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Sharkovskii order for non-wandering points

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Abstract. For a map $f : I \to I$, a point $x \in I$ is periodic with period $p \in \mathbb{N}$ if $f^p(x) = x$ and $f^j(x) \neq x$ for all $0 < j < p$. When f is continuous and I is an interval, a theorem due to Sharkovskii ([1]) states that there is an order in $\mathbb{N}$, say $\lhd$, such that if f has a periodic point of period p and $p \lhd q$, then f also has a periodic point of period q. In this work, we will see how an extension of the order $\lhd$ to sequences of positive integers yields a Sharkovskii-type result for non-wandering points of f.

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1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a continuous map. A point $x_0 \in [a, b]$ is *non-wandering* if, for each neighborhood $\mathscr{V}$ of x_0, there is a positive integer N such that $f^N(\mathscr{V}) \cap \mathscr{V} \neq \emptyset$. If moreover $f^k(\mathscr{V}) \cap \mathscr{V} = \emptyset$ for all $k \in \{1, 2, \dots, N-1\}$, we say that N is a *first return* of $\mathscr{V}$ to itself. This notion is a weak form of recurrence and gathers recurrent points (the ones that are accumulated by their orbits) and the periodic ones. The aim of this work is to generalize Sharkovskii's Theorem to non-wandering points, replacing periodic points by neighborhoods, and periods by first return times.

The main difficulty of such a formulation lies on the control of the speed of the returns and their nearness to the starting non-wandering point, parameters that, in the case of a periodic orbit, are not only elementary to express but completely determined by the period. A straight extension of Sharkovskii's result should state that, given two sequences $(R_n)_{n \in \mathbb{N}}$ and $(S_n)_{n \in \mathbb{N}}$ of positive integers such that $R_n \lhd S_n$ for all n, if $f : [a, b] \to \mathbb{R}$ has a non-wandering point with a fundamental system of neighborhoods whose first returns happen at times $(R_n)_{n \in \mathbb{N}}$, then f has a

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non-wandering point with a fundamental system of neighborhoods first returning at times $(S_n)_{n\in\mathbb{N}}$. This is true if the sequences $(R_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ are eventually constant (equal to c and d, respectively, with $c \lhd d$). In fact, if, for n big enough, each neighborhood $\mathscr{V}_n$ of a fundamental system of x_0 has a first return by the power f^c, then there is $y_n \in \mathscr{V}_n$ (so the sequence $(y_n)_{n\in\mathbb{N}}$ converges to x_0) such that $f^c(y_n) \in \mathscr{V}_n$ (thus the sequence $\left(f^c(y_n)\right)_{n\in\mathbb{N}}$ also converges to x_0), and, therefore, as f is continuous, x_0 is periodic with period c; hence, as $c \lhd d$, Sharkovskii's Theorem informs that f has a periodic point with period d, to whom we may easily find a fundamental system of neighborhoods first returning by f^d. (Similar reasoning holds if $(R_n)_{n\in\mathbb{N}}$ has a bounded subsequence.)

For more general sequences of returns, our argument demands some control of the size of the neighborhoods with respect to the amount of time a return needs to occur. Given $\varepsilon > 0$, we consider a fundamental system of dynamical balls centered at the non-wandering point x_0, say $(\mathscr{V}_n)_{n\in\mathbb{N}}$, determined by ε and the uniform continuity of f^j, for all $1 \le j \le S_n$ (see Lemma 3.1). The connection between $(\mathscr{V}_n)_{n\in\mathbb{N}}$, $(R_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ will be established by the following conditions, which essentially assert that R_n is a first return of $\mathscr{V}_n$ with respect to both the usual and Sharkovskii's order:

(F1) For each n, R_n is the first return of $\mathscr{V}_n$.

(F2) For each n, the neighborhood $\mathscr{V}_n$ does not return by the iteration S_n.

Therefore, we have:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, x_0 a non-wandering point of f, $(R_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ two sequences of positive integers such that, for all $n \in \mathbb{N}$, $R_n \lhd S_n$.*

(a) *If condition* (F1) *holds, then we may find a subsequence $(S_{n_k})_{k\in\mathbb{N}}$ and a point x_1 in $[a,b]$ with a neighborhood whose length is arbitrarily close to 2ε and returns at times S_{n_k} for all k.*

(b) *If, besides condition* (F1), *we assume* (F2), *then there is a periodic point by f with period S_n for all n.*

The proof goes as follows. Starting with a non-wandering point x_0, we create, through countably many small local perturbations of f (see Lemma 3.2), a sequence $(f_m)_{m\in\mathbb{N}}$ of continuous maps of $[a,b]$, converging to f and such that f_m has a periodic point with period R_m. To each of them we apply Sharkovskii's Theorem, ensuring the existence of another periodic point z_m of f_m with period S_m. Then, under assumption F1, we take an accumulation point x_1 of $(z_m)_{m\in\mathbb{N}}$ and verify that it has the sought neighborhood with the requested return times by f. If we assume both (F1) and (F2), then we guarantee that each z_m was created out of the neighborhoods of perturbation of f, and so conclude that it is a periodic point of f.

