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Global attractor for a nonlinear model with periodic boundary value condition*

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Abstract. In this paper, we consider the existence of a global attractor for a nonlinear model with periodic boundary value condition. Based on the iteration technique for regularity estimates and the classical existence theorem of global attractors, we prove that the equation possesses a global attractor on some affine space of H^k $(0 \leq k < +\infty)$.

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1. Introduction

Let $\Omega = [0, L] \times [0, L]$, where $L > 0$. Consider the nonlinear model describing the process growing of a crystal surface:

$$u_t = -a\Delta^2 u - \mu\nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2}\right), \qquad (x, t) \in \Omega \times \mathbb{R}^+. \tag{1}$$

On the basis of physical considerations, equation (1) is supplemented with

$$u \text{ is } L \text{ periodic}, \qquad \forall t \in \mathbb{R}^+,$$

and

$$u(x, 0) = u_0(x), \qquad \forall x \in \Omega. \tag{2}$$

Equation (1) describes the process growing of a crystal surface. Here, a and μ are positive constants, and $u(x, t)$ denotes a displacement of height of surface from

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the standard level ($u = 0$) at a position $x \in \Omega$. The term $-a\Delta^2 u$ in equation (1) denotes a surface diffusion of adatoms which is caused by the difference of the chemical potential. In the meantime, $-\mu\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ denotes the effect of surface roughening.

During the past years, many authors have paid much attention to equation (1). It was Johnson et al. [6] who presented this equation for describing the process of growing of a crystal surface on the basis of the BCF theory. Rost and Krug [9] studied the unstable epitaxy on singular surfaces using equation (1) with a prescribed slope dependent surface current. In their paper, they derived scaling relations for the late stage of growth, where power law coarsening of the mound morphology is observed. In [8], in the limit of weak desorption, O. Pierre-Louis et al. derived equation (1) for a vicinal surface growing in the step flow mode. This limit turned out to be singular, and nonlinearities of arbitrary order need to be taken into account.

Recently, H. Fujimura and A. Yagi [1], [2] also considered equation (1). In their papers, the uniqueness local solutions and the global solutions were obtained. A dynamical system determined from the initial-boundary value problem of the model equation was constructed, and the asymptotic behavior of trajectories of the dynamical system was also considered. In [4], M. Grasselli, G. Mola and A. Yagi showed that equation (1) endowed with no-flux boundary conditions generates a dissipative dynamical system under very general assumptions on $\partial\Omega$ on a phase-space of L^2-type. They proved that the system possesses a global as well as an exponential attractor. In addition, if $\partial\Omega$ is smooth enough, they showed that each trajectory converges to a single equilibrium by means of a suitable Lojasiewicz–Simon inequality. An estimate of the convergence rate was also obtained in [4].

There is much literature concerned with equation (1); for more results we refer the reader to [3], [5], [10] and the references therein.

The dynamical properties of the fourth-order nonlinear equation (1), such as global asymptotical behaviors of solutions and the global attractors, are important for the study of nonlinear diffusion systems, which ensures the stability of nonlinear diffusion phenomena and provides the mathematical foundation for the study of nonlinear dynamics. During the past years, many authors have studied the attractors of general nonlinear dynamical systems (see [11], [12], [13]). In [14], Zhao and Liu considered a fourth-order parabolic equation in a bounded domain $\Omega \subset \mathbb{R}^2$,

$$u_t + \gamma\Delta^2 u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0,$$

where $\gamma > 0$, $\frac{7}{2} < p \leq 4$. Based on the regularity estimates for the semigroups and the classical existence theorem of global attractors, the authors prove that the equation possesses a global attractor in H^k $(0 \leq k < 5)$ space, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.

In this paper, we are interested in the existence of global attractors for equation (1). Based on Ma and Wang's recent work [7], we shall prove that problem (1)–(2) possesses a global attractor in H^k $(0 \leq k < \infty)$ space, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.

It remains as an interesting open problem the question of showing whether the global attractor is a bounded subset of $H^k(\Omega)$. We recall that this is indeed the case for $k = 0$ and $k = 1$, as shown in [4].

2. Preliminaries

Using the same method as [1, 4], we summarize the result on the existence and uniqueness of global solution for problem (1)–(2).

