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Periodic solutions for a third-order differential equation without asymptotic behavior on the potential

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Abstract. We consider the third-order differential equation $u''' + au'' + g(u') + cu = p(t)$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. We prove the existence of ω-periodic solution for this equation, using coincidence degree theories.

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1. Introduction

We consider the third-order differential equation

$$u''' + au'' + g(u') + cu = p(t), \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous, $p \in C(\mathbb{R}, \mathbb{R})$ is ω-periodic, $a \in \mathbb{R}$, $c \in \mathbb{R}\backslash\{0\}$.

In [3], J. O. C. Ezeilo and P. Omari studied problem (1), with $p : [0, 2\pi] \to \mathbb{R}$ belongs to $L^1(0, 2\pi)$ and periodic boundary conditions

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \qquad u''(0) = u''(2\pi),$$

assuming that g satisfies the condition

$$m^2 + h^-(|s|) \le \frac{g(s)}{s} \le (m+1)^2 - h^+(|s|) \tag{2}$$

for $|s| \ge r > 0$, where $m \in \mathbb{N}$ and $h^\pm : [0, +\infty[\mapsto \mathbb{R}$ are two functions such that

$$\lim_{|s| \to +\infty} |s| h^\pm(|s|) = +\infty. \tag{3}$$

We observe that conditions (2) and (3) imply, for $|s|$ large enough, that $\frac{g(s)}{s}$ lies strictly between m^2 and $(m+1)^2$.

Moreover $\liminf_{|s|\to+\infty} \frac{g(s)}{s}$ or $\limsup_{|s|\to+\infty} \frac{g(s)}{s}$ main attain either m^2 or $(m+1)^2$ but "slowly" on account of condition (3).

In [5], F. M. Minhós studied problem (1), with $p : [0, 2\pi] \to \mathbb{R}$ belongs to $L^1(0, 2\pi)$ and periodic boundary conditions

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \qquad u''(0) = u''(2\pi),$$

assuming that g and G, defined by $G(u) = \int_0^u g(s)\,ds$, satisfies the conditions

$$m^2 \le \liminf_{|s|\to+\infty} \frac{g(s)}{s} \le \limsup_{|s|\to+\infty} \frac{g(s)}{s} \le (m+1)^2$$

and

$$m^2 < \limsup_{s\to+\infty} \frac{2G(s)}{s}, \qquad \liminf_{s\to+\infty} \frac{2G(s)}{s} < (m+1)^2.$$

F. M. Minhós [5] improves a result due to J. O. C. Ezeilo and P. Omari [3], weakening the condition on the oscillation of g.

In [1], P. Amster, P. De Nápoli and M. C. Mariani studied the problem (1) with condition corresponds to the resonant case in g, $p \in L^2(0, 2\pi)$ and periodic boundary conditions

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \qquad u''(0) = u''(2\pi).$$

Our main theorem is:

Theorem 1.1. *Consider $g : \mathbb{R} \to \mathbb{R}$ continuous, $a, c \in \mathbb{R}\backslash\{0\}$, such that*

$$\frac{c}{a} < 0. \tag{4}$$

Then the equation

$$u''' + au'' + g(u') + cu = p(t), \tag{5}$$

has a solution for every $p \in C(\mathbb{R}, \mathbb{R})$, ω-periodic.

Note that the hypothesis in Theorem 1.1 about the potential g is only the continuity.

Example 1.2. Consider the equation (1) with $a = -2$, $c = 2$, $g(s) = \sqrt{1 + s^4}$ and $p(t) = \cos(t)$. Theorem 1.1 states that the equation

$$u''' - 2u'' + \sqrt{1 + (u')^4} + 2u = \cos(t)$$

has a nontrivial 2π-periodic solution. Note that $g(s) = \sqrt{1 + s^4}$ does not satisfy the hypotheses of [3] and [5].

Corollary 1.3. *Consider the same assumptions of Theorem* 1.1, *with* $g(0) \neq 0$. *Then the equation*

$$u''' + au'' + g(u') + cu = 0,$$

has a nontrivial ω*-periodic solution.*

2. Some tools from coincidence degree theory

The method to be used in this paper involves the applications of operators of index zero and fixed points theories. In order to make this presentation as self-contained as possible we introduce a few concepts and results about the operators of index zero as follows. For more details see R. E. Gaines and J. L. Mawhin [4].