2. Examples

(1) The point x_1 just obtained may coincide with x_0. However, properties of either x_0 or the returns may prevent this to happen.

If $(S_n)_{n\in\mathbb{N}}$ is eventually periodic, then x_1 is periodic by f.

Suppose that $(S_n)_{n\in\mathbb{N}}$ is eventually periodic, say equal to $\overline{\beta_1\beta_2\ldots\beta_\ell}$. Then, taking a subsequence if necessary, we may assume that there is β_j such that each z_{n_k} is periodic by f_{n_k} with period β_j, for some $j \in \{1,2,\ldots,\ell\}$. Then, as $(f_{n_k})_k$ converges to f, $(z_{n_k})_k$ converges to x_1 and $f_{n_k}^{\beta_j}(z_{n_k}) = z_{n_k}$ for all k, we conclude that x_1 is periodic by f with period that divides β_j.

Thus, in particular, if the initial point x_0 is not periodic, or has a period that does not divide β_j, then $x_1 \neq x_0$.

For instance, assume that $(R_n)_{n\in\mathbb{N}} = (3^n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}} = \overline{24}$. As the R_n's are first returns of a fundamental system of neighborhoods of x_0, then x_0 cannot be a periodic point with period in $\{1,2,4\}$. Theorem 1.1 ensures that f has a periodic point x_1 with period that divides 4, hence $x_1 \neq x_0$.

If x_0 is not periodic, then $x_1 \neq x_0$.

Indeed, in this case $(R_n)_{n\in\mathbb{N}}$ cannot have a bounded subsequence, and so $\lim_{n\to\infty} R_n = \infty$. Therefore, fixing any $p \in \mathbb{N}$, we may consider the constant sequence $(S_n)_{n\in\mathbb{N}} = (2^p)_{n\in\mathbb{N}}$ and be sure that $(R_n)_{n\in\mathbb{N}} \lhd (S_n)_{n\in\mathbb{N}}$. Thus x_1 is a periodic point by f, with a period that divides 2^p, and so $x_1 \neq x_0$.

(2) Condition (F1) is hard to check but seems to be the natural extension to returns of the connection between periods described by Sharkovskii's order.

Yet, it would be interesting to unlink the size of the sets $(\mathscr{V}_n)_{n\in\mathbb{N}}$ from the sequence $(S_n)_{n\in\mathbb{N}}$ because what we need is that these neighborhoods have first returns $(R_n)_{n\in\mathbb{N}}$. This is achieved if we demand that the sequences $(R_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ are related by both the Sharkovskii's order and the usual one: if, for each n, we have $R_n \lhd S_n$ and $S_n < R_n$, then the definition of the neighborhoods $\mathscr{V}_n$ may use R_n instead of S_n. (And moreover condition (F2) holds since R_n is the first return of $\mathscr{V}_n$.)

(3) The assumption (F2) prevents the perturbations of f to destroy periodic orbits of f with periods given by the sequence $(S_n)_{n\in\mathbb{N}}$.

For instance, if $(R_n)_{n\in\mathbb{N}} = (5 \times 2^n)_{n\in\mathbb{N}}$, $(S_n)_{n\in\mathbb{N}} = (7 \times 2^n)_{n\in\mathbb{N}}$, the R_n's are first returns of the fundamental system of neighborhoods $(\mathscr{V}_n)_{n\in\mathbb{N}}$ and, for every

n, we have $f^{S_n}(\mathscr{V}_n) \cap \mathscr{V}_n = \emptyset$, then f has a periodic point with period 7 (and so infinitely many others).

3. The size of the neighborhoods

In the sequel we will consider the uniform norm $\|g\| = \max\{|g(x)| : a \leq x \leq b\}$ on the space of real continuous maps g defined on the interval $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be one of such maps. As f is uniformly continuous in $[a, b]$, given $\tau > 0$ there exists $\eta(\tau) > 0$ such that, if x and y belong to $[a, b]$ and $|x - y| < \eta(\tau)$, then $|f(x) - f(y)| < \frac{\tau}{2}$.

