbb{R}^N$ and allow one to drop the third term of the right-hand side of inequality (3.7). We make a suitable choice of the test function ξ by setting

$$\xi(x) = \prod_{1\le j\le N} \psi\Big(\frac{|x_j|}{R_j}\Big) \qquad \text{for all } x \in \mathbb{R}^N,$$

where $R_j = R_j(\varepsilon, \eta)$, ε, η are constants with $\eta \ge 1$. The same form of test function is taken in [16]. But here, we have to adapt the choice of R_j to the anisotropic case.

The function ψ is $\mathscr{C}^2$ almost everywhere on $\mathbb{R}$ and is subject to the following conditions:

(i) $\psi \in L^1(\mathbb{R}^+)$, we set $C_0 = \int_0^{+\infty} \psi(x)\,dx$;

(ii) $|\psi'| \le \psi$, $|\psi''| \le \psi$;

(iii) $0 \le \psi \le 1$ in $\mathbb{R}$;

(iv) $\psi \equiv 1$ in $[-1, 1]$.

Thanks to (iii) and (iv) one has

$$\xi(x) = 1 \qquad \text{for all } x \in D := \prod_{j=1}^{N} [-R_j, R_j] \text{ and } 0 \le \xi \le 1 \text{ in } \mathbb{R}^N. \tag{3.8}$$

(For example, one can take $\psi(r) = \exp\big(-(|r| - 1)^+\big)$.)

We divide the proof into two steps.

Step 1: $N > 2$. From (ii), we have

$$|\partial_{x_j} \xi| \le \frac{1}{R_j} \xi, \qquad |\partial^2_{x_i x_j} \xi| \le \frac{1}{R_i R_j} \xi.$$

Replacing the above inequalities in (3.7), we see that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big(W(r, .) + \int_0^r h^-(t, .)\,dt \Big) \xi\,dx \\
&\quad \le \int_{\mathbb{R}^N} W_0 \xi\,dx + \int_{Q_r} h^+ \xi\,dx\,dt \\
&\qquad + \sum_{j=1}^{N} \left(\sum_{i=1}^{N} \frac{1}{\varepsilon R_j} \left[\frac{\omega_{ij}(\varepsilon)}{R_i} + \alpha_j(\varepsilon) \right] \right) \int_{\tilde{D}} \Big(\int_0^r W(t, .)\,dt \Big) \xi\,dx \\
&\qquad + T \sum_{j=1}^{N} \left(\sum_{i=1}^{N} \frac{1}{R_j} \left[\frac{\omega_{ij}(\varepsilon)}{R_i} + \alpha_j(\varepsilon) \right] \right) \int_{\mathbb{R}^N} \xi\,dx,
\end{aligned} \tag{3.9}$$

where

$$\tilde{D} := \mathbb{R}^N \backslash D = \bigcup_{j=1}^{N} (\{|x_j| > R_j\}).$$

We choose R_j as follows. Let $\overline{R_{ij}}$ be such that

$$\frac{1}{\varepsilon \overline{R_{ij}}} \left(\frac{\omega_{ij}(\varepsilon)}{\overline{R_{ij}}} + \alpha_j(\varepsilon) \right) = \frac{1}{2}.$$

This implies that

$$\overline{R_{ij}} = \frac{\alpha_j(\varepsilon) + [\alpha_j(\varepsilon)^2 + 2\varepsilon\omega_{ij}(\varepsilon)]^{1/2}}{\varepsilon}.$$

According to (3.1), if we set

$$R_{ij}(\varepsilon,\eta) = \frac{\overline{R_{ij}}}{\eta} = \frac{1}{\eta\varepsilon}\left(\alpha_j(\varepsilon) + \left(\alpha_j(\varepsilon)^2 + 2\varepsilon\omega_{ij}(\varepsilon)\right)^{1/2}\right) \quad \text{and}$$
$$R_j(\varepsilon,\eta) = \sum_{i=1}^{N} R_{ij}(\varepsilon,\eta), \tag{3.10}$$

one can easily check that

$$\lim_{\varepsilon\downarrow 0} R_{ij}(\varepsilon,\eta) = +\infty, \qquad 0 \le \frac{1}{\varepsilon R_{ij}}\left(\frac{\omega_{ij}(\varepsilon)}{R_{ij}} + \alpha_j(\varepsilon)\right) \le \eta^2$$

and

$$0 \le \sum_{j=1}\left(\sum_{i=1}^{N}\frac{1}{\varepsilon R_{ij}}\left(\frac{\omega_{ij}(\varepsilon)}{R_{ij}} + \alpha_j(\varepsilon)\right)\right) \le (N\eta)^2.$$