Lemma 2.1. *Let $u_0 \in H^1_{\text{per}}(\Omega)$. Then problem* (1)–(2) *possesses a unique global solution $u(x,t)$ such that*

$$u \in \mathscr{C}\big([0,\infty); H^1_{\text{per}}(\Omega)\big) \cap \mathscr{C}^1\big((0,\infty); L^2(\Omega)\big) \cap \mathscr{C}\big((0,\infty); H^4_{\text{per}}(\Omega)\big).$$

Noticing the total mass of $u(x,t)$ for problem (1)–(2) is conserved, that is

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx, \qquad \forall t \geq 0.$$

We set a phase space

$$\mathscr{H} = \Big\{u \in H^1(\Omega); \int_\Omega u(x,t)\,dx = 0\Big\}.$$

Hence, for $u_0 \in \mathscr{H}$, let $S(t)u_0 = u(t,u_0)$, $0 \leq t < \infty$. It is easy to check that $S(t)$ defines a nonlinear semigroup acting on $\mathscr{H}$.

In [4], the authors proved the existence of global attractor in $H^1(\Omega)$ space for equation (1) endowed with no-flux boundary conditions. In fact, the main result in [4] holds true for functions with bounded—not necessarily zero—mean value. Using the same method as [4], we give the following result on the existence of global attractor for problem (1)–(2) in $H^1(\Omega)$ space.

Lemma 2.2. *Let $u_0 \in H^1_{\text{per}}(\Omega)$, then the semigroup associated with problem* (1)–(2) *possesses a global attractor in $\mathscr{H}$ which attracts all the bounded sets of $\mathscr{H}$ in the $\mathscr{H}$-norm.*

Now, for problem (1)–(2) we introduce the following spaces:

$$\begin{cases} H = \{u \in L^2(\Omega); \int_\Omega u(x,t)\,dx = 0\}, \\ H_{1/2} = H^2_{\text{per}}(\Omega) \cap H, \\ H_1 = H^4_{\text{per}}(\Omega) \cap H. \end{cases} \tag{3}$$

We define the linear operator L and the nonlinear operator G by

$$\begin{cases} Lu = -a\Delta^2 u, \\ Gu = \nabla \cdot g(u) = -\nabla \cdot \frac{\mu \nabla u}{1+|\nabla u|^2}. \end{cases} \tag{4}$$

It is easy to check that L given by (4) is a sectorial operator and the tractional power operator $(-L)^{1/2}$ is given by $(-L)^{1/2} = -\sqrt{a}\Delta$. The space $H_{1/2}$ is the same as (3), $H_{1/4}$ is given by $H_{1/4} =$ closure of $H_{1/2}$ in $H^1(\Omega)$ and $H_k = H^{4k} \cap H_1$ for $k \geq 1$.

We introduce the following Proposition 2.3, which can be found in [7], [11], [14].

Proposition 2.3. *Assume that $L : H \to H_\kappa$ is a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy $\operatorname{Re}\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathscr{L}^\kappa (\mathscr{L} = -L)$ we have*

(C1) *$T(t) : H \to H_\kappa$ is bounded for all $\kappa \in R^1$ and $t > 0$.*

(C2) *$T(t)\mathscr{L}^\kappa x = \mathscr{L}^\kappa T(t)x$ for all $x \in H_\kappa$.*

(C3) *For each $t > 0$, $\mathscr{L}^\kappa T(t) : H \to H$ is bounded and*

$$\|\mathscr{L}^\kappa T(t)\|_H \leq C_\kappa t^{-\kappa} e^{-\delta t},$$

where some $\delta > 0$ and $C_\kappa > 0$ is a constant depending only on κ.

(C4) *The H_κ-norm can be defined by $\|x\|_{H_\kappa} = \|\mathscr{L}^\kappa x\|_H$.*

Now we give the main result of this paper.

Theorem 2.4. *Let $u_0 \in H_\kappa(\Omega)$. Then, for any $\kappa \geq 0$, the semigroup associated with problem (1)–(2) possesses a global attractor in $H_\kappa(\Omega)$ which attracts all the bounded sets of $H_\kappa(\Omega)$ in the H_κ-norm.*

Remark 2.5. In [4], Grasselli, Mola and Yagi studied the longtime behavior of equation (1) endowed with the boundary conditions

$$\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} \Delta u = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+.$$

In their paper, they proved that equation (1) admits a global attractor in L^2 space and H^1 space. Here, we introduce a generalized space H_κ ($\kappa \in \mathbb{R}^+$), which is a fractional dimension space. Using the iteration technique for regularity estimates and Sobolev's embedding theorem, we obtain the result on the existence of global attractor in the generalized space H_κ.