Take a positive ε and consider a sequence $(S_n)_{n \in \mathbb{N}}$ of positive integers.

Definition 3.1. For each $n \in \mathbb{N}$, $\delta(\varepsilon, S_n)$ denotes the minimum of the set with $S_n + 1$ elements given by $\rho_1 = \frac{1}{n}$, $\rho_2 = \frac{\varepsilon}{2}$ and, for $3 \leq k \leq S_n + 1$,

$$\rho_k = \frac{1}{2}\eta(2\rho_{k-1}).$$

Lemma 3.1. *If $g : [a, b] \to \mathbb{R}$ is continuous and $\|f - g\| < \delta(\varepsilon, S_n)$, then, for all $k \in \{1, \ldots, S_n\}$, we have $\|f^k - g^k\| < \varepsilon$.*

Proof. Fix n and the corresponding S_n. If $k = 1$, the assertion is a direct consequence of the fact that $\delta(\varepsilon, S_n) < \varepsilon$. For $k = 2$, as $\|f - g\| < \delta(\varepsilon, S_n)$, we know that $\|f - g\| < \frac{\varepsilon}{2}$ and $\|f - g\| < \eta(\varepsilon)$. Therefore, by the definition of $\eta(\varepsilon)$, we have $\|f \circ f - f \circ g\| < \frac{\varepsilon}{2}$; besides, $\|f \circ g - g \circ g\| \leq \|f - g\| < \frac{\varepsilon}{2}$. So

$$\|f^2 - g^2\| \leq \|f \circ f - f \circ g\| + \|f \circ g - g \circ g\| < \varepsilon.$$

Similarly, from $\|f - g\| < \delta(\varepsilon, S_n)$, we deduce that $\|f - g\| < \frac{\varepsilon}{2}$, $\|f - g\| < \frac{\eta(\varepsilon)}{2}$ and $\|f - g\| < \eta(\eta(\varepsilon))$, which together imply, as just checked, that

$$\|f^2 - g^2\| \leq \|f \circ f - f \circ g\| + \|f \circ g - g \circ g\| < \frac{\eta(\varepsilon)}{2} + \frac{\eta(\varepsilon)}{2} = \eta(\varepsilon),$$

and so

$$\|f^3 - g^3\| \leq \|f \circ f^2 - f \circ g^2\| + \|f \circ g^2 - g \circ g^2\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

The argument proceeds inductively. □

Given x_0 and $n \in \mathbb{N}$, let $\mathscr{V}_n =]x_0 - \delta(\varepsilon, S_n), x_0 + \delta(\varepsilon, S_n)[$. Assume that each $\mathscr{V}_n$ first returns at time R_n. We may choose a sequence $(y_n)_{n \in \mathbb{N}}$ of elements of

$[a,b]$ such that, for all n, both y_n and $f^{R_n}(y_n)$ belong to $\mathscr{V}_n$. The next Lemma tells us how, by a small perturbation of f, we may bind the extremes of the finite block $\{y_n, f(y_n), \ldots, f^{R_n}(y_n)\}$ of the orbit of y_n, thus creating a periodic point for a close dynamics.

Lemma 3.2. *For each n, there is a continuous map $f_n : [a,b] \to \mathbb{R}$ such that $f_n^{R_n}(y_n) = y_n$ and $\|f - f_n\| < \delta(\varepsilon, S_n)$.*

Proof. As y_n and $f^{R_n}(y_n)$ belong to the open interval $\mathscr{V}_n$, we may find a $\zeta > 0$ such that the intervals I_n (the one that connects these two points inside $[a,b]$, which may be degenerate if $y_n = f^{R_n}(y_n)$) and J_n (which we obtain from I_n adding to it two short segments, with length ζ, on its extremes) are contained in $\mathscr{V}_n$. Consider a continuous bump-function $\phi_n : [a,b] \to \mathbb{R}$ so that the restriction of ϕ_n to I_n is constant and equal to 1, and the value of ϕ_n in $[a,b]\backslash J_n$ is zero. Denote by $T_n : [a,b] \to \mathbb{R}$ the map

$$T_n(t) = t + [y_n - f^{R_n}(y_n)] \times \phi_n(t).$$

The function T_n is the identity in the complement of J_n and translates the elements of J_n by an amount that does not exceed $|y_n - f^{R_n}(y_n)|$.

