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Diophantine equations with binary recurrences associated to the Brocard–Ramanujan problem

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Abstract. In this paper, applying the Primitive Divisor Theorem, we solve completely the diophantine equation

$$G_{n_1} G_{n_2} \dots G_{n_k} + 1 = G_m^2$$

in non-negative integers $k > 0$, m and $n_1 < n_2 < \cdots < n_k$ if the binary recurrence $\{G_n\}_{n=0}^{\infty}$ is either the Fibonacci sequence, or the Lucas sequence, or it satisfies the recurrence relation $G_n = AG_{n-1} - G_{n-2}$ with $G_0 = 0$, $G_1 = 1$ and an arbitrary positive integer A. The more specific case

$$G_n G_{n+1} \dots G_{n+k-1} + 1 = G_m^2$$

was investigated by Marques [3] in *Portugaliae Mathematica* in the case of the Fibonacci sequence.

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1. Introduction

Let F_n denote the n^{th} term of the Fibonacci sequence. The variant

$$F_n F_{n+1} \dots F_{n+k-1} + 1 = F_m^2 \tag{1}$$

of the Brocard–Ramanujan problem was investigated by Marques [3]. Applying the Primitive Divisor Theorem (in short, PDT), the author proved that (1) has no solution in positive integers n, k and m. Although the idea of the proof is correct and adequate for equation (1), unfortunately by some inaccuracy in the evaluation, the solutions $F_4 + 1 = F_3^2$ and $F_6 + 1 = F_4^2$ to (1) have not been noticed. Article [4] of the same author deals with a modification of (1).

Consider now the following generalization. Replace the Fibonacci sequence by any binary recurrence $\{G_n\}_{n=0}^{\infty}$, and suppose that the subscripts of the terms in the product on the left-hand side of (1) do not necessarily form an arithmetic progression with difference 1. More precisely, we will examine the diophantine equation

$$G_{n_1} G_{n_2} \dots G_{n_k} + 1 = G_m^2 \tag{2}$$

in integers $k \geq 1$, $m \geq 0$ and $0 \leq n_1 < n_2 < \cdots < n_k$.

In this paper, the complete solution to (2) is provided if the terms G_n are either the Fibonacci numbers, or the Lucas numbers or they satisfy the relation $G_n = AG_{n-1} - G_{n-2}$, where A is any positive integer and $G_0 = 0$, $G_1 = 1$.

Since the primitive divisors play a crucial role in the proofs, first we deal with them. Suppose that α and β are two algebraic numbers such that $\alpha + \beta$ and $\alpha\beta$ are non-zero coprime integers, and α/β is not a root of unity. The sequence

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{D}} \tag{3}$$

is called Lucas sequence linked to α and β, where $\sqrt{D} = \alpha - \beta$. It is known that the terms U_n satisfy the recurrence relation $U_n = (\alpha + \beta)U_{n-1} - (\alpha\beta)U_{n-2}$ for $n \geq 2$, further the initial values are $U_0 = 0$, $U_1 = 1$. Formula (3) is often called Binet formula corresponding to the sequence $\{U_n\}_{n=0}^{\infty}$. Obviously, the linear recursive sequence given previously by $G_n = AG_{n-1} - G_{n-2}$ and $G_0 = 0$, $G_1 = 1$ is included in Lucas sequences if $A \geq 3$.

The prime p is a primitive divisor of U_n if

$$p \mid U_n \quad \text{but} \quad p \nmid DU_1 \dots U_{n-1}. \tag{4}$$

The question of primitive divisors has more than one century of history. The most remarkable results are due to Bilu, Hanrot and Voutier [1], who completely described the sequences that have no primitive divisors at some term U_n, and these terms are exactly determined. For our purpose, it is sufficient to apply the weaker result of Carmichael [2], where he defined the primitive divisors without the factor D in (4).

Theorem 1.1 (Carmichael, PDT). *If α and β are real numbers and $n \neq 1, 2, 6$, then U_n contains a primitive divisor, except when $n = 12$, $\alpha + \beta = 1$, $\alpha\beta = -1$ (i.e. except the Fibonacci sequence).*

In the sequel we consider the posed problem for the three aforementioned sequences.

2. Fibonacci sequence

In this section we generalize the result of Marques [3]. It is well known that in the case of the Fibonacci sequence $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $D = 5$ hold in formula (3). Putting

$$\varepsilon = \varepsilon(m) = \begin{cases} 2 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd,} \end{cases}$$

we can formulate the result associated to the Fibonacci sequence.