3. Proof of Theorem 2.4

Based on [7], the solution $u(t, u_0)$ of the problem (1)–(2) can be written as

$$u(t, u_0) = e^{tL}u_0 + \int_0^t e^{(t-\tau)L} Gu \, d\tau = e^{tL}u_0 + \int_0^t (-L)^{1/4} e^{(t-\tau)L} g(u) \, d\tau. \tag{5}$$

In order to prove Theorem 2.4, we shall prove the following two lemmas.

Lemma 3.1. *For any $\kappa \geq 0$, the semigroup $S(t)$ generated by the problem* (1)–(2) *is uniformly compact in H_κ.*

Proof. It suffices to prove that for any bounded set $U \subset H_\kappa$, there exists a constant $C > 0$ such that

$$\|u(t, u_0)\|_{H_\kappa} \leq C, \qquad \forall t \geq 0, \, u_0 \in U \subset H_\kappa, \, \kappa \geq 0. \tag{6}$$

For $\kappa = \frac{1}{4}$, this follows form Lemma 2.2, i.e. for any bounded set $U \subset H_{1/4}$, there is a constant $C > 0$ such that

$$\|u(t, u_0)\|_{H_{1/4}} \leq C, \qquad \forall t \geq 0, \, u_0 \in U \subset H_{1/4}.$$

Then we shall prove (6) for any $\kappa > \frac{1}{4}$, which will be shown in the following steps.

Step 1. We prove that for any bounded set $U \subset H_\kappa$ $\left(\frac{1}{4} < \kappa < \frac{3}{4}\right)$ there is a constant $C > 0$ such that

$$\|u(t, u_0)\|_{H_\kappa} \leq C, \qquad \forall t \geq 0, \, u_0 \in U, \, \frac{1}{4} < \kappa < \frac{3}{4}. \tag{7}$$

Note that

$$\|g(u)\|_H = \int_\Omega |g(u)|^2 \, dx = \mu^2 \int_\Omega \frac{|\nabla u|^2}{(1 + |\nabla u|^2)^2} dx \leq C\|\nabla u\|_H^2 \leq C\|u\|_{H_{1/4}}^2,$$

which means $g : H_{1/4} \to H$ is bounded. Hence,

$$\begin{aligned}
\|u(t,u_0)\|_{H_\kappa} &= \left\| e^{tL}u_0 + \int_0^t (-L)^{1/4} e^{(t-\tau)L} g(u)\, d\tau \right\|_{H_\kappa} \\
&\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^{1/4+\kappa} e^{(t-\tau)L} g(u)\|_H \, d\tau \\
&\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^{1/4+\kappa} e^{(t-\tau)L}\| \cdot \|g(u)\|_H \, d\tau \\
&\le C\|u_0\|_{H_\kappa} + C\int_0^t \tau^{-\varepsilon} e^{-\delta\tau}\, d\tau \le C, \qquad \forall t \ge 0,\ u_0 \in U \subset H_\kappa,
\end{aligned}$$

where $\varepsilon = \frac{1}{4} + \kappa$ $(0 < \varepsilon < 1)$. Then (7) holds.

Step 2. We prove that for any bounded set $U \subset H_\kappa$ $\left(\frac{1}{2} < \kappa < 1\right)$ there is a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\kappa} \le C, \qquad \forall t \ge 0,\ u_0 \in U,\ \frac{1}{2} < \kappa < 1. \tag{8}$$

