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Normalization of the 1-stratum of the moduli space of stable curves

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Abstract. Let $\mathscr{M}_g(\Gamma)$ be the stack of stable curves of genus g with a given dual graph Γ and let $\bar{\mathscr{M}}_g(\Gamma)$ be its closure in $\bar{\mathscr{M}}_g$. We consider the normalization of $\bar{\mathscr{M}}_g(\Gamma)$ in order to classify the residual orbifolds of the normalizations of the irreducible components of the locus in $\bar{\mathscr{M}}_g$ corresponding to curves with at least $3g-4$ nodes.

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1. Introduction

In his famous paper [18], Mumford introduced a stratification of the stack $\bar{\mathscr{M}}_g$ given by the number of nodes. The stratum, $\mathscr{M}_g^n$, corresponding to curves with n nodes has pure dimension $3g-3-n$, but is not irreducible. The irreducible components of $\mathscr{M}_g^n$ are indexed by weighted stable graphs (see Definition 3.1). Precisely, if Γ is a weighted stable graph with n edges and (weighted) genus g, then the substack $\mathscr{M}_g(\Gamma)$ parametrizing curves with dual graph Γ is an irreducible component of $\mathscr{M}_g^n$. The substacks $\mathscr{M}_g(\Gamma)$ and their closures in $\bar{\mathscr{M}}_g$ give a combinatorial decomposition of the stack $\bar{\mathscr{M}}_g$, and a natural question is to describe the irreducible components of Mumford's stratification. Unfortunately, the combinatorics is rather complicated due to the rapid growth of the number of possible graphs Γ for increasing genus and number of nodes.

In Sections 3 and 4, we give a general overview of the moduli substacks of stable curves. We use a well known explicit desingularization of $\bar{\mathscr{M}}_g(\Gamma)$ in the category of Deligne–Mumford stacks (Theorem 4.7). This allows us to verify whether $\bar{\mathscr{M}}_g(\Gamma)$ is singular, knowing the graph Γ (see Lemma 4.10 and [11], Lemma 4.2).

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In Section 6 we focus on the irreducible components $\bar{\mathscr{M}}_g(\Gamma)$ of the 1-stratum of $\bar{\mathscr{M}}_g$ corresponding to curves with at least $3g-4$ nodes. The main result of this paper is to describe the normalization of $\bar{\mathscr{M}}_g(\Gamma)$ as a gerbe over an orbifold (Theorems 6.3 and 6.11). In particular we show that there are only 5 possible residual orbifolds of the normalizations of irreducible components of the 1-stratum (Remark 6.4 and Theorem 6.11). We also give a local presentation for the singular points of $\bar{\mathscr{M}}_g(\Gamma)$ (Proof of Proposition 6.5). We also show that the coarse moduli spaces of these components are all $\mathbb{P}^1$ (see Proposition 6.5 and Remark 6.12).

For an alternative approach to the same problem see Zintl [21], where a description of these components is given as a different quotient stack.

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2. Preliminaries

2.1. Deligne–Mumford stacks. We will work with stacks over a noetherian base scheme S. This means in particular that a stack $\mathscr{X}$ will be considered equipped with a morphism

$$\psi : \mathscr{X} \to S.$$

Stacks are defined as categories fibered in groupoids over a site (with some extra conditions). The base scheme S represents category of schemes of finite presentation over S equipped with the étale topology. For basic definitions of stacks we refer to [7], [3], [16], and Appendix in [20]. Here we gather together a few basic facts.

Definition 2.1. A morphism $F : \mathscr{X} \to \mathscr{Y}$ between two stacks over S is *representable* if for every scheme T and for every morphism $T \to \mathscr{Y}$, the fibered product $\mathscr{X} \times_{\mathscr{Y}} T \to T$ is a scheme.

Many concepts about morphisms of schemes may be applied to representable morphisms of stacks.

Definition 2.2. Let **P** be a property of morphisms of schemes that is stable under base change and of local nature on the target (e.g., flat, smooth, étale, surjective, unramified, normal, locally of finite type, locally of finite presentation). Then we say that a representable morphism of stacks $\mathscr{X} \to \mathscr{Y}$ has property **P** if for every morphism $T \to \mathscr{Y}$, the morphism of schemes deduced by base change $g : T \times_{\mathscr{Y}} \mathscr{X}$ has that property.

Definition 2.3. A stack $\mathscr{X}$ is a *Deligne–Mumford* (*DM*) *stack* if the following conditions are satisfied.

(1) The diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable, quasi-compact, and separated.

(2) There exists an étale surjective morphism $U \to \mathscr{X}$, where U is a scheme.

The scheme U is called an *atlas* for $\mathscr{X}$.

An *algebraic space* is a DM stack which is equivalent to a sheaf.

Remark 2.4. The representability of the diagonal implies that any morphism $U \to \mathscr{X}$ with U a scheme is representable and the morphism $U \to \mathscr{X}$ of Definition 2.3 is étale and surjective in the sense of Definition 2.2.

Remark 2.5. If $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of DM stacks then f has property **P** if for some (and hence every) étale atlas $U \to \mathscr{X}$ the morphism of schemes $U \times_{\mathscr{Y}} \mathscr{X} \to U$ has property **P**.

Let **P** be a property of schemes, local in the étale topology (e.g., regular, smooth, normal, reduced), then we say that a Deligne–Mumford stack has property **P** if and only if the atlas U satisfies **P**.

Remark 2.6. The structure morphism $\psi : \mathscr{X} \to S$ is not representable unless $\mathscr{X}$ is a scheme. So, according to the given definition, we cannot say that $\mathscr{X}$ satisfies **P** if and only if ψ satisfies **P**. However if **P** is a property of local nature, at source and target, for the étale topology (e.g., flat, smooth, étale, unramified, normal, locally of finite type, locally of finite presentation), then we can extend the definitions for morphisms of DM stacks which are not necessarily representable (see [7], p. 100).

Definition 2.7. A stack $\mathscr{X}$ is separated over S if the diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is a finite representable morphism.

By [8] Theorem 2.7, every Deligne–Mumford stack $\mathscr{X}$ admits a finite surjective morphism $Z \to \mathscr{X}$ with Z a scheme. Using this fact we can define the notions of proper and finite morphisms of DM stacks. Our definition of proper morphism is equivalent to the one given in [7].

Definition 2.8. A morphism of DM stacks $\mathscr{X} \to \mathscr{Y}$ is proper (resp. finite) if for some (and hence all) finite surjective morphism $Z \to \mathscr{X}$ with Z a scheme, the composite morphism $Z \to \mathscr{X} \to \mathscr{Y}$ is a representable proper (resp. finite) morphism.

Proposition 2.9. *Let $F : \mathscr{X} \to \mathscr{Y}$ be a finite surjective morphism of DM stacks which is faithful* (*i.e., the functor F is a faithful functor*) *then F is representable.*

To prove Proposition 2.9 we begin with a Lemma.

Lemma 2.10. *Let $F : \mathscr{X} \to \mathscr{Y}$ be a morphism between two DM stacks. Then F is weakly representable* (*a morphism is* weakly representable *if for every morphism $T \to \mathscr{Y}$ with T an algebraic space, the fiber product is represented by an algebraic space*) *if and only if F is faithful.*

Proof. From [16] Corollary (8.1.2), we have that F is weakly representable if and only if the diagonal morphism

$$\Delta_F : \mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X},$$
$$x \in \mathrm{Obj}(\mathscr{X}) \mapsto (x, x, \mathrm{id}_{F(x)}),$$
$$(f : x \to x') \mapsto \big(f, f, F(f)\big)$$

is fully faithful. This condition means exactly that two morphisms

$$f, g \in \mathrm{Hom}_{\mathscr{X}}(x, x')$$

are equal if and only if $F(f) = F(g)$. □

Proof. Proof of Proposition 2.9. By Lemma 2.10 the faithful functor $F : \mathscr{X} \to \mathscr{Y}$ is weakly representable. Let $T \to \mathscr{X}$ be a morphism from a scheme and let $\mathscr{X}_T$ denote the fiber product $T \times_{\mathscr{Y}} \mathscr{X}$. Since F is weakly representable we know that $\mathscr{X}_T$ is an algebraic space. To prove that F is representable we need to show that $\mathscr{X}_T$ is actually a scheme. Working locally on T we may assume that T is affine. Let $Z \to \mathscr{X}$ be a finite surjective morphism from a scheme. Since $Z \to \mathscr{Y}$ is finite, surjective and representable, the fiber product $Z_T = T \times_{\mathscr{Y}} Z$ is represented by a scheme and the map $Z_T \to T$ is finite and surjective. Since T is assumed affine, the scheme Z_T is also affine. Now the morphism $Z_T \to \mathscr{X}_T$ is, by base change, a finite surjective morphism of algebraic spaces. Chevalley's theorem for algebraic spaces [15], Chapter III, Theorem 4.1, implies that $\mathscr{X}_T$ is an affine scheme as well. □

Definition 2.11. Let $\mathscr{X}$ be a DM stack. A geometric point of $\mathscr{X}$ is a morphism

$$\mathrm{Spec}(K) \xrightarrow{x} \mathscr{X},$$

where K is an algebraically closed field. From any such map we can deduce an object ξ in $\mathscr{X}\big(\mathrm{Spec}(K)\big)$. Let G_x be the automorphism group of ξ, we have a monomorphism

$$\mathrm{B}G_x \xrightarrow{\mathrm{rg}_x} \mathscr{X}.$$

We call G_x the *stabilizer* of x and rg_x the *residual gerbe* of x. If G_x is not trivial, we say that x is a *stacky point* of $\mathscr{X}$ (in literature it is also called *twisted point*).