Let $\mu(\eta) = (N\eta)^2$. Since $0 < \frac{1}{R_j} \le \frac{1}{R_{ij}}$, we show that $\lim_{\varepsilon\downarrow 0} R_j(\varepsilon,\eta) = +\infty$ and

$$0 \le \sum_{j=1}^{N}\left(\sum_{i=1}^{N}\frac{1}{\varepsilon R_j}\left(\frac{\omega_{ij}(\varepsilon)}{R_i} + \alpha_j(\varepsilon)\right)\right) \le \mu(\eta). \tag{3.11}$$

Defining

$$K(\varepsilon) := \sum_{j=1}^{N}\left(\sum_{i=1}^{N}\frac{1}{R_j}\left(\frac{\omega_{ij}(\varepsilon)}{R_i} + \alpha_j(\varepsilon)\right)\right)\int_{\mathbb{R}^N} \xi\, dx, \tag{3.12}$$

one can see that

$$0 \le K(\varepsilon) \le (2C_0)^N \sum_{j=1}^{N}\left(\sum_{i=1}^{N}\frac{1}{R_{ij}}\left(\frac{\omega_{ij}(\varepsilon)}{R_{ij}} + \alpha_j(\varepsilon)\right)\right)\prod_{1\le j\le N} R_j. \tag{3.13}$$

From what has already been proved in (3.10) and (3.11), it follows that

$$\prod_{1\le j\le N} R_j = \frac{1}{\eta^N}\prod_{1\le j\le N}\left(\sum_{i=1}^{N}\overline{R_{ij}}\right) = \frac{1}{\varepsilon^N\eta^N}\prod_{1\le j\le N}\left(\alpha_j(\varepsilon) + \left(\alpha_j(\varepsilon)^2 + 2\varepsilon\omega_{ij}(\varepsilon)\right)^{1/2}\right)$$

and

$$0 \le K(\varepsilon) \le \frac{N^2(2C_0)^N}{\eta^{N-2}} \varepsilon^{1-N} \prod_{1 \le j \le N} \Big(\sum_{i=1}^{N} \big(\alpha_j(\varepsilon) + \big(\alpha_j(\varepsilon)^2 + 2\varepsilon\omega_{ij}(\varepsilon)\big)^{1/2}\big)\Big). \quad (3.14)$$

In view of assumption (3.2), we can extract a subsequence $\big(K(\varepsilon_k)\big)_{k \ge 1}$ of $\big(K(\varepsilon)\big)_{\varepsilon > 0}$ such that

$$\lim_{\varepsilon_k \to 0} K(\varepsilon_k) \le \frac{C}{\eta^{N-2}}. \quad (3.15)$$

Since $\lim_{\varepsilon \downarrow 0} R_j(\varepsilon, \eta) = +\infty$ and $W \in L^\infty(Q) \cap L^\infty\big(0, T; L^1(\mathbb{R}^N)\big)$,

$$\lim_{\varepsilon_k \to 0} \int_{\tilde{D}} \Big(\int_0^r W(t,.)\, dt \Big) \xi \, dx = 0. \quad (3.16)$$

Replacing (3.11) and (3.12) in (3.9), we send $\varepsilon_k \downarrow 0$; due to (iv), (3.15) and (3.16), we obtain

$$\int_{\mathbb{R}^N} \Big(W(r,.) + \int_0^r h^-(t,.)\, dt \Big)\, dx \le \int_{\mathbb{R}^N} W_0 \, dx + \int_{Q_r} h^+ \, dx\, dt + T \frac{C}{\eta^{N-2}}. \quad (3.17)$$

So this clearly forces h^- to be in $L^1(Q)$. We finally get (3.5) by sending $\eta \uparrow +\infty$ in the previous inequality.