In fact, by the embedding theorems of fractional order spaces, $\forall \kappa \ge \frac{1}{2}$, we have $H_\kappa \hookrightarrow H^2(\Omega)$. Then

$$\begin{aligned}
\|g(u)\|_{1/4}^2 &\le C\int_\Omega |\nabla g(u)|^2\, dx = \mu^2 C \int_\Omega \left(\frac{|\Delta u|}{1+|\nabla u|^2} - \frac{2|\nabla u|^2 |\Delta u|}{(1+|\nabla u|^2)^2} \right)^2 dx \\
&\le 2\mu^2 C \int_\Omega \left(\frac{|\Delta u|^2}{(1+|\nabla u|^2)^2} + \frac{4|\nabla u|^4 |\Delta u|^2}{(1+|\nabla u|^2)^4} \right) dx \\
&\le C\int_\Omega |\Delta u|^2\, dx \\
&\le C\|u\|_{H_\kappa}^2,
\end{aligned} \tag{9}$$

which means $g : H_\kappa \to H_{1/4}$ is bounded for $\kappa \ge \frac{1}{2}$. Hence,

$$\begin{aligned}
\|u(t,u_0)\|_{H_\kappa} &\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^{1/4} e^{(t-\tau)L} g(u)\|_{H_\kappa}\, d\tau \\
&\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^{\kappa} e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{1/4}}\, d\tau \\
&\le C\|u_0\|_{H_\kappa} + C\int_0^t \tau^{-\kappa} e^{-\delta\tau}\, d\tau \le C, \qquad \forall t \ge 0,\ u_0 \in U \subset H_\kappa,
\end{aligned}$$

where $0 < \kappa < 1$. Then (8) holds.

Step 3. We prove that for any bounded set $U \subset H_\kappa$ $\left(\frac{3}{4} < \kappa < \frac{5}{4}\right)$ there is a constant $C > 0$ such that

$$\|u(t, u_0)\|_{H_\kappa} \le C, \qquad \forall t \ge 0,\, u_0 \in U,\, \frac{3}{4} < \kappa < \frac{5}{4}. \tag{10}$$

In fact, by the embedding theorems of fractional order spaces, for all $\kappa \ge \frac{3}{4}$, we have

$$H_\kappa \hookrightarrow H^3(\Omega), \qquad H_\kappa \hookrightarrow W^{2,4}(\Omega).$$

Then

$$\begin{aligned}
\|g(u)\|_{H_{1/2}}^2 &\le C \int_\Omega |\Delta g(u)|^2\, dx \\
&\le \mu^2 C \int_\Omega \left(\frac{|\nabla \Delta u|}{1 + |\nabla u|^2} - \frac{6|\nabla u|\,|\Delta u|^2}{(1 + |\nabla u|^2)^2} - \frac{2|\nabla u|^2 |\nabla \Delta u|}{(1 + |\nabla u|^2)^2} + \frac{8|\nabla u|^3 |\Delta u|^2}{(1 + |\nabla u|^2)^3} \right)^2 dx \\
&\le C \int_\Omega (|\nabla \Delta u|^2 + |\Delta u|^4)\, dx \\
&\le C(\|u\|_{H_\kappa}^4 + \|u\|_{H_\kappa}^2),
\end{aligned} \tag{11}$$

which means $g : H_\kappa \to H_{1/2}$ is bounded for $\kappa \ge \frac{3}{4}$. Hence,

$$\begin{aligned}
\|u(t, u_0)\|_{H_\kappa} &\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^{1/4} e^{(t-\tau)L} g(u)\|_{H_\kappa}\, d\tau \\
&\le C\|u_0\|_{H_\kappa} + \int_0^t \|(-L)^{\kappa - 1/4} e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{1/2}}\, d\tau \\
&\le C\|u_0\|_{H_\kappa} + C\int_0^t \tau^{-\varepsilon} e^{-\delta\tau}\, d\tau \le C, \qquad \forall t \ge 0,\, u_0 \in U \subset H_\kappa,
\end{aligned}$$

where $\varepsilon = \kappa - \frac{1}{4}$ $(0 < \varepsilon < 1)$. Then (10) holds.

Step 4. We prove that for any bounded set $U \subset H_\kappa$ $\left(1 < \kappa < \frac{3}{2}\right)$, there is a constant $C > 0$ such that

$$\|u(t, u_0)\|_{H_\kappa} \le C, \qquad \forall t \ge 0,\, u_0 \in U,\, 1 < \kappa < \frac{3}{2}. \tag{12}$$

In fact, by the embedding theorems of fractional order spaces, $\forall \kappa \ge 1$, we have

$$H_\kappa \hookrightarrow H^4(\Omega), \qquad H_\kappa \hookrightarrow W^{2,4}(\Omega), \qquad H_\kappa \hookrightarrow W^{2,6}(\Omega), \qquad H_\kappa \hookrightarrow W^{3,4}(\Omega).$$