Definition 2.1. Let X, Y be real Banach spaces, $L : \mathrm{Dom}(L) \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L is said to be a Fredholm mapping of index zero if

$$\dim\big(\mathrm{Ker}(L)\big) = \mathrm{codim}\big(\mathrm{Im}(L)\big) < +\infty$$

and $\mathrm{Im}(L)$ is closed in Y.

Suppose that X and Y are Banach spaces, $L : \mathrm{Dom}(L) \subset X \to Y$ is a linear Fredholm operator of index 0 with range $\mathrm{Im}(L)$ and null space $\mathrm{Ker}(L)$, Y_2 is a complement of $\mathrm{Im}(L)$ in Y, X_2 is a complement of $\mathrm{Ker}(L)$ in X. Thus, we have the decomposition $X = \mathrm{Ker}(L) \oplus X_2$ and $Y = \mathrm{Im}(L) \oplus Y_2$. Let $P : X \to \mathrm{Ker}(L)$ be the projection parallel to X_2 and $Q : Y \to Y_2$ the projection parallel to Y_2. Hence $\mathrm{Im}(P) = \mathrm{Ker}(L)$ and $\mathrm{Im}(L) = \mathrm{Ker}(Q)$. Let $K_P : \mathrm{Im}(L) \to \mathrm{Dom}(L) \cap \mathrm{Ker}(P)$ be the inverse of L restricted to $\mathrm{Dom}(L) \cap \mathrm{Ker}(P)$ and $K_{P,Q} = K_P(I - Q)$.

Definition 2.2. The mapping N is said to be L-compact on $\overline{\Omega}$ if Ω is an open bounded subset of X, $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Since $\mathrm{Im}(Q)$ is isomorphic to $\mathrm{Ker}(L)$, there exists an isomorphism $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$.

We will be interested to prove the existence of solutions for the operator equation

$$Lu = Nu, \tag{6}$$

where $u \in \mathrm{Dom}(L) \cap \overline{\Omega}$ verifying the equation (6).

The following result is due to R. E. Gaines and J. L. Mawhin [4].

Proposition 2.3 (Mawhin's Continuation Theorem). *Let L be a Fredholm mapping of index 0 and let N be L-compact on $\overline{\Omega}$. Suppose that*

(1) *for each $\lambda \in (0,1)$, $u \in \partial\Omega$,*

$$Lu \neq \lambda Nu,$$

(2) *$QNu \neq 0$ for each $u \in \mathrm{Ker}\, L \cap \partial\Omega$ and*

$$\deg(JQN, \Omega \cap \mathrm{Ker}\, L, 0) \neq 0,$$

where $J : \mathrm{Im}\, Q \to \mathrm{Ker}\, L$ is an isomorphism.

Then the equation $Lu = Nu$ has at least one solution in $\mathrm{Dom}\, L \cap \overline{\Omega}$.

In the following lemma we recall some inequalities which will be very useful for obtaining an important a priori estimation for the Mawhin's continuation theorem. See C. Bereanu [2].

Lemma 2.4. (i) *For any $u \in C^2(\mathbb{R}, \mathbb{R})$, ω-periodic one has*

$$\int_0^\omega (u'(t))^2\, dt \leq \left(\frac{\omega}{\pi}\right)^2 \int_0^\omega (u''(t))^2\, dt.$$

(ii) *For any $u \in C^2(\mathbb{R}, \mathbb{R})$, ω-periodic one has*

$$|u'|_\infty \leq \int_0^\omega |u''(t)|\, dt.$$

3. Proof of Theorem 1.1

Proof. Consider the following Banach spaces

$$X = \{u \mid u \in C^2(\mathbb{R}, \mathbb{R}), u(t+\omega) = u(t) \text{ for all } t \in \mathbb{R}\}$$

and

$$Y = \{u \mid u \in C(\mathbb{R}, \mathbb{R}), u(t+\omega) = u(t) \text{ for all } t \in \mathbb{R}\},$$

with the norms

$$|u|_2 = \max\{|u|_\infty, |u'|_\infty, |u''|_\infty\},$$

where $|u|_\infty = \max_{t \in [0,\omega]} |u(t)|$.

If $\mathrm{Dom}(L) = \{u \in X \mid u \in C^3(\mathbb{R}, \mathbb{R})\}$, we define a linear operator $L : \mathrm{Dom}(L) \subset X \to Y$ by

$$Lu = u'''.$$

We also define a nonlinear operator $N : X \to Y$ by setting

$$Nu = -au'' - g(u') - cu + p(t).$$

It is not difficult to see that

$$\mathrm{Ker}(L) = \mathbb{R} \quad \text{and} \quad \mathrm{Im}(L) = \left\{u \mid u \in Y, \int_0^\omega u(s)\, ds = 0\right\}.$$

Thus the operator L is a Fredholm operator with index zero.