Step 2: $N = 2$. Note that in this case, the constant C in step 1 is 0 and we can conclude in the same way. □

Remark 3.3. The choice of the finite sequence $(R_j)_{1 \le j \le N}$ in (3.10) can be improved without affecting the hypothesis and the conclusion of the Lemma 3.1. To be more precise, if we set

$$M(\varepsilon) = \varepsilon \sqrt{\prod_{j=1}^{N} \max\bigg\{ \bigg(\Big(\frac{\alpha_j(\varepsilon)}{\varepsilon}\Big)^2, \frac{\omega_{1j}(\varepsilon)}{\varepsilon}, \ldots, \frac{\omega_{Nj}(\varepsilon)}{\varepsilon} \bigg) \bigg\}}, \quad (3.18)$$

it is a simple matter to show, with the equivalence between the three standard norms of $\mathbb{R}^N$, that the conditions (3.2) and (3.3) are respectively equivalent to

$$\liminf_{\varepsilon \to 0} M(\varepsilon) < +\infty \qquad \text{and} \qquad \liminf_{\varepsilon \to 0} M(\varepsilon) = 0.$$

These last conditions are easier to check in practice.

Notice that if (3.3) holds but (3.1) fails, it is easy to modify α_i and ω_{ij} so that (3.3) and (3.1) hold.

For instance, if

$$\varepsilon^{1-N} \prod_{1\le j\le N} \Big[\sum_{i=1}^{N} \big(\alpha_j(\varepsilon) + \big(\alpha_j(\varepsilon)^2 + 2\varepsilon\omega_{ij}(\varepsilon)\big)^{1/2}\big)\Big] = K\varepsilon^{\theta}$$

with $K > 0$ and $\theta \in (0,1)$, it is enough to replace respectively every α_i and ω_{ij} by $\tilde{\alpha}_i$ and $\tilde{\omega}_{ij}$ with

$$\tilde{\alpha}_i(\varepsilon) = \alpha_i(\varepsilon)\varepsilon^{-\theta/2N} \ge \alpha_i(\varepsilon) \quad \text{and} \quad \tilde{\omega}_{ij}(\varepsilon) = \omega_{ij}(\varepsilon)\varepsilon^{-\theta/N} \ge \omega_{ij}(\varepsilon).$$

Unfortunately, (3.2) is not necessarily preserved under such change of α_i and ω_{ij}. Therefore, in the context of notations and definitions given in Section 2, we state the final result as follows:

Theorem 3.4. *Suppose that* (2.1) *holds and the modulus of continuity* α_{F_i}, $\omega_{A_{ij}}$ *satisfy* $\liminf_{\varepsilon\to 0} M(\varepsilon) = 0$ *where* $M(\varepsilon)$ *is defined by* (3.18). *Then the Cauchy problem* (CP) *has a unique entropy solution in the sense of Definition* 2.1.

Proof. The proof falls naturally into two parts:

(1) Uniqueness. We have divided this first part into steps and sequence of lemmas:

Step 1. In the following lemma, we provide the so called Kato's Inequality for two entropy solutions in the anisotropic case.

Lemma 3.5. *Let* (u_0, f), (v_0, g) *satisfy* (2.1). *Let* u, v *be entropy solutions of* (CP)(A, F, f, u_0), (CP)(A, F, g, v_0) *respectively. Then*

$$\begin{aligned}\int_Q H_0(u-v)\bigg(&\sum_{i=1}^{N}\big(F_i(u) - F_i(v)\big)\partial_{x_i}\xi \\ &- \sum_{i,j=1}^{N}\big(A_{ij}(u) - A_{ij}(v)\big)\partial^2_{x_i x_j}\xi - (u-v)\partial_t\xi\bigg)\,dx\,dt \\ \le \int_{\mathbb{R}^N} (u_0 - v_0)^+\xi(0)\,dx &+ \int_Q \kappa\big(f(u) - g(v)\big)\xi\,dx\,dt \qquad (3.19)\end{aligned}$$

for any $\kappa \in H(u-v)$ *a.e. and* $0 \le \xi \in D(Q)$.

See the Appendix for the proof of Lemma 3.5.