Define now the map $f_n : [a,b] \to \mathbb{R}$ by $f_n = T_n \circ f$. This is a continuous function and, as R_n is the first return of $\mathscr{V}_n$ to itself, we have, for any integer ℓ such that $1 \leq \ell < R_n$,

$$f_n^l(y_n) = T_n\big(f^l(y_n)\big) = f^l(y_n) \neq y_n,$$

and

$$f_n^{R_n}(y_n) = (T_n \circ f)^{R_n}(y_n) = T_n\big(f^{R_n}(y_n)\big) = f^{R_n}(y_n) + [y_n - f^{R_n}(y_n)] \times 1 = y_n.$$

Moreover, f_n coincides with f in $[a,b]\backslash f^{-1}(\mathscr{V}_n)$ since, if $t \notin f^{-1}(\mathscr{V}_n)$, then $\phi\big(f(t)\big) = 0$ and therefore

$$f_n(t) = T_n\big(f(t)\big) = f(t) + [y_n - f^{R_n}(y_n)] \times \phi\big(f(t)\big) = f(t).$$

Furthermore, if $t \in f^{-1}(\mathscr{V}_n)$, then

$$\begin{aligned} |f_n(t) - f(t)| &= \big|f(t) + [y_n - f^{R_n}(y_n)] \times \phi\big(f(t)\big) - f(t)\big| \\ &= \big|[y_n - f^{R_n}(y_n)] \times \phi\big(f(t)\big)\big| \leq |y_n - f^{R_n}(y_n)| \\ &< \delta(\varepsilon, S_n). \end{aligned}$$

So $\|f_n - f\| < \delta(\varepsilon, S_n)$. □

Corollary 3.3. *The functions just defined verify:*

(a) *The sequence $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f.*

(b) *For each $n \in \mathbb{N}$, the map f_n has a periodic point z_n with period S_n.*

Remark 3.1. If, for each n, the point y_n is already periodic by f with period R_n, then $f_n = f$ and we know, without using the previous lemmas, that f has a periodic point z_n with period S_n for all n.

4. Proof of Theorem 1.1

Proof. (a) Take any accumulation point $x_1 \in [a, b]$ of $(z_n)_{n \in \mathbb{N}}$ and consider a sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ verifying the condition

$$z_{n_k} \in \left]x_1 - \frac{1}{k}, x_1 + \frac{1}{k}\right[\quad \text{for all } k \in \mathbb{N}.$$

By Lemma 3.2, for each n_k, we have $\|f_{n_k} - f\| < \delta(\varepsilon, S_{n_k})$, and so, by Lemma 3.1, we get $\|f_{n_k}^{S_{n_k}} - f^{S_{n_k}}\| < \varepsilon$. Consequently, as z_{n_k} is a periodic point with period S_{n_k} for the map f_{n_k}, the neighborhood $\mathscr{W}_k = \left]x_1 - \frac{1}{k} - \varepsilon, x_1 + \frac{1}{k} + \varepsilon\right[$ of x_1 returns to itself by $f^{S_{n_k}}$. Hence x_1 and the neighborhoods $\mathscr{W}_k$ are the ones we were looking for.

(b) As the neighborhood $\mathscr{V}_n$ does not return by the iterate S_n, the orbit of the point z_n cannot cross it since any point of this orbit is periodic with period S_n. So, the perturbation f_n coincides with f along this orbit, and therefore z_n is already periodic by f. □

5. Motivation

This result emerged from the nonstandard version of the Theorem of Sharkovskii, conveyed to the hyperreals through the known transfer principles. We started using nonstandard analysis and an ultrafilter in $\mathbb{N}$ that contains all the co-finite sets to extend the order $\lhd$ to an ultrapower of the positive integers (see [2] and [3] for more details). Then we reformulated the infinitely many first returns, of a countable family of neighborhoods of a non-wandering point by the map f, as a periodic point, with a hyperinteger period corresponding to the class of $(R_n)_{n \in \mathscr{K}}$ for some *big* set $\mathscr{K} \subseteq \mathbb{N}$, associated to a suitable continuous (and internal) dynamical system acting on an interval of the hyperreals. Applying the transfered version of Sharkovskii's Theorem, we obtained another periodic point for the same dynamics, with period given by the hyperinteger represented by $(S_n)_{n \in \mathscr{K}}$. Finally, this information was projected on f and $[a, b]$, thus arising the requested point x_1 and the corresponding neighborhoods with return times given by a subsequence of $(S_n)_{n \in \mathscr{K}}$. The argument presented in Sections 3 and 4 is the standard statement of the previous lines.

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