Theorem 2.1. *The diophantine equation*

$$F_{n_1}F_{n_2}\dots F_{n_k} + 1 = F_m^2 \tag{5}$$

in positive integers k, m and $3 \le n_1 < n_2 < \cdots < n_k$ has an infinite family of solutions given by

$$F_{m-\varepsilon}F_{m+\varepsilon} + 1 = F_m^2, \qquad m \ge 5.$$

Moreover, there exist only two sporadic solutions: $F_4 + 1 = F_3^2$ and $F_6 + 1 = F_4^2$.

Remark 2.2. Note that the relation $F_1 + 1 \ne F_m^2$ makes it possible to avoid $F_1 = 1$ on the left-hand side of (5). This does not change if one replaces F_1 with $F_2 = 1$ or F_1F_2. Thus we can assume that $3 \le n_1$.

On the other hand, the two sporadic solutions can be included in the infinite family by allowing F_1 or F_2 among the factors of the product: $F_2F_4 + 1 = F_3^2$ and $F_2F_6 + 1 = F_4^2$.

In the proof we will use

Lemma 2.3. $F_m^2 - 1 = F_{m-\varepsilon}F_{m+\varepsilon}$ *holds for any positive integer m.*

Proof. This is an immediate consequence of the identities (3) in [3]. Or one can show the result by the direct application of the Binet formula for the Fibonacci numbers. □

Proof of Theorem 2.1. Taking a solution to (5), by Lemma 2.3 we get

$$F_{n_1}F_{n_2}\dots F_{n_k} = F_{m-\varepsilon}F_{m+\varepsilon}. \tag{6}$$

Suppose that $m \geq 15$. Then $13 \leq m - \varepsilon < m + \varepsilon$, therefore we can apply Theorem 1.1. Since $F_{m+\varepsilon}$ has a primitive divisor, $n_k = m + \varepsilon$ and (6) reduces to

$$F_{n_1} F_{n_2} \ldots F_{n_{k-1}} = F_{m-\varepsilon}. \tag{7}$$

Now $F_{m-\varepsilon} > 1$ entails $k \geq 2$. Using the same arguments linked to primitive divisors as above, (7) provides $n_{k-1} = m - \varepsilon$. Consequently, $k = 2$, i.e., there are no more terms on the left-hand side of (7). Thus we obtain an infinite family of solutions $F_{m-\varepsilon} F_{m+\varepsilon} + 1 = F_m^2$, $m \geq 15$. Clearly, this can be extended for $m \geq 5$ by Lemma 2.3.

Assume that $3 \leq m \leq 14$ is fixed. Since $F_{2m} > F_m^2$ for $m > 1$, it is easy to check all the candidates of the solution to (5) by running over on the possibilities

$$F_3^{\delta_3} F_4^{\delta_4} \ldots F_{2m-1}^{\delta_{2m-1}} + 1 = F_m^2$$

with $\delta_i \in \{0, 1\}$. The verification yields the aforementioned two sporadic solutions.

Clearly, neither $m = 1$ nor $m = 2$ is possible in (5). □

3. The companion sequence of Fibonacci numbers

As usual, the companion sequence (or associate sequence) of U_n defined by (3) is the sequence $V_n = \alpha^n + \beta^n$. In the case of Fibonacci sequence, its companion is denoted by $\{L_n\}_{n=0}^{\infty}$ and often called also Lucas sequence. In order to avoid the ambiguousness in U_n and L_n, we always call the terms L_n Fibonacci–Lucas numbers. At the first sight the application of PDT in solving the analogous problem of (2) for Fibonacci–Lucas numbers is impossible since $\{L_n\}_{n=0}^{\infty}$ is not a Lucas sequence in sense of (3). But the identities in the following lemma allow us to transform the problem

$$L_{n_1} L_{n_2} \ldots L_{n_k} + 1 = L_m^2 \tag{8}$$

into an equivalent form containing only Fibonacci numbers.

Lemma 3.1.

$$L_m^2 - 1 = \begin{cases} F_{3m}/F_m & \textit{if } m \textit{ is even}, \\ 5F_{m-1}F_{m+1} & \textit{if } m \textit{ is odd}. \end{cases} \tag{9}$$

Proof. This can be easily obtained by using the explicit formulae for F_n and L_n. □

Theorem 3.2. *The diophantine equation*

$$L_{n_1} L_{n_2} \dots L_{n_k} + 1 = L_m^2 \tag{10}$$

in non-negative integers $k > 0$, m *and* $n_1 < n_2 < \dots < n_k$ $(n_i \neq 1)$ *has only the single solution* $L_2 + 1 = L_0^2$.

Before turning to the details of the proof, we note that $L_1 + 1 \neq L_m^2$. Therefore we may suppose that $n_i \neq 1$ $(i = 1, \dots, k)$. We also remark that $L_2 + 1 = L_0^2$ trivially satisfies the weaker equation $L_n L_{n+1} \dots L_{n+k-1} + 1 = L_m^2$.