Then,

$$\begin{aligned}
\|g(u)\|_{H_{3/4}}^2 &\le C\int_\Omega |\nabla\Delta g(u)|^2\,dx\\
&\le \mu^2 C\int_\Omega \left(\frac{|\Delta^2 u|}{1+|\nabla u|^2}-\frac{16|\nabla u\Delta u\nabla\Delta u|}{(1+|\nabla u|^2)^2}-\frac{6|\Delta u|^3}{(1+|\nabla u|^2)^2}-\frac{2|\nabla u|^2\Delta^2 u}{(1+|\nabla u|^2)^2}\right.\\
&\qquad\left.+\frac{48|\nabla u|^2|\Delta u|^3}{(1+|\nabla u|^2)^3}+\frac{24|\nabla u|^3|\Delta u\nabla\Delta u|}{(1+|\nabla u|^2)^3}-\frac{48|\nabla u|^4|\Delta u|^3}{(1+|\nabla u|^2)^4}\right)^2 dx\\
&\le C\int_\Omega (|\Delta^2 u|^2+|\Delta u|^4+|\nabla\Delta u|^4+|\Delta u|^6)\,dx\\
&\le C(\|u\|_{H_\kappa}^2+\|u\|_{H_\kappa}^4+\|u\|_{H_\kappa}^6),
\end{aligned}$$

which means $g: H_\kappa \to H_{3/4}$ is bounded for $\kappa \ge 1$. Hence,

$$\begin{aligned}
\|u(t,u_0)\|_{H_\kappa} &\le C\|u_0\|_{H_\kappa}+\int_0^t \|(-L)^{1/4}e^{(t-\tau)L}g(u)\|_{H_\kappa}\,d\tau\\
&\le C\|u_0\|_{H_\kappa}+\int_0^t \|(-L)^{\kappa-1/2}e^{(t-\tau)L}\|\cdot\|g(u)\|_{H_{3/4}}\,d\tau\\
&\le C\|u_0\|_{H_\kappa}+C\int_0^t \tau^{-\varepsilon}e^{-\delta\tau}\,d\tau \le C, \qquad \forall t\ge 0,\ u_0\in U\subset H_\kappa,
\end{aligned}$$

where $\varepsilon = \kappa - \frac{1}{2}$ $(0<\varepsilon<1)$. Then (12) holds.

Using the same method as above, by iteration we can prove that for any bounded set $U \subset H_\kappa$ $(\kappa \ge 0)$ there exists a constant $C>0$ such that (6) holds. i.e., for all $\kappa \ge 0$ the semigroup $S(t)$ generated by problem (1)–(2) is uniformly compact in H_κ. □

Lemma 3.2. *For any $\kappa \ge 0$, problem* (1)–(2) *has a bounded absorbing set in H_κ.*

Proof. It suffices to prove that for any bounded set $U \subset H_\kappa$ $(\kappa \ge 0)$, there exist $T>0$ and a constant $C>0$ independent of u_0, such that

$$\|u(t,u_0)\|_{H_\kappa} \le C, \qquad \forall t \ge T,\ u_0 \in U \subset H_\kappa. \tag{13}$$

For $\kappa = \frac{1}{4}$, this follows from Lemma 2.2. So we shall prove (13) for any $\kappa > \frac{1}{4}$. We prove it in the following steps:

Step 1. We prove that for any $\frac{1}{4} < \kappa < \frac{3}{4}$, problem (1)–(2) has a bounded absorbing set in H_κ. By (5), we deduce that

$$u(t,u_0) = e^{(t-T)L}u(T,u_0) + \int_T^t e^{(t-\tau)L}g(u)\,d\tau. \tag{14}$$

Assume B is the bounded absorbing set of problem (1)–(2) and satisfy $B \subset H_{1/4}$, we also assume $T_0 > 0$ such that

$$u(t, u_0) \in B, \qquad \forall t > T_0,\ u_0 \in U \subset H_\kappa,\ \kappa > \frac{1}{4}. \tag{15}$$