Define the continuous projector $P : X \to \mathrm{Ker}(L)$ and the averaging projector $Q : Y \to Y$ by setting

$$Pu(t) = u(0)$$

and

$$Qu(t) = \frac{1}{\omega}\int_0^\omega u(s)\, ds.$$

Hence, $\mathrm{Im}(P) = \mathrm{Ker}(L)$, $\mathrm{Ker}(Q) = \mathrm{Im}(L)$ and $K_P : \mathrm{Im}(L) \to \mathrm{Dom}(L) \cap \mathrm{Ker}\, P$ the inverse of $L|_{\mathrm{Dom}(L) \cap \mathrm{Ker}\, P}$, is give by

$$K_P v(t) = \left(\frac{t}{2} - \frac{t^2}{2\omega}\right) \int_0^\omega \int_0^{s_1} v(s_2)\, ds_2\, ds_1 - \frac{t}{\omega}\int_0^\omega \int_0^{s_1} \int_0^{s_2} v(s_3)\, ds_3\, ds_2\, ds_1$$
$$+ \int_0^t \int_0^{s_1} \int_0^{s_2} v(s_3)\, ds_3\, ds_2\, ds_1.$$

Then $QN : X \to Y$ and $K_P(I-Q)N : X \to X$ read

$$QNu(t) = -\frac{1}{\omega}\int_0^\omega g(u')\,ds - \frac{c}{\omega}\int_0^\omega u\,ds + \frac{1}{\omega}\int_0^\omega p(s)\,ds,$$

$$K_P(I-Q)Nu(t) = \left(\frac{t}{2}-\frac{t^2}{2\omega}\right)\int_0^\omega\int_0^{s_1} Nu(s_2)\,ds_2\,ds_1 - \frac{t}{\omega}\int_0^\omega\int_0^{s_1}\int_0^{s_2} Nu(s_3)\,ds_3\,ds_2\,ds_1$$
$$+\int_0^t\int_0^{s_1}\int_0^{s_2} Nu(s_3)\,ds_3\,ds_2\,ds_1 - QNu\left[\frac{t\omega^2}{12}-\frac{t^2\omega}{4}+\frac{t^3}{6}\right].$$

Clearly, QN and $K_P(I-Q)N$ are continuous. By using the Arzela–Ascoli theorem, it is not difficult to prove that $K_P(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

We will show that there is some positive number R_0, to be given later, such that the assumptions of Proposition 2.3 are satisfied in set

$$\Omega_0 := \{u \in X \mid |u|_2 < R_0\}.$$

For each $\lambda \in (0,1)$, consider u solution of the equation

$$u''' + a\lambda u'' + \lambda g(u') + c\lambda u = \lambda p(t). \tag{7}$$

Multiplying equation (7) by u'', using integration by parts and the boundary conditions $u'(0) = u'(\omega)$ and $u''(0) = u''(\omega)$ we conclude that

$$\int_0^\omega \frac{1}{2}\frac{d}{dt}\left(\left(u''(t)\right)^2\right)dt + a\lambda\int_0^\omega \left(u''(t)\right)^2 dt + \lambda\int_0^\omega \frac{d}{dt}\int_0^{u'(t)} g(s)\,ds\,dt - c\lambda\int_0^\omega \left(u'(t)\right)^2 dt$$
$$= \lambda\int_0^\omega p(t)u''(t)\,dt,$$
$$\lambda\int_0^\omega \left(u''(t)\right)^2 dt - \frac{c}{a}\lambda\int_0^\omega \left(u'(t)\right)^2 dt = \lambda\frac{1}{a}\int_0^\omega p(t)u''(t)\,dt.$$

From $0 < \lambda < 1$ and (4) we conclude that

$$\int_0^\omega \left(u''(t)\right)^2 dt \le \frac{1}{|a|}\int_0^\omega |p(t)|\,|u''(t)|\,dt \le \frac{1}{2a^2}\int_0^\omega |p(t)|^2\,dt + \frac{1}{2}\int_0^\omega |u''(t)|^2\,dt.$$