Let $x : \mathrm{Spec}(K) \to \mathscr{X}$ be a geometric point of a Deligne–Mumford stack $\mathscr{X}$, and let U_x be an étale scheme-theoretic neighborhood of x. We have a lifting of x to U_x, so we define

$$\hat{\mathcal{O}}_{x,\mathscr{X}} := \hat{\mathcal{O}}_{x,U_x}.$$

We recall the following fundamental Lemma (see [1], 2.2.2 and 2.2.3). It states that, locally in the étale topology, every separated Deligne–Mumford stack is a quotient stack by a finite group.

Lemma 2.12. *Let $\mathscr{X}$ be a separated Deligne–Mumford stack, and X its coarse moduli space. There is an étale covering $\{X_\alpha \to X\}_{\alpha \in I}$ such that for each $\alpha \in I$ there is a scheme U_α and a finite group G_α, acting on U_α, with the property that the pullback $\mathscr{X} \times_X X_\alpha$ is isomorphic to the quotient stack $[U_\alpha/G_\alpha]$. Moreover, if X is noetherian, X_α is the coarse moduli space of $[U_\alpha/G_\alpha]$ (therefore it is isomorphic to the geometric quotient U_α/G_α).*

Remark 2.13. All the stacks we will consider are proper, therefore separated.

Proposition 2.14. *Let $\mathscr{X}$ be a separated Deligne–Mumford stack with a noetherian coarse moduli space X. Let $x : \mathrm{Spec}(K) \to \mathscr{X}$ be a geometric point and G_x its stabilizer. Then*

(1) *there is a natural action of G_x on $\hat{\mathcal{O}}_{x,\mathscr{X}}$;*

(2) $\hat{\mathcal{O}}_{x,X} \cong (\hat{\mathcal{O}}_{x,\mathscr{X}})^{G_x}$.

Proof.

(1) Following the notation of Lemma 2.12, we consider an étale neighborhood $[U_\alpha/G_\alpha]$ of x in $\mathscr{X}$. G_x is a subgroup of G_α which stabilizes a lifting of x to U_α, therefore we have an induced action of G_x on $\hat{\mathcal{O}}_{x,\mathscr{X}} = \hat{\mathcal{O}}_{x,U_\alpha}$.

(2) We have that an étale neighborhood of x in X is the geometric quotient U_α/G_α. We can also choose $U_\alpha = \mathrm{Spec}(R)$ for some ring R. A groupoid which represents $[\mathrm{Spec}(R)/G]$ is

$$W = \big(\mathrm{Spec}(R) \times_k G\big) \underset{\gamma}{\overset{\pi_1}{\rightrightarrows}} \mathrm{Spec}(R)$$

where π_1 is the first projection and γ is the action. Now, from [13] Proposition 5.1, we have that the geometric quotient is $\mathrm{Spec}(R^W)$. In our case $R^W = R^G$, so the coarse moduli space of $[\mathrm{Spec}(R)/G]$ is $\mathrm{Spec}(R^G)$. In order to conclude we must show that the natural morphism

$$\mathrm{Spec}(R^{G_x}) \to \mathrm{Spec}(R^G)$$

is étale on x, and this is given by [9], Exposé V, Proposition 2.2. □

2.2. Quotients of DM stacks. In the following we will consider quotients of DM stacks by an action of a finite group acting on it. More precisely by *group* we mean a sheaf in groups over the base category. Usually the base is the category of schemes over a base scheme S, which we simply call S (in this case the base category is the stack structure of S). The main reference is [19]. Here we recall some basic definitions (see (loc. cit.) Definitions 2.1 and 2.3).

Definition 2.15. Let $\mathscr{M}$ be a stack over a base scheme S, and let G be a sheaf in groups over S. Let m be the multiplication of G and e its unit section. An *action* of G on $\mathscr{M}$ is a morphism of stacks $\mu : G \times \mathscr{M} \to \mathscr{M}$ with strictly commutative diagrams.

$$\begin{array}{ccc} G \times G \times \mathscr{M} & \xrightarrow{m \times \mathrm{id}} & G \times \mathscr{M} \\ {\scriptstyle \mathrm{id}\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times \mathscr{M} & \xrightarrow{\mu} & \mathscr{M}, \end{array} \qquad \begin{array}{ccc} G \times \mathscr{M} & \xrightarrow{\mu} & \mathscr{M} \\ {\scriptstyle e\times\mathrm{id}}\uparrow & \nearrow_{\mathrm{id}} & \\ \mathscr{M} & & \end{array}$$

We say that $\mathscr{M}$ is a G-stack.

Any stack over S can be seen as a G-stack over S through the trivial action.

Definition 2.16. Let $\mathscr{M}$ and $\mathscr{N}$ be two G-stacks and $\psi : \mathscr{N} \to \mathscr{M}$ a morphism. We say that ψ is a G-morphism ($\psi \in \hom_{G\text{-stacks}}(\mathscr{N}, \mathscr{M})$) if the diagram

$$\begin{array}{ccc} G \times \mathscr{N} & \xrightarrow{\mu_{\mathscr{N}}} & \mathscr{N} \\ {\scriptstyle \mathrm{id}\times\psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ G \times \mathscr{M} & \xrightarrow{\mu_{\mathscr{M}}} & \mathscr{M} \end{array}$$

is strictly commutative.

Remark 2.17. If we consider groupoids over S instead of stacks, the above diagrams are 2-diagrams satisfying some "higher associativity" condition (see [19], Definition 1.3). In the case of stacks we require that the action is strict.

Definition 2.18. Let G be a sheaf in groups over S and let $\mathscr{M}$ be a G-stack over S. A *quotient stack* $\mathscr{M}/G$ is a stack that 2-represents the 2-functor

$$F(\mathscr{N}) = \hom_{G\text{-stacks}}(\mathscr{N}, \mathscr{M}).$$

Proposition 2.19 ([19], Theorem 3.3). *Let G be a sheaf in groups over S, and $\mathscr{M}$ a G-stack over S. Then there exists a quotient stack $\mathscr{M}/G$ and its formation commutes with base change on S.*

Under some hypothesis on G we can extend the above proposition to DM stacks (see [19], Theorem 4.1).

Theorem 2.20. *Let G be an étale group scheme over S. Let $\mathcal{M}$ be a DM G-stack over S. Then the quotient stack $\mathcal{M}/G$ is a DM stack.*

3. Topological classes of stable curves

Definition 3.1. We call *weighted graph*, a graph with a natural number on each vertex. Given a weighted graph, we call $V(\Gamma)$ the set of vertices, $E(\Gamma)$ the set of edges and $w : V(\Gamma) \to \mathbb{N}$ the assignment of weights. Let $c = |V(\Gamma)|$, $n = |E(\Gamma)|$ and $h = \sum_{v \in V(\Gamma)} w(v)$. For each vertex v in $|V(\Gamma)|$ we call *degree* of v ($\deg(v)$) the number of edges starting from v (loops counting twice). We also define *multiplicity* of v to be the number $\operatorname{mult}(v) = 3w(v) + \deg(v)$. We call *weighted genus* of a weighted graph the number

$$g = h + n - c + 1. \tag{1}$$

This is not the standard genus of a graph that we can find in literature, more precisely we are adding the total weight h.

We say that a weighted graph Γ is *stable* if Γ is connected, its genus is ≥ 2 and for every $v \in V(\Gamma)$ we have $\operatorname{mult}(v) \geq 3$.

We call loop graph a graph whose cycles are only loops.

Remark 3.2. There is a subtlety that should be considered at this point. We are thinking of graphs as unlabeled because we want to consider a unique topological class of a stable curve that does not depend on labeling. However in the definition of graph and in the following definition of automorphism, we need to label vertices and edges by fixing the sets $V(\Gamma)$ and $E(\Gamma)$.