Step 2. We express the comparison and contraction principles as below.

Lemma 3.6. *Let* (u_0, f), (v_0, g) *satisfy* (2.1) *and* u, v *be entropy solutions with respect to* $(\mathrm{CP})(A, F, f, u_0)$, $(\mathrm{CP})(A, F, g, v_0)$. *Assume that* (3.3) *holds. Then*

$$\int_{\mathbb{R}^N} \big(u(t) - v(t)\big)^+ \, dx \le \int_{\mathbb{R}^N} (u_0 - v_0)^+ \, dx + \int_{Q_t} \kappa\big(f(u) - g(v)\big) \, dx \, ds;$$

for $\kappa \in H(u - v)$ *a.e. so that*

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^N)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|f(u) - g(v)\|_{L^1(\mathbb{R}^N)} \, ds.$$

Proof. Set $W = (u - v)^+$, $W_0 = (u_0 - v_0)^+$, $h = \big(f(u) - g(v)\big)^+$.

Since u, v are entropy solutions of (CP), by Lemma 3.5, we have

$$0 \le \int_Q \Big(\sum_{i=1}^N |F_i(u) - F_i(v)| \, |\partial_{x_i} \xi| + \sum_{i,j=1}^N |A_{ij}(u) - A_{ij}(v)| \, |\partial^2_{x_i x_j} \xi| + W \partial_t \xi + h\xi \Big) \, dx \, dt,$$

for any $\xi \in D(Q)$, $\xi \ge 0$ and $\xi(0) = 0$.

From the subadditivity of modulus of continuity, for all $i, j \in I_N$, we have

$$|F_i(u) - F_i(v)| \le (W + \varepsilon) \frac{\alpha_{F_i}(\varepsilon)}{\varepsilon} \quad \text{and}$$

$$|A_{ij}(u) - A_{ij}(v)| \le (W + \varepsilon) \frac{\omega_{A_{ij}}(\varepsilon)}{\varepsilon}, \tag{3.20}$$

The using of these inequalities in the previous one leads to (3.4). Therefore, the first inequality follows by application of Lemma 3.1. □

Step 3. We deduce from Lemma 3.6 uniqueness of entropy solution of $(\mathrm{CP})(A, F, f, u_0)$.

Lemma 3.7. *Let* u *and* v *be entropy solutions with respect to* $(\mathrm{CP})(A, F, f, u_0)$, $(\mathrm{CP})(A, F, f, v_0)$.

If $u_0 \le v_0$ *a.e. in* $\mathbb{R}^N$ *then* $u \le v$ *a.e. on* Q.

If u *is an entropy solution of* $(CP)(A, F, f, u_0)$, *then* u *is unique.*

Proof. Suppose that $u_0 \le v_0$ a.e. in $\mathbb{R}^N$. Since $f \in \mathrm{Lip}(\mathbb{R})$, there exists a positive constant $c(f)$ such that

$$\int_{Q_t} \kappa\big(f(u) - f(v)\big) \, dx \, ds \le c(f) \int_0^t \int_{\mathbb{R}^N} (u - v)^+ \, dx \, ds$$

for any $\kappa \in H(u-v)$ a.e. so from (25) we get

$$\int_{\mathbb{R}^N} \big(u(t)-v(t)\big)^+ \, dx \le c(f) \int_0^t \int_{\mathbb{R}^N} (u-v)^+ \, dx \, ds.$$

We deduce by Gronwall's inequality that $u \le v$ a.e. on Q and uniqueness of entropy solution of $(CP)(A, F, f, u_0)$ follows. □

(2) Existence. Existence is given in the same way as in [11]. □

Remark 3.8. The result of Theorem 3.4 remains true under the slightly weaker assumption $\liminf_{\varepsilon \to 0} M(\varepsilon) < +\infty$, provided $N > 2$ and all the components F_i and coefficients A_{ij} are non-Lipschitz continuous functions. This latter assumption is clearly a technical one and could be removed (see [16] for the isotropic case).