Therefore,

$$\frac{1}{2}\int_0^\omega \left(u''(t)\right)^2 dt \le \frac{1}{2a^2}\int_0^\omega |p(t)|^2\,dt,$$

that is,

$$\int_0^\omega \left(u''(t)\right)^2 dt \le \frac{1}{a^2}\int_0^\omega |p(t)|^2\, dt =: M_1. \tag{8}$$

By Lemma 2.4, we have

$$\int_0^\omega \left(u'(t)\right)^2 dt \le \left(\frac{\omega}{\pi}\right)^2 \int_0^\omega \left(u''(t)\right)^2 dt \le \left(\frac{\omega}{\pi}\right)^2 M_1 =: M_2 \tag{9}$$

and

$$|u'|_\infty \le \int_0^\omega |u''(t)|\, dt \le \omega^{1/2} M_1^{1/2}.$$

Now we take

$$g_{\omega^{1/2}M_1^{1/2}} = \sup_{\substack{s\in[-\omega^{1/2}M_1^{1/2} \\ \omega^{1/2}M_1^{1/2}]}} |g(s)|. \tag{10}$$

Multiplying the equation (7) by u, using integration by parts and boundary conditions $u(0) = u(\omega)$ and $u''(0) = u''(\omega)$ we conclude that

$$\begin{aligned}
&-\int_0^\omega u''(t)u'(t)\, dt - a\lambda \int_0^\omega \left(u'(t)\right)^2 dt + \lambda \int_0^\omega g\left(u'(t)\right)u(t)\, dt + c\lambda \int_0^\omega \left(u(t)\right)^2 dt \\
&\quad = \lambda \int_0^\omega p(t)u(t)\, dt.
\end{aligned}$$

Note that

$$\int_0^\omega u''(t)u'(t)\, dt = \frac{1}{2}\int_0^\omega \frac{d}{dt}\left(u'(t)\right)^2 dt = \frac{1}{2}\left(\left(u'(\omega)\right)^2 - \left(u'(0)\right)^2\right) = 0.$$

So

$$-a\lambda \int_0^\omega \left(u'(t)\right)^2 dt + \lambda \int_0^\omega g\left(u'(t)\right)u(t)\, dt + c\lambda \int_0^\omega \left(u(t)\right)^2 dt = \lambda \int_0^\omega p(t)u(t)\, dt.$$

From $0 < \lambda < 1$, the Cauchy–Schwarz inequality, (9) and (10) and the last equality we conclude that

$$\int_0^\omega \left(u(t)\right)^2 dt \le M_3. \tag{11}$$

Now, multiplying equation (7) by u''' and using that $u''(0) = u''(\omega)$, we have

$$\int_0^\omega \big(u'''(t)\big)^2\,dt + \lambda\int_0^\omega g\big(u'(t)\big)u'''(t)\,dt + c\lambda\int_0^\omega u(t)u'''(t)\,dt = \lambda\int_0^\omega p(t)u'''(t)\,dt.$$

From $0 < \lambda < 1$, the Cauchy–Schwarz inequality, (10) and (11), we conclude that

$$\int_0^\omega \big(u'''(t)\big)^2\,dt \le M_4, \tag{12}$$

where M_4 is independent of λ.

If u is a solution of (7), by (8), (9), (11) and (12) we conclude that

$$\begin{aligned}\|u\|^2_{H^3(0,\omega)} &= \int_0^\omega \big(u(t)\big)^2\,dt + \int_0^\omega \big(u'(t)\big)^2\,dt + \int_0^\omega \big(u''(t)\big)^2\,dt + \int_0^\omega \big(u'''(t)\big)^2\,dt\\ &\le M_1 + M_2 + M_3 + M_4,\end{aligned}$$

with $M_1 + M_2 + M_3 + M_4$ independent of λ, i.e., $u \in H^3(0,\omega)$ and

$$\|u\|_{H^3(0,\omega)} \le (M_1 + M_2 + M_3 + M_4)^{1/2}. \tag{13}$$

As $H^3(0,\omega)$ is immersed continuously in $C^2(0,\omega)$, there exists a constant $C > 0$ such that

$$|u|_2 \le C\|u\|_{H^3(0,\omega)} \qquad \text{for all } u \in H^3(0,\omega). \tag{14}$$

Therefore, if u is a solution of (7), it follows from (13) and (14) that

$$|u|_2 \le C(M_1 + M_2 + M_3 + M_4)^{1/2}. \tag{15}$$

Take $M_5 > 0$ large enough such that

$$\left|\frac{1}{c}\left(g(0) - \frac{1}{\omega}\int_0^\omega p(t)\,dt\right)\right| < M_5. \tag{16}$$