Definition 3.3 (see [2], Chap. X, Definition 2.16). An automorphism of a weighted graph Γ is the following set of data:

(1) a one-to-one correspondence $f : V(\Gamma) \to V(\Gamma)$ such that $w(f(v)) = w(v)$ for all vertices of Γ;

(2) a one-to-one correspondence $g : E(\Gamma) \to E(\Gamma)$ such that for all $e \in E(\Gamma)$ the two vertices connected by $g(e)$ are images of the vertices connected by e;

(3) an element of $\mathbb{Z}_2^{l(\Gamma)}$ where $l(\Gamma)$ is the number of the loops of Γ. We are thinking that we can flip each loop and in general that there are two ways to send a loop into another one.

We call $\operatorname{Aut}(\Gamma)$ the group of automorphisms of Γ.

Given a stable curve C of weighted genus g, we can associate a weighted stable graph Γ of genus g by setting

$$V(\Gamma) = \{\text{irreducible components of } C\},$$
$$E(\Gamma) = \{\text{nodes of } C\}$$

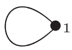
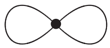
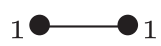
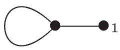
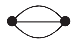
and

$$w : V(\Gamma) \to \mathbb{N}, \qquad w(v) := \text{genus of the normalization of } v.$$

Notice that a node belongs to at most two components, and we set a loop when a component has self intersection.

We call Γ the *topological class* of C.

As an example we give now the table that we get for genus $g = 2$. Each graph corresponds to a topological class of a curve of genus 2. We put in columns graphs with a given total weight h and in lines we fix the number of components.

$g = 2$	$h = 2$	$h = 1$	$h = 0$
$c = 1$			
$c = 2$			 

In general the total weight runs from 0 to g while the number of components is at most $2g - 2$. We add here also the case $g = 3$ but we just write the number of stable weighted graphs in each square.

$g = 3$	$h = 3$	$h = 2$	$h = 1$	$h = 0$
$c = 1$	1	1	1	1
$c = 2$	1	3	4	4
$c = 3$	1	3	6	5
$c = 4$	1	2	3	5

Remark 3.4. Instead of the total weight we can also use for labeling the columns the standard genus g_s of the graph, since $h + g_s = g$.

Definition 3.5. For any integer $g \geq 2$ and any weighted stable graph Γ of genus g, we call $\bar{\mathcal{M}}_g(\Gamma)$ the closed substack of $\bar{\mathcal{M}}_g$ parametrizing curves from which a curve with topological class Γ can be obtained through smoothing a subset (possibly empty) of nodes.

Notice that this $\bar{\mathcal{M}}_g(\Gamma)$ is the closed substack $\mathcal{D}_\Gamma$ of [2], Chapter XII, (10.5) (with $P = \emptyset$). We denote by $\mathcal{M}_g(\Gamma)$ the locally closed substack of $\bar{\mathcal{M}}_g$ parametrizing stable curves with topological class Γ.

We also call i-stratum of $\bar{\mathcal{M}}_g$ the union of $\bar{\mathcal{M}}_g(\Gamma)$ such that $\dim(\bar{\mathcal{M}}_g(\Gamma)) = i$. This is the stratum of $\bar{\mathcal{M}}_g$ consisting of curves with exactly $3g - 3 - i$ nodes

Remark 3.6. Given a stable weighted graph Γ we have

$$\dim(\mathcal{M}_g(\Gamma)) = \sum_{v \in V(\Gamma)} (\mathrm{mult}(v) - 3).$$

Notice also that in the above tables the anti-diagonals preserve the number of edges (i.e. of nodes), therefore graphs in the same anti-diagonal correspond to components of the same dimension.

Proposition 3.7. *For every stable graph Γ of genus g, the stack $\mathcal{M}_g(\Gamma)$ is smooth over* $\mathrm{Spec}(\mathbb{Z})$.

Proof. We know by [7] that $\bar{\mathcal{M}}_g$ is a smooth stack over $\mathrm{Spec}(\mathbb{Z})$. To prove that the substack $\mathcal{M}_g(\Gamma)$ is smooth it suffices to show that the substack $\mathcal{M}_g(\Gamma)$ is formally smooth. Let $x : \mathrm{Spec}(k) \to \mathcal{M}_g(\Gamma)$ be a geometric point corresponding to a stable curve C_x of topological class Γ.

The complete local ring of $\bar{\mathcal{M}}_g$ at x is the complete local ring of the universal deformation space $\mathcal{M}$ of the curve C_x. By [7] this ring is $\mathfrak{o}_k[[t_1, \ldots, t_{3g-3}]]$. Here $\mathfrak{o}_k = k$ if the characteristic is 0 and $\mathfrak{o}_k$ is the unique complete regular local ring with residue field k and maximal ideal $p\mathfrak{o}_k$ if the characteristic is p.

Moreover, we may choose the t_i's such that if $\mathscr{C} \to \mathcal{M}$ is the universal curve, then the complete local ring of $\mathscr{C}$ at the nodes of C_x is isomorphic to

$$\mathfrak{o}_k[[u_i, v_i, t_1, \ldots, t_{3g-3}]]/(u_i v_i - t_i).$$

The complete local ring of $\mathcal{M}_g(\Gamma)$ at x is the quotient of $\mathcal{O}_{x, \bar{\mathcal{M}}_g}$ by the ideal corresponding to deformations that preserve the nodes. From the description of the complete local rings to $\mathscr{C}$ at the nodes of C we see that this ideal is $(t_1, \ldots, t_r)$ where r is the number of edges of Γ. Hence $\mathcal{M}_g(\Gamma)$ is smooth. □

4. Normalization of the substacks $\bar{\mathcal{M}}_g(\Gamma)$

Given a vertex v in a graph Γ, we call $\hat{E}(v)$ the (ordered) set of edges meeting v and considering loops twice.

Definition 4.1. Given a stable weighted graph Γ of weighted genus g, we define the stack

$$\mathscr{N}_g(\Gamma) := \Big(\prod_{v \in V(\Gamma)} \mathscr{M}_{w(v),\, \hat{E}(v)} \Big)$$

and the natural 1-morphism

$$\mathscr{N}_g(\Gamma) \xrightarrow{\pi_\Gamma} \mathscr{M}_g(\Gamma)$$

induced by gluing sections corresponding to the same edge.

Moreover we define $\overline{\mathscr{N}}_g(\Gamma)$ as the stack

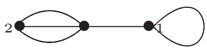
$$\overline{\mathscr{N}}_g(\Gamma) := \Big(\prod_{v \in V(\Gamma)} \overline{\mathscr{M}}_{w(v),\, \hat{E}(v)} \Big)$$

and extend π_Γ:

$$\overline{\mathscr{N}}_g(\Gamma) \xrightarrow{\bar{\pi}_\Gamma} \overline{\mathscr{M}}_g(\Gamma).$$

Remark 4.2. The stack $\overline{\mathscr{N}}_g(\Gamma)$ defined above, is the same of the stack $\overline{\mathscr{M}}_\Gamma$ given in [2], Chapter XII, (10.2).

Example 4.3. Consider the graph

$\Gamma :=$  .

This graph represents a curve with three irreducible components: a genus-2, a genus-0, and a genus-1 curve. The genus-2 and the genus-0 curves intersect each other in three points. The genus-1 curve intersects the genus-0 curve in one point and intersects itself once. The genus of this curve is

$$g = h + n - c + 1 = 3 + 5 - 3 + 1 = 6.$$

From the above definition we have

$$\mathscr{N}_6(\Gamma) := \mathscr{M}_{2,3} \times \mathscr{M}_{0,4} \times \mathscr{M}_{1,3},$$
$$\overline{\mathscr{N}}_6(\Gamma) := \overline{\mathscr{M}}_{2,3} \times \overline{\mathscr{M}}_{0,4} \times \overline{\mathscr{M}}_{1,3}.$$

Notice that even if the genus-1 curve has 2 nodes, we consider $\mathscr{M}_{1,3}$ because of the loop which is counted twice. The morphism π_Γ glues

– the three sections of $\mathscr{M}_{2,3}$ respectively with the first three sections of $\mathscr{M}_{0,4}$;
– the last section of $\mathscr{M}_{0,4}$ with the first section of $\mathscr{M}_{1,3}$;
– the last two sections of $\mathscr{M}_{1,3}$ together.

The morphism $\bar{\pi}_\Gamma$ glues the sections in the same way.

Proposition 4.4. *The* 1-*morphisms* π_Γ *and* $\bar{\pi}_\Gamma$ *are* (a) *representable;* (b) *finite;* (c) *unramified;* (d) *surjective. Moreover,* (e) *the* 1-*morphism* π_Γ *is ètale.*

Proof. The morphism $\bar{\pi}_\Gamma$ is a composition of clutching morphisms in the sense of Knudsen [14]. Knudsen proved that clutching morphisms are representable, finite and unramified [14], Corollary 3.9. To prove (d) we have to check that for all curves $C \to T$ in $\mathscr{M}_g(\Gamma)$, there exists an étale covering $T' \to T$ such that $C_{T'} \to T'$ (obtained by base change) is isomorphic to the image of some object in $\mathscr{N}_g(\Gamma)$.