The assumptions on the modulus of continuity are optimal for the purely hyperbolic case. Some relevant counterexamples are indicated in [14] where authors formulate also anisotropic conditions. In the same way, it has been proved in [3] that if such assumptions do not hold, there is no uniqueness of entropy solution in the case $A \equiv 0$, $N = 2$, $F_i = r^{\alpha_i}$, $i = 1, 2$ with $\alpha_i \ge 0$ and $\alpha_1 + \alpha_2 < 1$. In forthcoming works, in the same framework but with L^1-data, we will deal with the well-posedness of this problem in the sense of renormalized solution. We will also be interested by the question of the continuous dependence of the solution with respect to the data.

Appendix

Proof of Lemma 3.5. As in [10] we do the proof of Lemma 3.5 by using the Kruzhkov's method of doubling variables. For any test function $\xi \in D(Q)$ with $\xi \ge 0$, there exists some open ball $B(0, R)$ in $\mathbb{R}^N$ such that $\operatorname{supp}(\xi) \subset (0, T) \times B(0, R)$. Let u, v be entropy solutions of (CP) and

$$Q_1 = (0, T) \times B(0, R_1), \qquad Q_2 = (0, T) \times B(0, R_2).$$

Note also that by density, (2.11) is still true with $\gamma \in W^{2,\infty}(\mathbb{R})$. Besides, since we are working with test functions which have compact supports, we can only ask γ to be $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$. In particular, (2.11) still holds with γ^1_ε, γ^2_ε which are in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$. So we use:

- $\gamma_\varepsilon^1(u,k) = \gamma_\varepsilon^1\big(u(\tau, y), k\big)$ in (2.11); we get

$$\int_{Q_1} \Big[-\gamma_\varepsilon^1(u,k)\partial_\tau\xi - \sum_{i=1}^N \theta_\varepsilon^{i1}(u,k)\partial_{y_i}\xi + \sum_{i,j=1}^N H_\varepsilon^1(u-k)\partial_{y_i}A_{ij}(u)\partial_{y_j}\xi - \gamma_\varepsilon^{1'}(u,k)f(u)\xi\Big]\,dy\,d\tau \le -\int_{Q_1} H_\varepsilon^{1'}(u-k)\sum_{l=1}^N\big(\operatorname{div}\rho_l\big(u(t,x)\big)\big)^2\xi\,dy\,d\tau \tag{3.21}$$

for any $k \in \mathbb{R}$ and $\xi = \xi(\tau, y) \in D(Q)$ with $\xi \ge 0$, $\operatorname{supp}(\xi) \subset Q_1$.

- $\gamma_\varepsilon^2(v,k) = \gamma_\varepsilon^2\big(v(s,z),k\big)$ in (2.11); we get

$$\int_{Q_2} \Big[-\gamma_\varepsilon^2(v,k)\partial_s\xi - \sum_{i=1}^N \theta_\varepsilon^{i2}(v,k)\partial_{z_i}\xi + \sum_{i,j=1}^N H_\varepsilon^2(v-k)\partial_{z_i}A_{ij}(v)\partial_{z_j}\xi - \gamma_\varepsilon^{1'}(v,k)g(v)\xi\Big]\,dz\,ds \le -\int_{Q_2} H_\varepsilon^{2'}(v-k)\sum_{l=1}^N\big(\operatorname{div}\rho_l\big(v(s,z)\big)\big)^2\xi\,dz\,ds \tag{3.22}$$

for any $k \in \mathbb{R}$ and $\xi = \xi(s,z) \in D(Q)$ with $\xi \ge 0$, $\operatorname{supp}(\xi) \subset Q_2$.

We replace k by v in (3.21), k by u in (3.22), we integrate respectively over (s,z) and (τ, y).

In (2.4) and (2.5), we have for all $r, k \in \mathbb{R}$ and $i = 1, \dots, N$,

$$\begin{aligned} H_\varepsilon^1(r) &= -H_\varepsilon^2(-r), & \gamma_\varepsilon^1(r,k) &= \gamma_\varepsilon^2(k,r), \\ \theta_\varepsilon^{i1}(r,k) &= \theta_\varepsilon^{i2}(k,r), & v_\varepsilon^{i1}(r,k) &= v_\varepsilon^{i2}(k,r). \end{aligned} \tag{3.23}$$