Proof of Theorem 3.2. Considering a solution to (10), assume first that m is a positive even integer. Then (8), together with (9) and the well-known identity $F_n L_n = F_{2n}$, implies that

$$\frac{F_{2n_1}}{F_{n_1}} \frac{F_{2n_2}}{F_{n_2}} \dots \frac{F_{2n_k}}{F_{n_k}} = \frac{F_{3m}}{F_m}. \tag{11}$$

Suppose now that $m \geq 14$. Then F_{3m} has a primitive divisor by Theorem 1.1. Consequently, $2n_k = 3m$, i.e., $n_k = 3m/2 > m$. Obviously, $k \neq 1$, otherwise (11) reduces to $F_m = F_{n_1}$, and we arrive at a contradiction to $m = n_1$. Assuming $k = 2$, (11) simplifies to

$$\frac{F_{2n_1}}{F_{n_1}} F_m = F_{n_2}.$$

Since $n_2 > m$, F_{n_2} contains a primitive divisor. Then $n_2 = 2n_1$ and $m = n_1$, which contradicts $n_2 = 3m/2$. If $k \geq 3$, observe that $n_{k-1} < m$ holds, otherwise it would contradict $L_{n_{k-1}} L_{n_k} < L_m^2$. Thus, again by $n_k = 2n_{k-1}$, we conclude that

$$\frac{F_{2n_1}}{F_{n_1}} \dots \frac{F_{2n_{k-2}}}{F_{n_{k-2}}} F_m = F_{n_{k-1}}, \tag{12}$$

which has no solution since $m \geq 14$. Therefore F_m has a primitive divisor on the left-hand side of (12), which cannot exist on the right-hand side.

In the next part we must check the cases when the even m is at most 12. Factorizing the candidates $L_m^2 - 1$, none of them is a product of other Lucas numbers, except $L_0^2 - 1 = L_2$.

Assume now that the subscript m is odd. Then again by (8) and (9), we obtain

$$L_{n_1} L_{n_2} \dots L_{n_k} = 5 F_{m-1} F_{m+1}. \tag{13}$$

Since 5 divides no Lucas numbers, (13) is impossible. □

4. The sequence $U_n = AU_{n-1} - U_{n-2}$

Let A denote a positive integer. Assume that the terms of the linear recurrence $\{U_n\}_{n=0}^{\infty}$ satisfy the recurrence relation $U_n = AU_{n-1} - U_{n-2}$ with the initial values $U_0 = 0$ and $U_1 = 1$. Thus the Binet formula (3) is valid with

$$\alpha = \frac{A + \sqrt{A^2 - 4}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{A^2 - 4}}{2}. \tag{14}$$

For example, the sequence of Balancing numbers is corresponding to $A = 6$.

In this section we consider a more general question than (2) for the sequence $\{U_n\}_{n=0}^{\infty}$. Namely, for $A \geq 3$ we study and solve the diophantine equation

$$U_{n_1} U_{n_2} \dots U_{n_k} + U_t^2 = U_m^2$$

in positive integers k, t, m and $2 \leq n_1 < n_2 < \dots < n_k$.

Note that if $A = 1$ then the sequence $\{U_n\}_{n=0}^{\infty}$ is the periodic repetition of the cycle

$$0,\ 1,\ 1,\ 0,\ -1,\ -1$$

and the posed problem is of no interest anymore. The case $A = 2$ is still beyond reach since it is exactly the original Brocard–Ramanujan problem if we prescribe $t = 1$, $n_1 = 1$ and $n_i + 1 = n_{i+1}$ for $i = 1, \dots k - 1$. In order to avoid the trivial and the very difficult cases above, we suppose that $A \geq 3$. Then $D = A^2 - 4 \geq 5$. Thus α and β are positive real numbers, moreover,

$$\{U_n\}_{n=0}^{\infty} : U_0 = 0, \quad U_1 = 1, \quad U_2 = A, \quad U_3 = A^2 - 1, \quad U_4 = A^3 - 2A,$$
$$U_5 = A^4 - 3A^2 + 1, \quad U_6 = A^5 - 4A^3 + 3A, \dots$$

is strictly monotonically increasing.

The precise result is the following.

Theorem 4.1. *Suppose that $A \geq 3$. All solutions to the diophantine equation*

$$U_{n_1} U_{n_2} \dots U_{n_k} + U_t^2 = U_m^2 \tag{15}$$

in positive integers k, t, m and $2 \leq n_1 < n_2 < \dots < n_k$ are determined by the two infinite families of solutions

- $U_{2t+1} + U_t^2 = U_{t+1}^2$,
- $U_{m-t} U_{m+t} + U_t^2 = U_m^2$, $m \geq t + 2$.