First of all fix a geometric point $\mathrm{Spec}(k) \to T$. We can choose a smooth point on each irreducible component of the fiber C_k defining sections $\{s_v\}_{v \in V(\Gamma)}$. We know that there exists an étale covering $T_1 \to T$ where the sections $\{s_v\}_{v \in V(\Gamma)}$ extend. So we get a curve $C_1 \to T_1$ of topological class Γ where we have labeled irreducible components with $V(\Gamma)$.

Let us now consider the normalization $\widehat{C_1} \to C_1 \to T_1$. The pre-image of the relative singular locus of C_1 defines a divisor $\widehat{D_1} \subset \widehat{C_1}$ and an étale covering $\widehat{D_1} \to T_1$ of degree $\sum_{v \in V(\Gamma)} |\hat{E}(v)|$. If this covering is trivial, then we can choose $T' := T_1$ and $C_{T'} = C_1$ (the sections $\{s_v\}_{v \in V(\Gamma)}$ rigidify the components, but not necessarily the nodes).

Otherwise let $H_1, H_2, \ldots, H_r$ be all the irreducible components of $\widehat{D_1}$ such that each morphism $H_i \to T_1$ is not trivial. For every $1 \le i \le r$, let q_1 be the degree of $H_i \to T_1$. We call $q = \sum_{i=1}^r q_i$ the excess covering number of $\pi_1 : C_1 \to T_1$. Let

$$T_2 := \coprod_{i=1}^{r} H_i$$

and consider the cartesian diagram

$$\begin{array}{ccc}
\widehat{C_2} & \longrightarrow & \widehat{C_1} \\
\downarrow & \square & \downarrow \\
C_2 & \longrightarrow & C_1 \\
{\scriptstyle \pi_2}\downarrow & \square & \downarrow{\scriptstyle \pi_1} \\
T_2 & \xrightarrow{\sigma_1} & T_1.
\end{array}$$

The morphism $\widehat{C_2} \to T_2$ admits at least the identity section, therefore $\pi_2 : C_2 \to T_2$ has a strictly smaller excess covering number.

After a finite number of steps we get the required T' and $C_{T'}$.

(e) We have that $\mathcal{N}_g(\Gamma)$ is smooth because it is the product of smooth stacks, moreover we have proved in proposition 3.7 that $\mathcal{M}_g(\Gamma)$ is smooth, therefore π_Γ is a finite representable surjective morphism between a smooth (hence Cohen-Macaulay) stack and a smooth (hence regular) stack of the same dimension. Hence it is flat [17], Remark 3.11. From [10], Proposition 17.6.1, we have that a flat and unramified morphism of relative dimension 0 is necessarily étale. □

There is a natural action of Aut(Γ) on $\mathcal{N}_g(\Gamma)$ (that extends to $\overline{\mathcal{N}}_g(\Gamma)$) which is consistent with the action on Γ, so the quotient stacks

$$\mathcal{N}_g(\Gamma)/\mathrm{Aut}(\Gamma) \qquad \overline{\mathcal{N}}_g(\Gamma)/\mathrm{Aut}(\Gamma)$$

are well defined by Theorem 2.20.

Let us consider a geometric point of $\mathcal{M}_g(\Gamma)$, that is to say a stable curve C over an algebraically closed field K of topological class Γ. We have the diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) \times_{\mathcal{M}_g(\Gamma)} \mathcal{N}_g(\Gamma) & \longrightarrow & \mathcal{N}_g(\Gamma) \\ \downarrow & \square & \downarrow \pi_\Gamma \\ \mathrm{Spec}(K) & \longrightarrow & \mathcal{M}_g(\Gamma). \end{array}$$

Proposition 4.5. *There is a natural isomorphism*

$$\mathrm{Spec}(K) \times_{\mathcal{M}_g(\Gamma)} \mathcal{N}_g(\Gamma) \cong \mathrm{Aut}(\Gamma).$$

Proof. It is enough to check the isomorphism over Spec(K). Let C be the curve over K with topological class Γ defined by $\mathrm{Spec}(K) \to \mathcal{M}_g(\Gamma)$. The objects of the groupoid

$$\mathcal{N}_g(\Gamma)_K\big(\mathrm{Spec}(K)\big) := \big(\mathrm{Spec}(K) \times_{\mathcal{M}_g(\Gamma)} \mathcal{N}_g(\Gamma)\big)\big(\mathrm{Spec}(K)\big)$$

are pairs $(\hat{C}, \alpha)$ where $\hat{C}$ is an object in $\mathcal{N}_g(\Gamma)\big(\mathrm{Spec}(K)\big)$ and α is an isomorphism between C and $\pi_\Gamma(\hat{C})$.

The isomorphisms between two objects $(\hat{C}, \alpha)$ and $(\hat{C}', \alpha')$, are isomorphisms $g : \hat{C} \to \hat{C}'$ such that the diagram

$$\begin{array}{ccc} \pi_\Gamma(\hat{C}) & \xrightarrow{\pi_\Gamma(g)} & \pi_\Gamma(\hat{C}') \\ & {}_{\alpha}\nwarrow \quad \nearrow_{\alpha'} & \\ & C & \end{array}$$

commutes. That is to say $\pi_\Gamma(g) = \alpha'\alpha^{-1}$. By representability of π_Γ we have at most one isomorphism g having this property. In particular this means, as we already know from Proposition 4.4 (b), that $\mathcal{N}_g(\Gamma)_K$ is a set of points.

Let us now fix an object $(\hat{C}, \alpha)$ and take $\gamma \in \mathrm{Aut}(\Gamma)$ different from the identity. Let us write

$$\hat{C} = \coprod_{v \in V(\Gamma)} C_v$$

where C_v is a curve in $\mathcal{M}_{w(v), \hat{E}(v)}$. Let us define

$$\gamma(\hat{C}) := \coprod_{v \in V(\Gamma)} C_{\gamma(v)}$$

where now $C_{\gamma(v)}$ is a curve in $\mathcal{M}_{w(\gamma(v)), \gamma(\hat{E}(v))}$.

Notice that $\hat{C}$ and $\gamma(\hat{C})$ are two isomorphic curves just labeled differently in $\mathcal{N}_g(\Gamma)$. There is an isomorphism β between $\pi_\Gamma(\hat{C})$ and $\pi_\Gamma(\gamma(\hat{C}))$ sending each component v to $\gamma(v)$ and each node e to $\gamma(e)$. Define $\gamma(\alpha) := \beta\alpha : C \to \pi_\Gamma(\gamma(\hat{C}))$. There are no isomorphisms in $\mathcal{N}_g(\Gamma)$ between $\hat{C}$ and $\gamma(\hat{C})$ whose image through π_Γ is β, since isomorphisms in $\mathcal{N}_g(\Gamma)$ preserve the labelling. Therefore we have that $(\gamma(\hat{C}), \gamma(\alpha))$ defines a different point in $\mathcal{N}_g(\Gamma)_K$.

On the other hand let $(\hat{C}_2, \alpha_2)$ be another object in $\mathcal{N}_g(\Gamma)_K$ such that $\alpha_2\alpha^{-1}$ cannot be lifted to an isomorphism $\hat{\beta} : \hat{C}_2 \to \hat{C}$.

Let us now consider the diagram

$$\begin{array}{ccc}
\hat{C} & \xrightarrow{\ \delta\ } & \hat{C}_2 \\
\downarrow & & \downarrow \\
\pi_\Gamma(\hat{C}) & \xrightarrow{\ \alpha_2\alpha^{-1}\ } & \pi_\Gamma(\hat{C}_2) \\
{\scriptstyle\alpha}\nwarrow & & \nearrow{\scriptstyle\alpha_2} \\
& C &
\end{array}$$

where vertical maps are normalization morphisms which are consistent with the labelling of Γ. There exists a unique isomorphism δ making the diagram commute. Notice that δ may exchange two points coming from a loop. This is why we need part (3) in Definition 3.3. Moreover, since δ cannot be an isomorphism in $\mathcal{N}_g(\Gamma)_K$, it cannot preserve all nodes and components. Therefore δ defines an element γ in $\mathrm{Aut}(\Gamma)$ different from identity.

So we can conclude that (C_2, α_2) is isomorphic to $(\gamma(\hat{C}), \gamma(\alpha))$. □

Proposition 4.6. *There is an isomorphism*

$$[\mathcal{N}_g(\Gamma)/\mathrm{Aut}(\Gamma)] \cong \mathcal{M}_g(\Gamma).$$

Proof. From the proof of Proposition 4.5 we get that the induced morphism

$$[\mathcal{N}_g(\Gamma)/\mathrm{Aut}(\Gamma)] \to \mathcal{M}_g(\Gamma)$$

is faithful, that is to say representable (from Lemma 2.10). From Proposition 4.4 we have that it is an étale covering and using Proposition 4.5 we get it is of degree one. Hence it is an isomorphism. □

We still have a morphism

$$\varphi_\Gamma : [\overline{\mathcal{N}}_g(\Gamma)/\mathrm{Aut}(\Gamma)] \to \overline{\mathcal{M}}_g(\Gamma),$$

but it is in general far from being an isomorphism. The main reason is that we could get extra automorphisms at the points in the closure.