When ε goes to 0,

$$\begin{aligned} H_\varepsilon^1(r) &\to H_0(r) \text{ a.e.}, \\ \gamma_\varepsilon^1(r,k) &\to \gamma(r,k) = H_0(r-k)(r-k) \text{ a.e.}, \\ \theta_\varepsilon(r,k) &\to \theta(r,k) = H_0(r-k)\big(F(r)-F(k)\big) \text{ a.e.}, \\ v_\varepsilon(r,k) &\to v(r,k) = H_0(r-k)\big(A(r)-A(k)\big) \text{ a.e.} \end{aligned} \tag{3.24}$$

Keeping in mind (3.23), and adding the two resulting inequalities yields:

$$\int_{Q_1\times Q_2}\Big\{-\gamma_\varepsilon^1(u,v)(\partial_\tau+\partial_s)\xi-\sum_{i=1}^N\theta_\varepsilon^{i1}(u,v)(\partial_{y_i}+\partial_{z_i})\xi-H_\varepsilon^1(u-v)\big(f(u)-g(v)\big)\xi$$
$$+\sum_{i,j=1}^N H_\varepsilon^1(u-v)[\partial_{y_i}A_{ij}(u)\partial_{y_j}\xi-\partial_{z_i}A_{ij}(v)\partial_{z_j}\xi]\Big\}\,dy\,d\tau\,dz\,ds$$
$$\le-\int_{Q_1\times Q_2}H_\varepsilon^{1'}(u-v)\sum_{l=1}^N\Big[\big(\operatorname{div}\rho_l\big(u(\tau,y)\big)\big)^2+\big(\operatorname{div}\rho_l\big(v(s,z)\big)\big)^2\Big]\xi\,dy\,d\tau\,dz\,ds$$
$$\le-2\int_{Q_1\times Q_2}H_\varepsilon^{1'}(u-v)\sum_{l=1}^N\operatorname{div}\rho_l\big(u(\tau,y)\big)\operatorname{div}\rho_l\big(v(s,z)\big)\xi\,dy\,d\tau\,dz\,ds,\tag{3.25}$$

for any $k\in\mathbb{R}$ and $\xi=\xi(\tau,y,s,z)\in D(Q\times Q)$ with $\operatorname{supp}(\xi)\subset Q_1\times Q_2$, $\xi\ge0$.

An elementary decomposition of the last term of the left-hand side of (3.25) gives us

$$\int_{Q_1\times Q_2}\Big\{-\gamma_\varepsilon^1(u,v)(\partial_\tau+\partial_s)\xi-\sum_{i=1}^N\theta_\varepsilon^{i1}(u,v)(\partial_{y_i}+\partial_{z_i})\xi-H_\varepsilon^1(u-v)\big(f(u)-g(v)\big)\xi$$
$$+\sum_{i,j=1}^N H_\varepsilon^1(u-v)[\partial_{y_i}A_{ij}(u)-\partial_{z_i}A_{ij}(v)](\partial_{y_j}+\partial_{z_j})\xi\Big\}\,dy\,d\tau\,dz\,ds$$
$$\le J_1^\varepsilon(u,v)+J_2^\varepsilon(u,v)+J_3^\varepsilon(u,v),\tag{3.26}$$

where $J_i^\varepsilon(u,v)=\int_{Q_1\times Q_2}j_i^\varepsilon(u,v)\,dy\,d\tau\,dz\,ds$, $i=1,2,3$, with

$$j_1^\varepsilon(u,v)=-2H_\varepsilon^{1'}(u-v)\sum_{l=1}^N\operatorname{div}\rho_l\big(u(\tau,y)\big)\operatorname{div}\rho_l\big(v(s,z)\big)\xi$$
$$j_2^\varepsilon(u,v)=\sum_{i,j=1}^N H_\varepsilon^1(u-v)\partial_{y_i}A_{ij}(u)\partial_{z_j}\xi$$
$$j_3^\varepsilon(u,v)=-\sum_{i,j=1}^N H_\varepsilon^1(u-v)\partial_{z_i}A_{ij}(v)\partial_{y_j}\xi.$$