We consider

$$R_0 > \max\{M_5, C(M_1 + M_2 + M_3 + M_4)^{1/2}\}. \tag{17}$$

We define

$$\Omega_0 := \{u \in X \mid |u|_2 < R_0\}.$$

By (15) and (17), for each $\lambda \in (0,1)$, $u \in \partial\Omega_0$,

$$Lu \ne \lambda Nu.$$

Now we have two cases:

First case, when $c > 0$. Take $u \in \operatorname{Ker} L \cap \partial\Omega_0$; we have $u = R_0$ or $u = -R_0$. If $u = R_0$, by (16) and (17) we have

$$\begin{aligned} QNu(t) &= -\frac{1}{\omega}\int_0^\omega g(u')\,ds - \frac{c}{\omega}\int_0^\omega u\,ds + \frac{1}{\omega}\int_0^\omega p(s)\,dt \\ &= -g(0) - cR_0 + \frac{1}{\omega}\int_0^\omega p(s)\,dt < 0. \end{aligned} \tag{18}$$

If $u = -R_0$, by (16) and (17) we have

$$QNu(t) = -g(0) + cR_0 + \frac{1}{\omega}\int_0^\omega p(s)\,dt > 0. \tag{19}$$

Furthermore, define a continuous function $H(u, \mu)$ by

$$H(u,\mu) = -(1-\mu)u + \mu\Big(-\frac{1}{\omega}\int_0^\omega g\big(u'(t)\big)\,dt - \frac{c}{\omega}\int_0^\omega u\,dt + \frac{1}{\omega}\int_0^\omega p(s)\,dt\Big), \ \mu \in [0,1].$$

It follows from (18) and (19) that

$$H(u,\mu) \neq 0 \quad \text{for all } u \in \partial\Omega_0 \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned} \deg(QN, \Omega_0 \cap \operatorname{Ker} L, 0) &= \deg\Big(-\frac{1}{\omega}\int_0^\omega g\big(u'(t)\big)\,dt - \frac{c}{\omega}\int_0^\omega u\,dt \\ &\qquad + \frac{1}{\omega}\int_0^\omega p(s)\,dt, \Omega_0 \cap \operatorname{Ker} L, 0\Big) \\ &= \deg(-u, \Omega_0 \cap \operatorname{Ker} L, 0) = -1 \neq 0. \end{aligned}$$

Second case, when $c < 0$. We have to $u \in \operatorname{Ker} L \cap \partial\Omega_0$, that $u = R_0$ or $u = -R_0$. If $u = R_0$, by (16) and (17) we have

$$\begin{aligned} QNu(t) &= -\frac{1}{\omega}\int_0^\omega g(u')\,ds - \frac{c}{\omega}\int_0^\omega u\,ds + \frac{1}{\omega}\int_0^\omega p(s)\,dt \\ &= -g(0) - cR_0 + \frac{1}{\omega}\int_0^\omega p(s)\,dt > 0. \end{aligned} \tag{20}$$

If $u = -R_0$, by (16) and (17) we have

$$QNu(t) = -g(0) + cR_0 + \frac{1}{\omega}\int_0^\omega p(s)\,dt < 0. \tag{21}$$

Furthermore, define a continuous function $H(u,\mu)$ by

$$H(u,\mu) = (1-\mu)u + \mu\Big(-\frac{1}{\omega}\int_0^\omega g\big(u'(t)\big)\,dt - \frac{c}{\omega}\int_0^\omega u\,dt + \frac{1}{\omega}\int_0^\omega p(s)\,dt\Big), \quad \mu \in [0,1].$$

It follows from (20) and (21) that

$$H(u,\mu) \neq 0 \qquad \text{for all } u \in \partial\Omega_0 \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned}\deg(QN, \Omega_0 \cap \operatorname{Ker} L, 0) &= \deg\Big(-\frac{1}{\omega}\int_0^\omega g\big(u'(t)\big)\,dt - \frac{c}{\omega}\int_0^\omega u\,dt \\ &\qquad + \frac{1}{\omega}\int_0^\omega p(s)\,dt, \Omega_0 \cap \operatorname{Ker} L, 0\Big) \\ &= \deg(u, \Omega_0 \cap \operatorname{Ker} L, 0) = 1 \neq 0.\end{aligned}$$

In view of the discussion above, we conclude from Proposition 2.3 that equation (5) has a solution in $\operatorname{Dom}(L) \cap \overline{\Omega}_0$. □

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