However, $\overline{\mathcal{N}}_g(\Gamma)$ is normal, since it is the product of normal stacks, so also the quotient stack is normal. This means that φ_Γ factorizes through the normalization of $\overline{\mathcal{M}}_g(\Gamma)$.

Theorem 4.7. *The morphism*

$$\varphi_\Gamma : [\overline{\mathcal{N}}_g(\Gamma)/\mathrm{Aut}(\Gamma)] \to \overline{\mathcal{M}}_g(\Gamma)$$

is a normalization for $\overline{\mathcal{M}}_g(\Gamma)$.

Proof. The map φ_Γ is faithful on automorphism groups by Proposition 4.5. It is also finite and surjective so by Proposition 2.9 it is representable. Since normalization commutes with étale base change [10], Proposition 18.12.15, we may assume that φ_Γ is a map of schemes. Since the source is normal and the map is finite and generically an isomorphism, φ_Γ must be the normalization by the universal property of normalizations. □

Remark 4.8. The description of the normalization of $\mathcal{D}_\Gamma := \overline{\mathcal{M}}_{g,P}(\Gamma)$ can be found in [2], Chap. XII, Proposition 10.11.

Remark 4.9. Note also that since $[\overline{\mathcal{N}}_g(\Gamma)/\mathrm{Aut}(\Gamma)]$ is non-singular, the map φ_Γ is a desingularization of $\overline{\mathcal{M}}_g(\Gamma)$. Moreover $\overline{\mathcal{M}}_g(\Gamma)$ is smooth if and only if φ_Γ is an isomorphism. In particular we have the following Lemma.

Lemma 4.10. *The stack* $\overline{\mathcal{M}}_g(\Gamma)$ *is smooth if and only if for every* Γ_0 *such that* $\mathcal{M}_g(\Gamma_0) \subseteq \overline{\mathcal{M}}_g(\Gamma)$, *there is only one subset* F *of edges of* Γ_0 *such that* Γ *can be obtained from* Γ_0 *by shrinking the edges in* F.

Proof. This is [11], Lemma 4.2. We now follow their proof.

Let Γ_0 be a graph such that $\mathcal{M}_g(\Gamma_0) \subseteq \overline{\mathcal{M}}_g(\Gamma)$ and let $x : \mathrm{Spec}(k) \to \mathcal{M}_g(\Gamma_0)$ be a geometric point of $\mathcal{M}_g(\Gamma_0)$. From Remark 4.9, if $\overline{\mathcal{M}}_g(\Gamma)$ is smooth, then the pre-image of x through φ_Γ consists of a single point, this is equivalent to the fact that there is only one subset of nodes of a curve corresponding to Γ_0 which need to be smoothed in order to obtain a curve corresponding to Γ.

On the other hand, if the pre-image of x through φ_Γ consists of a single point, then, by using the notation of Proposition 3.7, the complete local ring of $\bar{\mathcal{M}}_g(\Gamma)$ at x, is the quotient of $\mathcal{O}_{x,\bar{\mathcal{M}}_g}$ by the ideal corresponding to deformations that preserve the nodes corresponding to Γ which can be chosen in only one way. □

5. Topological considerations on the 1-stratum

Throughout this section all graphs will be weighted and stable.

We want to prove that the 1-stratum of genus $g \geq 2$ stable curves is connected.

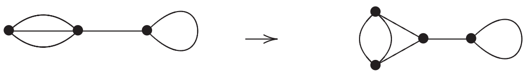
First of all notice that each of the irreducible components of the 1-stratum comes from a graph Γ having either of the following properties:

(1) all vertices of Γ have weight 0, exactly one of them has degree 4 and all the others have degree 3;

(2) exactly one among the vertices of Γ has weight 1 (and degree 1) and all the others have degree 3 and weight 0.

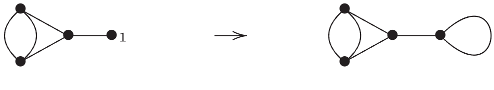
Fix a graph Γ coming from the 1-stratum, then the geometric points we have to add to $\bar{\mathcal{M}}_g(\Gamma)$ correspond to the graphs obtained from Γ in the following ways:

(1) if we are in the first case we split the vertex of degree 4 into two vertices and add an edge between them as in the following example:



we can do this in at most three different ways depending on the symmetries of the graph;

(2) if we are in the second case we change the weight 1 into 0 and add a loop. For example,



Definition 5.1. We call a transformation of graphs like the two above *pop*. On the other hand we call *shrink* the inverse of pop, that is to say shrinking any edge of graph from the 0-stratum and adding a weight 1 if we shrink a loop.

Remark 5.2. It is a straightforward computation to check that pop and shrink preserve stability and genus.

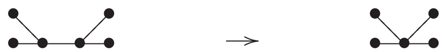
Theorem 5.3. *For any $g \geq 2$, the* 1*-stratum of* $\overline{\mathcal{M}}_g$ *is connected.*

Proof. This fact follows straightforward from the results mentioned in Remark 5.7. We give now a proof for reader's sake.

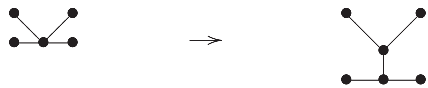
By applying a pop we see that any curve in $\overline{\mathcal{M}}_g(\Gamma)$ contains a geometric point of the 0-stratum. In order to prove connectedness of the 1-stratum it is enough to show that the 0-stratum is connected through components of the 1-stratum. If a graph Γ of the 0-stratum can be turned into another graph Γ' of the 0-stratum through a finite sequence of shrinks and pops, then $\mathcal{M}_g(\Gamma)$ is connected with $\mathcal{M}_g(\Gamma')$ through the 1-stratum. We do this by proving two claims. □

Claim 5.4. *Any graph of the* 0*-stratum can be turned into a loop graph of the* 0*-stratum by applying a finite sequence of shrink-pop transformations.*

Proof. Take a graph Γ of the 0-stratum and a cycle of Γ which is not a loop. Shrink one of the edges of the cycle getting a smaller cycle. The shrink locally looks as follows:



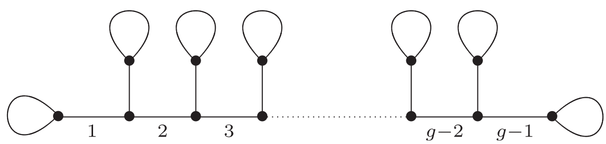
where the horizontal edges belong to the cycle we are focusing on. Then apply the only pop that preserves the shorter cycle



We continue until we get a loop. □

Remark 5.5. We cannot simply take the largest cycle because it is not guaranteed that in the process other cycles are not enlarged. Actually in the process we could also obtain isomorphic graphs at different steps: the relevant part is keeping track of the cycle we are reducing in order to take out a loop.

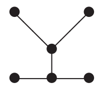
Claim 5.6. *Any loop graph of* (*weighted*) *genus $g \geq 2$ can be turned into the following loop graph*



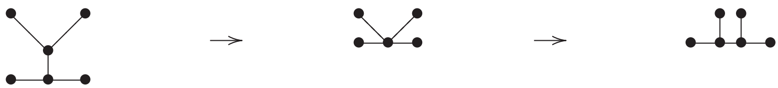
(2)

by applying a finite sequence of shrink-pop transformations.

Proof. Choose a path of maximal length of the graph. If the graph is not like (2), then we have somewhere the following subgraph (here the horizontal edges belong to the longest path)



that we can transform as



obtaining a graph whose horizontal edges belong to the (now unique) maximal path. It is clear that after a finite numbers of shrink-pop transformations we get the graph (2). □

Remark 5.7. Claims 5.4 and 5.6 imply that every pair of 3-regular graphs can be linked via a sequence of pop and shrinks of type (1). This statement is not new. A topological proof was originally given by Hatcher and Thurston [12]. More recently Caporaso [6] used combinatorial methods to prove that any two 3-connected p-regular graphs may be linked via a sequence of 3-connected p-regular graphs.

Remark 5.8. As a trivial consequence, we have that, for any $1 \leq i \leq 3g - 4$, the i-stratum of $\bar{\mathcal{M}}_g$ is connected. An analogous result for the moduli space of tropical curves, was proved by Brannetti, Melo, and Viviani [5], Proposition 3.2.59 (ii). In the language of tropical algebraic geometry, the stack $\bar{\mathcal{M}}_g$ with its stratification by dual graphs is *connected through codimension one*. Indeed there is a bijective correspondence between the stratification of $\bar{\mathcal{M}}_g$ by dual graphs and the strata of the tropical moduli space [6], Proposition 5.5.