Consider λ_0, $\lambda>0$. We take

$$\xi(\tau,y,s,z)=\phi\Big(\frac{\tau+s}{2},\frac{y+z}{2}\Big)\delta_{\lambda_0}\Big(\frac{\tau-s}{2}\Big)\omega_\lambda\Big(\frac{y-z}{2}\Big)\tag{3.27}$$

where

$$0 \le \phi \in D(Q), \qquad \operatorname{supp}(\phi) \subset Q_1 \cap Q_2, \qquad \omega_\lambda(x) = \frac{1}{\lambda^N}\delta\left(\frac{x_1}{\lambda}\right)\dots\delta\left(\frac{x_N}{\lambda}\right)$$

with

$$0 \le \delta \in D(\mathbb{R}), \qquad \delta(s) = \delta(-s), \qquad \delta(s) = 0 \quad \text{for } |s| \ge 1 \qquad \text{and} \qquad \int_{\mathbb{R}} \delta(s)\,ds = 1.$$

Using (2.10) and after integration by parts and also the fact that $\partial_{y_i}\omega = -\partial_{z_i}\omega$, we get

$$\begin{aligned}
&\int_{Q_1\times Q_2} \Big\{-\gamma_\varepsilon^1(u,v)(\partial_\tau + \partial_s)\phi - \sum_{i=1}^N \theta_\varepsilon^{i1}(u,v)(\partial_{y_i} + \partial_{z_i})\phi - H_\varepsilon^1(u-v)\big(f(u) - g(v)\big)\phi \\
&\qquad - \sum_{i,j=1}^N \nu_\varepsilon^{i,j,1}(u,v)[(\partial^2_{y_iy_j} + \partial^2_{z_jy_i})\phi + (\partial^2_{y_jz_i} + \partial^2_{z_iz_j})\phi]\Big\}\delta_{\lambda_0}\omega_\lambda\,dy\,d\tau\,dz\,ds \\
&\quad \le J_1^\varepsilon(u,v) + J_2^\varepsilon(u,v) + J_3^\varepsilon(u,v). \qquad (3.28)
\end{aligned}$$

We give the following lemma that we will prove later.

Lemma 3.9. *As $\varepsilon \to 0$,*

$$j_1^\varepsilon(u,v) + j_2^\varepsilon(u,v) + j_3^\varepsilon(u,v) \to 0 \qquad \text{in } L^1(Q_1 \times Q_2). \qquad (3.29)$$

Now let ε go to 0 in (3.28). With (3.24), (3.27) and (3.29), we obtain

$$\begin{aligned}
&\int_{Q_1\times Q_2} \Big\{-\gamma(u,v)(\partial_\tau + \partial_s)\phi - \sum_{i=1}^N \theta_i(u,v)(\partial_{y_i} + \partial_{z_i})\phi - H_0(u-v)\big(f(u) - g(v)\big)\phi \\
&\qquad - \sum_{i,j=1}^N \nu^{i,j1}(u,v)[(\partial^2_{y_iy_j} + \partial^2_{z_iy_j})\phi + (\partial^2_{y_iz_j} + \partial^2_{z_iz_j})\phi]\Big\} \\
&\qquad \times \delta_{\lambda_0}\omega_\lambda\,dy\,d\tau\,dz\,ds \le 0. \qquad (3.30)
\end{aligned}$$

To end this proof, we make the change of variables

$$x = \frac{y+z}{2}, \qquad t = \frac{\tau+s}{2}, \qquad \tilde{x} = \frac{y-z}{2}, \qquad \tilde{t} = \frac{\tau-s}{2}. \qquad (3.31)$$

We get the desired inequality in Lemma 3.5 when $\lambda_0 \downarrow 0$, $\lambda \downarrow 0$ respectively in (3.30). □

We now give the proof of Lemma 3.9.

Proof of Lemma 3.9. Notice that

$$J_1^\varepsilon(u,v) = -2\int_{Q_1\times Q_2} H_\varepsilon^{1'}(u-v)\sum_{l=1}^{N} \operatorname{div}\rho_l\big(u(\tau,y)\big)\operatorname{div}\rho_l\big(v(s,z)\big)\xi\,dy\,d\tau\,dz\,ds$$

$$= -2\int_{Q_1\times Q_2}\sum_{l=1}^{N}\sum_{i,j=1}^{N}\partial_{y_i}\rho_{il}(u)\Big(\partial_{z_j}\int_u^v H_\varepsilon^{1'}(u-\zeta)\sigma_{jl}(\zeta)\,d\zeta\Big)\xi\,dy\,d\tau\,dz\,ds.$$

Set $G_{jl}^\varepsilon(r) = \int_r^v H_\varepsilon^{1'}(r-\zeta)\sigma_{jl}(\zeta)\,d\zeta$.