The condition $2 \le n_1$ is prescribed in order to avoid the trivial factor $B_1 = 1$ on the left-hand side of (15). It does not cause any problem since if the product there contains only one term then $1 + U_t^2 = U_m^2$ is not soluble.

In the proof we will use the following identity.

Lemma 4.2. *If* $n \ge k$, *then* $U_n^2 - U_k^2 = U_{n-k} U_{n+k}$.

Proof. By (14), we have $\alpha\beta = 1$, and then

$$\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 - \left(\frac{\alpha^k - \beta^k}{\alpha - \beta}\right)^2 = \frac{\alpha^{2n} + \beta^{2n} - \alpha^{2k} - \beta^{2k}}{(\alpha - \beta)^2} = \frac{\alpha^{n-k} - \beta^{n-k}}{\alpha - \beta} \frac{\alpha^{n+k} - \beta^{n+k}}{\alpha - \beta}.$$

□

We also need

Lemma 4.3. *If* $A \ge 3$, *then* $U_{2n} > U_n^2$ *holds for any positive integer n.*

Proof. Since (14) implies that $\alpha > \beta > 0$, and $\alpha - \beta > 1$ also holds, we conclude that

$$\frac{U_{2n}}{U_n} = \frac{\alpha^{2n} - \beta^{2n}}{\alpha^n - \beta^n} = \alpha^n + \beta^n > \frac{\alpha^n - \beta^n}{\alpha - \beta} = U_n.$$

□

Proof of Theorem 4.1. Suppose that (15) has a solution. Clearly, $m + t \ge 3$. By Lemma 4.2,

$$U_{n_1} U_{n_2} \dots U_{n_k} = U_{m-t} U_{m+t} \tag{16}$$

holds.

I. Assume first that $m + t > 6$. Now, by Theorem 1.1, U_{m+t} has a primitive divisor, consequently $n_k = m + t$. Simplifying (16) by U_{m+t}, we obtain

$$U_{n_1} U_{n_2} \dots U_{n_{k-1}} = U_{m-t}. \tag{17}$$

Now we distinguish two principal cases.

I/1. If $m - t = 1$, then $U_{n_1} U_{n_2} \dots U_{n_{k-1}} = 1$ is not soluble since $2 \le n_i$, $A \ge 3$ and the sequence $\{U_n\}_{n=0}^{\infty}$ is strictly monotonically increasing. Therefore $k = 1$, which together with the conditions $m = t + 1$ and $n_1 = m + t$ leads to the solution $U_{2t+1} + U_t^2 = U_{t+1}^2$, $t \ge 3$. Obviously, we can extend this family for $t = 1, 2$ as well.

I/2. Supposing $m - t \ge 2$, first consider the case $m - t > 6$. By (17) and PDT, $n_{k-1} = m - t$ follows. Thus we arrive at the second infinite family of solutions given by $U_{m-t} U_{m+t} + U_t^2 = U_m^2$, $m \ge t + 7$. Clearly, we may even write $m \ge t + 2$

to extend the validity by (4.2). Assume now that $m - t = 6$. Solving all the possible equations

$$A^{\delta_1}(A^2-1)^{\delta_2}(A^3-2A)^{\delta_3}(A^4-3A^2+1)^{\delta_4} = A^5-4A^3+3A$$

with $\delta_i \in \{0,1\}$, we find no integer solution in $A \geq 3$.

Finally, for $2 \leq m - t \leq 5$ a similar investigation to the case $m - t = 6$ shows no further solution.

II. Let $m + t = 6$. Observe that $m + t$ and $m - t$ have the same parity. Therefore the right-hand side of (16) may contain only one of the terms U_4U_6 and U_2U_6 as a polynomial of A. Obviously, by Lemma 4.3, $n_k \leq 11$ in both cases. The verifications

$$\prod_{i=2}^{11} U_i^{\delta_i} = U_jU_6$$

for $j = 2$ and then for $j = 4$ do not yield further solutions.

III. Let $4 \leq m + t \leq 5$. Now, by Theorem 1.1 and the analogous equation to (17), we must consider only $U_2 = U_3$. But the roots of $A = A^2 - 1$ are not integer.

IV. Finally, $m + t = 3$ gives $t = 1$ and $m = 2$. Here, due to Lemma 4.3, we conclude the following possibilities:

$$U_2^{\delta_2}U_3^{\delta_3} + 1 = U_2^2, \tag{18}$$

with $\delta_i \in \{0,1\}$. Since (18) does not possess any additional solution, the proof of Theorem 4.1 is complete. □

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