6. The irreducible components of the 1-stratum

In this section we work over a base field $S = \mathrm{Spec}(k)$. We assume that $\mathrm{char}(k) \neq 2$ and that the field k contains the cubic roots of -1. These are made in order to consider standard results of the action of S_4 on $\bar{\mathcal{M}}_{0,4}$.

Let Γ be a stable weighted graph of weighted genus g and $3g - 4$ edges. As explained in Section 5, we have two possibilities:

(1) all vertices of Γ have weight 0, exactly one of them has degree 4 and all the others have degree 3 (total weight $h = 0$);

(2) exactly one among the vertices of Γ has weight 1 (and degree 1) and all the others have degree 3 and weight 0 (total weight $h = 1$).

6.1. Cross ratio. Before starting with the case $h = 0$ we need to point out some considerations about the cross ratio. More precisely we define a morphism

$$\mathbb{P}^1 \backslash \{0, 1, \infty\} \cong \mathscr{M}_{0,4} \xrightarrow{\text{cr}} \mathbb{P}^1 \backslash \{0, 1, \infty\}$$

which sends $\lambda \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ to the cross ratio of $\{0, 1, \infty, \lambda\}$.

This map is an isomorphism and can be extended to an isomorphism to

$$\mathbb{P}^1 \cong \bar{\mathscr{M}}_{0,4} \xrightarrow{\text{cr}} \mathbb{P}^1.$$

So once we have chosen the order of the marked points of a rational curve in $\bar{\mathscr{M}}_{0,4}$ there is a natural way to associate the cross ratio. Even if we should consider the isomorphism *cr* we call $\lambda \in \bar{\mathscr{M}}_{0,4}$ the cross ratio of the associated marked curve.

Let us now consider the action of S_4 on $\bar{\mathscr{M}}_{0,4}$ that permutes the marked points. It is known that the orbit of λ for the action of S_4 is generically of order 6:

$$\lambda, \quad \frac{1}{\lambda}, \quad \frac{\lambda - 1}{\lambda}, \quad \frac{\lambda}{\lambda - 1}, \quad 1 - \lambda, \quad \frac{1}{1 - \lambda},$$

and the generic stabilizer is the normal subgroup $V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by (12)(34) and (13)(24). Therefore, we have an induced action of $S_3 \cong S_4/V_4$ which, up to isomorphisms, corresponds to the permutation of $\{0, 1, \infty\}$ (that is to say the first three points).

On $\mathbb{P}^1$ we have the exceptional orbits:

$$o_1 := \{0, 1, \infty\}, \qquad o_2 := \left\{\tfrac{1}{2}, 2, -1\right\}, \qquad o_3 := \{\xi, \xi^{-1}\},$$

where $\xi \neq -1$ and $\xi^3 = -1$. The generic stabilizer for the action of S_3 is the identity. The stabilizers of the two exceptional orbits o_1 and o_2 are the three subgroups generated respectively by $(1, 2)$, $(1, 3)$, $(2, 3)$. Finally the stabilizer of the orbit o_3 is the subgroup generated by (123).

Remark 6.1. Two curves with topological class Γ are isomorphic if and only if the four nodes on the same component have the same cross ratio for some order consistent with Γ.

6.2. The case $h(\Gamma) = 0$. Let us call v_0 the point with degree 4 and let e_1, e_2, e_3, e_4 the four edges in $\hat{E}(\Gamma)$ (where loops are counted twice) ending in it.

Let $\sigma : \mathrm{Aut}(\Gamma) \to S_4$ be the map that sends every automorphism $\alpha \in \mathrm{Aut}(\Gamma)$ to the corresponding permutation $\sigma_\alpha \in S_4$ of the four edges. Clearly we have

$$\sigma(\alpha^{-1}) = \sigma(\alpha)^{-1}, \qquad \sigma(\alpha\alpha') = \sigma(\alpha)\sigma(\alpha').$$

Therefore, σ is a group homomorphism.

Definition 6.2. Let us define

$$R(\Gamma) := \mathrm{Aut}(\Gamma)/\sigma^{-1}(V_4) \subseteq S_3.$$

Theorem 6.3. *The normalization of $\bar{\mathcal{M}}_g(\Gamma)$ is a $\sigma^{-1}(V_4)$-gerbe, in the sense of* [4], *Proposition* 4.6, *over the orbifold*

$$[\bar{\mathcal{M}}_{0,4}/R(\Gamma)].$$

Proof. We have

$$\bar{\mathcal{N}}_g(\Gamma) \cong \bar{\mathcal{M}}_{0,4}$$

since all the other factors are $\mathcal{M}_{0,3} = \mathrm{Spec}(k)$.

From Theorem 4.7 we get that the normalization of $\bar{\mathcal{M}}_g(\Gamma)$ is $[\bar{\mathcal{M}}_{0,4}/\mathrm{Aut}(\Gamma)]$ where the generic stabilizer is the normal subgroup $\sigma^{-1}(V_4)$. We conclude by using the same argument of [4], Proposition 4.6. □

Remark 6.4. The smooth Deligne–Mumford stack $[\bar{\mathcal{M}}_{0,4}/R(\Gamma)]$ is the residual orbifold of the normalization of $\bar{\mathcal{M}}_g(\Gamma)$. In order to define uniquely such orbifolds, it is enough to give the order of the orbifold points, since the coarse moduli space is $\mathbb{P}^1$ for all of them (see [20], Proposition 2.11).

Now, $R(\Gamma)$ is a subgroup of S_3, so we have only four possibilities: $\{\mathrm{id}\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, S_3. For each possibility, we write $[\bar{\mathcal{M}}_{0,4}/R(\Gamma)] = [\underline{a} \,|\, \underline{b}]$,where $\underline{a}$ is the set of orders of the points in the closure and $\underline{b}$ is the set of the orders of the remaining orbifold points. We summarize everything in the following chart:

$R(\Gamma)$	$[\bar{\mathcal{M}}_{0,4}/R(\Gamma)]$
$\{\mathrm{id}\}$	$[1,1,1 \,|\, \emptyset]$
$\mathbb{Z}/2\mathbb{Z}$	$[1,2 \,|\, 2]$
$\mathbb{Z}/3\mathbb{Z}$	$[1 \,|\, 3,3]$
S_3	$[2 \,|\, 2,3]$

Proposition 6.5. *The coarse moduli space of $\bar{\mathcal{M}}_g(\Gamma)$ is $\mathbb{P}^1$.*

It may seem counterintuitive that the non-normal (and hence singular) stacks $\bar{\mathcal{M}}_g(\Gamma)$ can have smooth coarse moduli spaces. However, as the following the example shows, it is possible for the geometric quotient of a non-normal scheme by a finite group to be normal.

Example 6.6. There is an obvious $\mathbb{Z}_2$ action on $\mathbb{A}^2$ which exchanges the coordinates. If $X = \operatorname{Spec} k[x, y]/(xy)$ then the action of $\mathbb{Z}_2$ on the invariant subscheme X exchanges the two irreducible components. The $\mathbb{Z}_2$-invariant subring of $k[x, y]/(xy)$ is isomorphic to the ring $k[t]$ where $t = x + y$. Thus $\mathbb{A}^1$ is the coarse moduli space of the non-normal stack $[X/\mathbb{Z}_2]$.

Proof of Proposition 6.5. Let $\overline{M}_g(\Gamma)$ be the coarse moduli space of $\bar{\mathcal{M}}_g(\Gamma)$, it is a compact algebraic curve. From the universal property of coarse moduli spaces, we have a (non constant) scheme morphism from the coarse moduli space of $[\bar{\mathcal{N}}_g(\Gamma)]$, that is to say a morphism $\mathbb{P}^1 \to \overline{M}_g(\Gamma)$.

We now want to conclude by proving that $\overline{M}_g(\Gamma)$ is smooth. We use the notation of the proof of Proposition 3.7.