After an integration by parts over z,

$$J_1^\varepsilon(u,v) = 2\sum_{l=1}^{N}\sum_{i,j=1}^{N}\int_{Q_1\times Q_2}\partial_{y_i}\rho_{il}(u)G_{jl}^\varepsilon(u)\partial_{z_j}\xi\,dy\,d\tau\,dz\,ds.$$

Since $H_\varepsilon^{1'}\in L^\infty(\mathbb{R})$ and $\sigma_{jl}\in L^\infty_{\mathrm{loc}}(\mathbb{R})$, we have $G_{jl}^\varepsilon\in C(\mathbb{R})$. Thus we can use (2.10) and integrate by parts over y to obtain

$$J_1^\varepsilon(u,v) = -2\sum_{l=1}^{N}\sum_{i,j=1}^{N}\int_{Q_1\times Q_2}\int_v^u G_{jl}^\varepsilon(r)\sigma_{il}(r)\,dr\partial^2_{y_iz_j}\xi\,dy\,d\tau\,dz\,ds.$$

Let ε go to 0. We can proof the following lemma.

Lemma 3.10. *Let $h:\mathbb{R}\to\mathbb{R}$ be a locally integrable function. For each fixed $b\in\mathbb{R}$,*

$$\lim_{\varepsilon\downarrow 0}\int_a^b H_\varepsilon^{1'}(s-a)h(s)\,ds = H_0(b-a)h(a)\qquad \textit{for a.e } a\in\mathbb{R}.$$

Proof. Set $I =]a,b[$. Since $C^\infty(I)$ is dense in $L^1(I)$, let $h_\varepsilon\in C^\infty(I)$ such that $h_\varepsilon\to h$ in $L^1(I)$ when $\varepsilon\downarrow 0$. After integration by parts of $\int_a^b H_\varepsilon^{1'}(s-a)h_\varepsilon(s)\,ds$, result follows by using the convergence dominated theorem. □

From Lemma 3.10,

$$G_{jl}^\varepsilon(r)\to -H_0(r-v)\sigma_{jl}(r)\qquad \text{for a.e } r\in\mathbb{R}.$$

By the convergence dominated theorem, we deduce that

$$\lim_{\varepsilon\downarrow 0} J_1^\varepsilon(u,v) = 2\sum_{i=1}^{N}\sum_{i,j=1}^{N}\int_{Q_1\times Q_2}\Big(\int_v^u H_0(r-v)\sigma_{il}(r)\sigma_{jl}(r)\,dr\Big)\partial^2_{y_iz_j}\xi\,dy\,d\tau\,dz\,ds.$$

In the same way, we can show that

$$\lim_{\varepsilon\downarrow 0} J_2^\varepsilon(u,v) = -\sum_{l=1}^{N}\sum_{i,j=1}^{N}\int_{Q_1\times Q_2}\Big(\int_v^u H_0(r-v)\sigma_{il}^2(r)\sigma_{jl}^2(r)\,dr\Big)\partial^2_{y_iz_j}\xi\,dy\,d\tau\,dz\,ds$$

and

$$\lim_{\varepsilon\downarrow 0} J_3^\varepsilon(u,v) = -\sum_{l=1}^{N}\sum_{i,j=1}^{N}\int_{Q_1\times Q_2}\Big(\int_v^u H_0(r-v)\sigma_{il}(r)\sigma_{jl}(r)\,dr\Big)\partial^2_{y_iz_j}\xi\,dy\,d\tau\,dz\,ds.$$

Thus

$$\lim_{\varepsilon\downarrow 0}\big(J_1^\varepsilon(u,v)+J_2^\varepsilon(u,v)+J_3^\varepsilon(u,v)\big)=0. \qquad \square$$

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Received February 14, 2012; revised March 26, 2012

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