It is easy to check that

$$\|e^{tL}\| \le Ce^{-\lambda_1^2 t},$$

where $\lambda_1 > 0$ is the first eigenvalue of the equation

$$\begin{cases} -\sqrt{a}\Delta u = \lambda u, \\ u \text{ is periodic}, \\ \int_\Omega u\,dx = 0. \end{cases} \tag{16}$$

Thus, for any given $T > 0$ and $u_0 \in U \subset H_\kappa$ $\left(\kappa > \frac{1}{4}\right)$, we deduce that

$$\lim_{t\to\infty} \|e^{(t-T)L}u(T, \varphi)\|_{H_\kappa} = 0. \tag{17}$$

Using (14), (15) and (17), we have

$$\begin{aligned} \|u(t, u_0)\|_{H_\kappa} &\le \|e^{(t-T_0)L}u(T_0, u_0)\|_{H_\kappa} + \int_{T_0}^{t} \|(-L)^{1/4+\kappa}e^{(t-\tau)L}\| \cdot \|g(u)\|_H\,d\tau \\ &\le \|e^{(t-T_0)L}u(T_0, u_0)\|_{H_\kappa} + C\int_{T_0}^{t} \|(-L)^{1/4+\kappa}e^{(t-\tau)L}\|\,dx \\ &\le \|e^{(t-T_0)L}u(T_0, u_0)\|_{H_\kappa} + C\int_0^{T-T_0} \tau^{-(1/4+\kappa)}e^{-\delta\tau}\,d\tau \\ &\le \|e^{(t-T_0)L}u(T_0, u_0)\|_{H_\kappa} + C, \end{aligned}$$

where $C > 0$ is a constant independent of u_0. Then, (13) holds for all $\frac{1}{4} < \kappa < \frac{3}{4}$.

Step 2. We shall show that for any $\frac{1}{2} < \kappa < 1$, problem (1)–(2) has a bounded absorbing set in H_κ. Using (14) and (9), we deduce that

$$\begin{aligned} \|u(t, u_0)\|_{H_\kappa} &\le \|e^{(t-T_0)L}u(T_0, u_0)\|_{H_\kappa} + \int_{T_0}^{t} \|(-L)^{\kappa}e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{1/4}}\,d\tau \\ &\le \|e^{(t-T_0)L}u(T_0, u_0)\|_{H_\kappa} + C\int_{T_0}^{t} \|(-L)^{\kappa}e^{(t-\tau)L}\|\,dx \\ &\le \|e^{(t-T_0)L}u(T_0, u_0)\|_{H_\kappa} + C\int_0^{T-T_0} \tau^{-(1/4+\kappa)}e^{-\delta\tau}\,d\tau \\ &\le \|e^{(t-T_0)L}u(T_0, u_0)\|_{H_\kappa} + C, \end{aligned}$$

where $C > 0$ is a constant independent of u_0. Then (13) holds for all $\frac{1}{2} < \kappa < 1$.

Step 3. We shall show that for any $\frac{3}{4} < \kappa < \frac{5}{4}$, problem (1)–(2) has a bounded absorbing set in H_κ. Using (14) and (11), we deduce that

$$\begin{aligned}
\|u(t,u_0)\|_{H_\kappa} &\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + \int_{T_0}^{t} \|(-L)^{\kappa-1/4}e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{1/2}}\,d\tau \\
&\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C\int_{T_0}^{t} \|(-L)^{\kappa-1/4}e^{(t-\tau)L}\|\,dx \\
&\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C\int_{0}^{T-T_0} \tau^{-(1/4+\kappa)}e^{-\delta\tau}\,d\tau \\
&\le \|e^{(t-T_0)L}u(T_0,u_0)\|_{H_\kappa} + C,
\end{aligned}$$

where $C > 0$ is a constant independent of u_0. Then (13) holds for all $\frac{3}{4} < \kappa < \frac{5}{4}$.

By the iteration method, we have that (13) holds for all $\kappa > \frac{1}{4}$. Thus the proof is completed. □

Now we give the proof of the main result.

Proof of Theorem 2.4. By Lemma 3.1 and Lemma 3.2, the proof is complete. □

Remark 3.3. In this section we proved that there exists a global attractor in the space H_κ which attracts all the bounded sets in the H_κ-norm. It is easy to check that the global attractor in the space H_κ also attracts the bounded subsets of each less regular subspace $H_{\kappa'}$, with $\kappa' \le \kappa$.

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