Let $x : \operatorname{Spec}(k) \to \overline{M}_g(\Gamma)$ be a geometric point. From properties of coarse moduli spaces, it lifts to a geometric point of $\bar{\mathcal{M}}_g(\Gamma)$. Moreover we can assume that (the lift of) x belongs to $\bar{\mathcal{M}}_g(\Gamma) \backslash \mathcal{M}_g(\Gamma)$. Let Γ_x be the topological class of the curve C_x associated to x. The complete local ring of $\overline{M}_g(\Gamma)$ at x is the quotient of $\hat{\mathcal{O}}_{x, \bar{\mathcal{M}}_g} = \mathfrak{o}_k[[t_1, \ldots, t_{3g-3}]]$ by the ideal corresponding to deformations smoothing a node that allows to get a curve with topological class Γ (that is to say a shrink from Γ_x to Γ). After a possible reorder of the variables, we can assume that $t_1, \ldots, t_k$ correspond to the nodes $p_1, \ldots, p_k$ smoothing which we get a curve of topological class Γ. Therefore, the ideal we are looking for, is

$$I_x := \{t_i t_j\}_{i<j;\, i,j=1,\ldots,k} \cup \{t_{k+1}, \ldots, t_{3g-3}\}.$$

Notice that a node belongs to at most two components and we set a loop when a component has self intersection. So we get

$$\hat{\mathcal{O}}_{x, \bar{\mathcal{M}}_g(\Gamma)} = \mathfrak{o}_k[[t_1, \ldots, t_k]]/(\{t_i t_j\}_{i<j;\, i,j=1,\ldots,k}).$$

In order to compute the local ring $\hat{\mathcal{O}}_{x, \overline{M}_g(\Gamma)}$ of x in the moduli space $\overline{M}_g(\Gamma)$ we have to consider the invariant part of the action of the group $\operatorname{Aut}(\Gamma_x)$ (see Proposition 2.14).

Since this action comes from a transitive permutation of the nodes $p_1, \ldots, p_k$ (see Lemma 6.7), by direct computation we get

$$\hat{\mathcal{O}}_{x, \overline{M}_g(\Gamma)} = \mathfrak{o}_k[[s]]$$

for some generator s, and we conclude that $\overline{M}_g(\Gamma)$ is smooth. □

Lemma 6.7. *Let Γ be a graph of genus g and $3g-4$ edges. Let Γ_x be a graph obtained by a pop at the vertex of Γ with multiplicity 4. Then $\mathrm{Aut}(\Gamma_x)$ acts transitively on the edges $p_1, \dots, p_k$ shrinking which we get the graph Γ.*

Proof. Without loss of generality we show that there exists an element of $\mathrm{Aut}(\Gamma_x)$ exchanging p_1 and p_2.

We know that there exists an isomorphism γ between Γ_x shrunk at p_1 and Γ_x shrunk at p_2. Clearly γ lifts to an automorphism of Γ_x. □

Definition 6.8. Let Γ be a stable graph of genus g with total weight 0 and $3g-4$ edges. Let

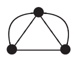
$$\mathrm{Spec}(k) \xrightarrow{x} \bar{\mathscr{M}}_g(\Gamma) \backslash \mathscr{M}_g(\Gamma)$$

be a geometric point and Γ_x its associated graph. We call order of x in $\bar{\mathscr{M}}_g(\Gamma)$ (written $\mathrm{ord}_x(\Gamma)$) the number of edges in Γ_x shrinking which we get the graph Γ. From the proof of Proposition 6.5 we get that $\mathrm{ord}_x(\Gamma)$ is equal to the number of generators for the complete local ring $\hat{\mathscr{O}}_{x, \bar{\mathscr{M}}_g(\Gamma)}$.

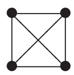
Remark 6.9. The point x is smooth in $\bar{\mathscr{M}}_g(\Gamma)$ if and only if $\mathrm{ord}_x(\Gamma) = 1$.

Remark 6.10. It is tempting to describe $\bar{\mathscr{M}}_g(\Gamma)$ as an H-gerbe over a Deligne–Mumford stack with only finite stacky points for some finite group. However, because $\bar{\mathscr{M}}_g(\Gamma)$ is not normal this is not the case, as the following example shows.

Let Γ be



and Γ_x be



We have $\mathrm{Aut}(\Gamma) = D_8$. The group H should be isomorphic to $\sigma^{-1}(V_4)$ (see Theorem 6.3), which in this case is $V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Now we have $\mathrm{Aut}(\Gamma_x) \cong S_4$, but the group $\sigma^{-1}(V_4)$ seen in $\mathrm{Aut}(\Gamma_x)$ is the subgroup fixing an edge which is not normal in $\mathrm{Aut}(\Gamma_x)$.

6.3. The case $h(\Gamma) = 1$. In this case we have $\mathscr{N}_g(\Gamma) = \mathscr{M}_{1,1}$. It is known that $\bar{\mathscr{M}}_{1,1}$ is a $\mathbb{Z}/2\mathbb{Z}$-gerbe over a $[1 \,|\, 2, 3]$ orbifold (the coarse moduli space is still $\mathbb{P}^1$).

Moreover in this case the stabilizer of points is the entire group $\mathrm{Aut}(\Gamma)$ and it acts trivially on $\mathscr{M}_{1,1}$ as it must fix the vertex of weight 1. So we have

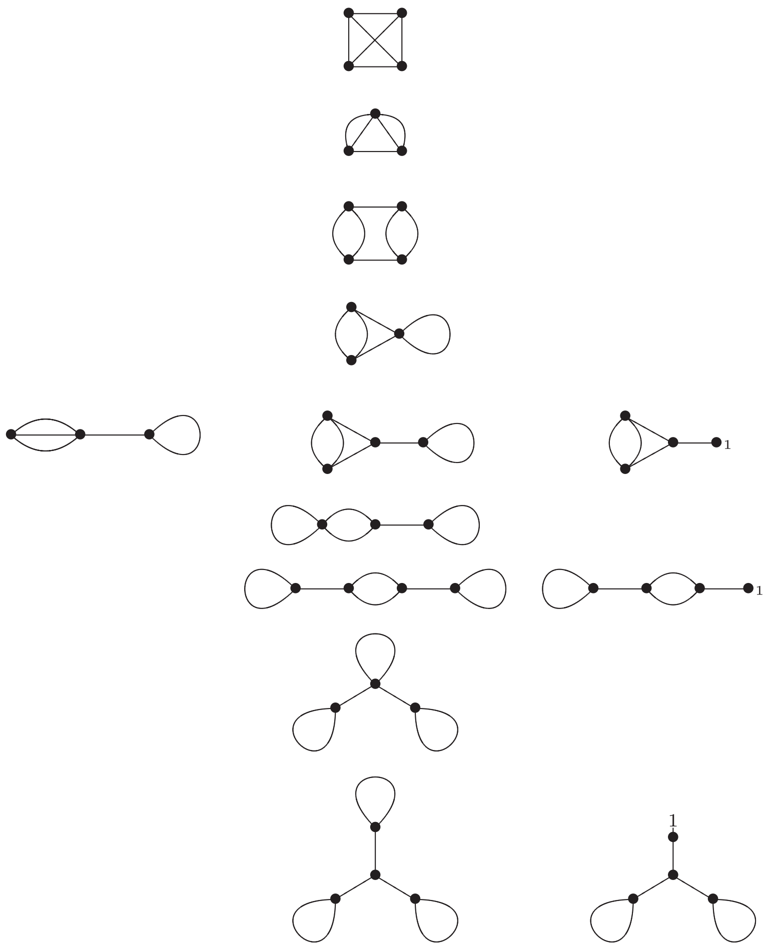
$$[\bar{\mathscr{M}}_{1,1}/\mathrm{Aut}(\Gamma)] = \bar{\mathscr{M}}_{1,1} \times \mathrm{B}\,\mathrm{Aut}(\Gamma)$$

and we get

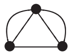
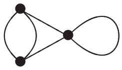
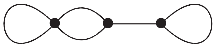
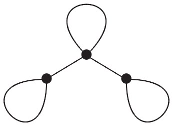
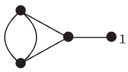
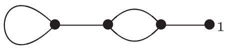
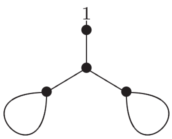
Theorem 6.11. *The normalization of* $\bar{\mathscr{M}}_g(\Gamma)$ *is a* $(\mathbb{Z}/2\mathbb{Z} \times \mathrm{Aut}(\Gamma))$*-gerbe over the orbifold* $[1\,|\,2,3]$.

Remark 6.12. Proposition 6.5 still holds and we can also extend Definition 6.8.

6.4. The 1-stratum of $\bar{\mathscr{M}}_3$. As an example we present the case $g = 3$. In the following picture we alternate the graphs with 5 and 6 nodes.

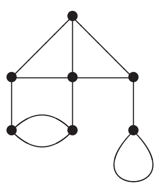


In the following table we describe the stacks $\bar{\mathscr{M}}_3(\Gamma)$. We use the notation of 6.4 and Definition 6.8.

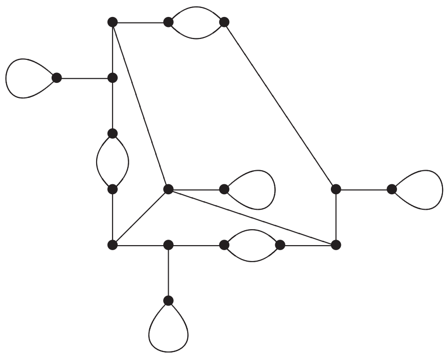
Γ	$\sigma^{-1}(V_4)$	$[\bar{\mathscr{N}}_3(\Gamma)/\mathrm{Aut}(\Gamma)]_{\mathrm{orb}}$	order of points in the 0-stratum
	V_4	$[1,2\,\vert\,2]$	$\mathrm{ord}_{x_1}(\Gamma)=6$ $\mathrm{ord}_{x_2}(\Gamma)=2$
	$(\mathbb{Z}/2\mathbb{Z})^2$	$[1,2\,\vert\,2]$	$\mathrm{ord}_{x_1}(\Gamma)=4$ $\mathrm{ord}_{x_2}(\Gamma)=1$
	$(\mathbb{Z}/2\mathbb{Z})^2$	$[1,2\,\vert\,2]$	$\mathrm{ord}_{x_1}(\Gamma)=2$ $\mathrm{ord}_{x_2}(\Gamma)=2$
	$(\mathbb{Z}/2\mathbb{Z})^3$	$[1,2\,\vert\,2]$	$\mathrm{ord}_{x_1}(\Gamma)=2$ $\mathrm{ord}_{x_2}(\Gamma)=3$
	$\mathbb{Z}/2\mathbb{Z}$	$[2\,\vert\,2,3]$	$\mathrm{ord}_{x_1}(\Gamma)=2$
	$(\mathbb{Z}/2\mathbb{Z})^3$	$[1\,\vert\,2,3]$	$\mathrm{ord}_{x_1}(\Gamma)=1$
	$(\mathbb{Z}/2\mathbb{Z})^3$	$[1\,\vert\,2,3]$	$\mathrm{ord}_{x_1}(\Gamma)=2$
	$(\mathbb{Z}/2\mathbb{Z})^4$	$[1\,\vert\,2,3]$	$\mathrm{ord}_{x_1}(\Gamma)=3$

Example 6.13. We want to show that all cases for residual orbifolds exist.

A graph Γ such that the residual orbifold of $[\bar{\mathscr{N}}_g(\Gamma)/\mathrm{Aut}(\Gamma)]$ is $[1,1,1\,|\,\emptyset]$, is



A graph Γ such that the residual orbifold of $[\bar{\mathscr{N}}_g(\Gamma)/\mathrm{Aut}(\Gamma)]$ is $[1\,|\,3,3]$, is



6.5. Remarks on higher dimension. The above techniques can be generalized to higher dimension. The main obstacle to a complete classification, is the rapid growth of the number of possibilities and the rather complicated combinatorics involved. However, we are able to formulate the following fact.

Proposition 6.14. *Let $N_k(g)$ be the set of isomorphism classes of residual orbifolds of the normalizations of the irreducible components of the k-stratum of $\bar{\mathcal{M}}_g$. Then, for every nonnegative integer k, the set*

$$N_k := \bigcup_{g=2}^{\infty} N_k(g)$$

is finite.

Proof. Following the notation of Definition 3.1, for a stable graph Γ, we have

$$k = \sum_{v \in \Gamma} \big(\mathrm{mult}(v) - 3\big).$$

This means that for every graph Γ, appearing in a k-stratum, there are at most k vertices with multiplicity greater than 3 and the multiplicity of a vertex is at most $k + 3$. Therefore, there are only finitely many possible

$$\bar{\mathcal{N}}_g(\Gamma) := \prod_{v \in V(\Gamma)} \bar{\mathcal{M}}_{w(v), \hat{E}(v)}$$

since all vertices with multiplicity exactly 3 contribute to the product with a single point.

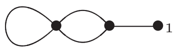
Now, let F be the normal subgroup of $\mathrm{Aut}(\Gamma)$ which fixes all vertices with multiplicity greater that 3 and their edges. The action of F on $\overline{\mathcal{N}}_g(\Gamma)$ is trivial. So, without loss of generality, we can consider the residual gerbes of

$$[\overline{\mathcal{N}}_g(\Gamma)/(\mathrm{Aut}(\Gamma)/F)].$$

But $\mathrm{Aut}(\Gamma)/F$ is a subgroup of the group G of all possible permutations of vertices of multiplicity greater than 3 and of their edges, moreover the action of G on $\overline{\mathcal{N}}_g(\Gamma)$ is canonical. Therefore we have finitely many possibilities. □

Remark 6.15. Clearly $|N_0| = 1$ since every irreducible stack of dimension 0 is a gerbe over a point. Throughout our description of the 1-stratum, we got $|N_1| = 5$.

Example 6.16. We provide an easy example from the 2-stratum of $\bar{\mathcal{M}}_3$

$\Gamma :=$ 

In this case we have

$$\overline{\mathcal{N}}_3(\Gamma) = \bar{\mathcal{M}}_{0,4} \times \bar{\mathcal{M}}_{1,1}$$

and

$$[\overline{\mathcal{N}}_3(\Gamma)/\mathrm{Aut}(\Gamma)]_{\mathrm{orb}} = [1,2\,|\,2] \times [1\,|\,2,3].$$

References

[1] D. Abramovich and A. Vistoli, Compactifying the space of stable maps. *J. Amer. Math. Soc.* **15** (2002), 27–75. Zbl 0991.14007 MR 1862797

[2] E. Arbarello, M. Cornalba, and P. A. Griffiths, *Geometry of algebraic curves*. Vol. II, Grundlehren Math. Wiss. 268, Springer, Heidelberg 2011. Zbl 1235.14002 MR 2807457

[3] M. Artin, Versal deformations and algebraic stacks. *Invent. Math.* **27** (1974), 165–189. Zbl 0317.14001 MR 0399094

[4] K. Behrend and B. Noohi, Uniformization of Deligne-Mumford curves. *J. Reine Angew. Math.* **599** (2006), 111–153. Zbl 1124.14004 MR 2279100

[5] S. Brannetti, M. Melo, and F. Viviani, On the tropical Torelli map. *Adv. Math.* **226** (2011), 2546–2586. Zbl 1218.14056 MR 2739784

[6] L. Caporaso, Geometry of tropical moduli spaces and linkage of graphs. *J. Combin. Theory Ser. A* **119** (2012), 579–598. Zbl 1234.14043 MR 2871751

[7] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), 75–109. Zbl 0181.48803 MR 0262240

[8] D. Edidin, B. Hassett, A. Kresch, and A. Vistoli, Brauer groups and quotient stacks. *Amer. J. Math.* **123** (2001), 761–777. Zbl 1036.14001 MR 1844577

[9] A. Grothendieck, Revêtements étales et groupe fondamental. Séminaire de Géometrie Algébrique du Bois Marie 1960–61.

[10] A. Grothendieck and J. Dieudonné, Étude locale des shémas et des morphismes de schémas. *Publ. Math. Inst. Hautes Études Sci.* **20** (1964), **24** (1964), 28 (1965), 32 (1966).

[11] P. Hacking, S. Keel, and J. Tevelev, Stable pair, tropical, and log canonical compactifications of moduli spaces of del Pezzo surfaces. *Invent. Math.* **178** (2009), 173–227. Zbl 1205.14012 MR 2534095

[12] A. Hatcher and W. Thurston, A presentation for the mapping class group of a closed orientable surface. *Topology* **19** (1980), 221–237. Zbl 0447.57005 MR 579573

[13] S. Keel and S. Mori, Quotients by groupoids. *Ann. of Math.* (2) **145** (1997), 193–213. Zbl 0881.14018 MR 1432041

[14] F. F. Knudsen, The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.* **52** (1983), 161–199. Zbl 0544.14020 MR 702953

[15] D. Knutson, *Algebraic spaces*. Lecture Notes in Math. 203, Springer-Verlag, Berlin 1971. Zbl 0221.14001 MR 0302647

[16] G. Laumon and L. Moret-Bailly, *Champs algébriques*. Ergeb. Math. Grenzgeb. (3) 39, Springer-Verlag, Berlin 2000. Zbl 0945.14005 MR 1771927

[17] Q. Liu, *Algebraic geometry and arithmetic curves*. Oxford Grad. Texts in Math. 6, Oxford University Press, Oxford 2002. Zbl 0996.14005 MR 1917232

[18] D. Mumford, Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry*, Vol. II, Progr. Math. 36, Birkhäuser Boston, Boston 1983, 271–328. Zbl 0554.14008 MR 717614

[19] M. Romagny, Group actions on stacks and applications. *Michigan Math. J.* **53** (2005), 209–236. Zbl 1100.14001 MR 2125542

[20] A. Vistoli, Intersection theory on algebraic stacks and on their moduli spaces. *Invent. Math.* **97** (1989), 613–670. Zbl 0694.14001 MR 1005008

[21] J. Zintl, The one-dimensional stratum in the boundary of the moduli stack of stable curves. *Nagoya Math. J.* **196** (2009), 27–66. Zbl 1209.14025 MR 2591090

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