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The classical solvability of the contact angle problem for generalized equations of mean curvature type

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Abstract. Mean curvature equations of general quasilinear type in connection with contact-angle boundary conditions are considered in this paper. We investigate the existence, uniqueness and continuous dependence of the solution in classical function spaces. On the one hand, a survey of techniques and ideas developed in the 1970s and 1980s, mainly by Uraltseva, is presented. On the other hand, extensions of these results are also proposed: we formulate growth conditions for the general dependence of the potential on the x_{N+1}-variable, and we extend the existence and uniqueness statements to this case. Moreover, the regularity assumptions on the right-hand side are relaxed, and alternative proofs for the higher-order estimates and the existence result are provided.

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1. Introduction

We consider the problem to determine in a domain $\Omega \subset \mathbb{R}^{N+1}$ ($N \geq 2$ the space dimension) a N-dimensional hypersurface $S \subset \Omega$, obeying the relation

$$\operatorname{div}_S \sigma_q(x, \nu) + \sigma_x(x, \nu) \cdot \nu = \Phi(x, \nu), \tag{1}$$

where div_S is the surface divergence operator, and ν denotes a unit normal to S. The potential $\sigma : \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$, $(x, q) \mapsto \sigma(x, q)$ is given and one-homogeneous in the q-variable. The right-hand side $\Phi : \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$ is a given function. In the case of isotropic data $\sigma(x, q) = \sigma(x)|q|$ and $\Phi(x, q) = \Phi(x)$, the equation (1)

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reduces to the problem of surfaces with prescribed mean curvature. We consider on the boundary $S \cap \partial\Omega$ the generalized contact-angle condition

$$\sigma_q(x, v) \cdot n(x) = \kappa(x), \tag{2}$$

where n is the outward unit normal to $\partial\Omega$, and $\kappa : \partial\Omega \to \mathbb{R}$ is a given function.

More specifically, we are interested in graph-solutions to the problem (1), (2). A graph-solution can be defined (after a suitable change of coordinates) if $\Omega = G \times \mathbb{R}$ with a bounded domain $G \subset \mathbb{R}^N$, and if S is represented as the graph of a function $\psi : G \to \mathbb{R}$. The problem (1), (2) on the manifold S reduces to boundary value problem posed in the domain G. Define for $(\bar{x}, x_{N+1}) \in G \times \mathbb{R}$ and for $p \in \mathbb{R}^N$

$$\bar{\sigma}(\bar{x}, x_{N+1}, p) := \sigma(\bar{x}, x_{N+1}, -p, 1). \tag{3}$$

and introduce a function $\bar{\Phi} : G \times \mathbb{R} \times \mathbb{R}^N$ via

$$\bar{\Phi}(\bar{x}, x_{N+1}, p) := \Phi\big(\bar{x}, x_{N+1}, v(p)\big), \qquad v_i(p) := \begin{cases} \frac{-p_i}{\sqrt{1+|p|^2}} & (i = 1, \ldots, N), \\ \frac{1}{\sqrt{1+|p|^2}} & i = N+1. \end{cases} \tag{4}$$

The problem (1), (2) is equivalent to the contact-angle problem

$$-\operatorname{div} \bar{\sigma}_p(\bar{x}, \psi, \nabla\psi) = \bar{\Phi}(\bar{x}, \psi, \nabla\psi) \quad \text{in } G, \tag{5}$$

$$-\bar{\sigma}_p(\bar{x}, \psi, \nabla\psi) \cdot n(\bar{x}) = \kappa(\bar{x}, \psi) \quad \text{on } \partial G. \tag{6}$$

Physical applications of the model (1), (2) respectively (5), (6) are to find in thermodynamical contexts, where (1) is to interpret as the first variation of a surface free energy. The equation (1) is known as generalized Gibbs-Thomson relation: The surface S typically represents a phase transition, and σ is the tensor of surface tension on S; The right-hand side Φ in (1) may involve quantities such as chemical potential, temperature and mechanical stresses on S: see the book [Vis96], Ch. IV for models in crystallization. Technical applications for the model (1) are for instance processes in industrial crystal growth, where curvature effects on the crystallization interface are assumed to be responsible for the formation of defects (cf. [DDEN08]).

Equations of mean curvature type were thouroughly studied in the seventies, in connection both with the Dirichlet and the contact-angle problem: see [Gia74], [Ger74], [Giu76] among others for the *BV* approach, see [Fin65], [Ser69], [Ura73], [Ura75], [Ura82], [SS76] a. o. for the classical approach, which also retains our attention in this paper.

The existence of graph-solutions essentially relies on the gradient estimate for the function ψ. To our knowledge *local* estimates were obtained first in [Mir67], [BDM69] for the problem of minimal surfaces ($\Phi = 0$, $\sigma(q) = |q|$). For general quasilinear equations, the local boundedness of the gradient was proved in [LU70] on the basis of profound results of geometric measure theory. Local estimates employing other methods were also derived early (cf. [Tru73]) by the authors of [GT01] (see Chapter 16). It is to note that the *a priori* estimate derived in these papers for C^2 solutions being local, they did not lead to the solvability of (5), (6).

The global estimate on the gradient for the contact angle problem (5), (6) was first obtained in the papers [Ura71], [Ura73], [Ura75] for general $\sigma = \sigma(q)$, mainly via extension of the methods of [LU70]. In [Ura71] the validity of these results was restricted to (strictly) convex $\mathscr{C}^{2,\alpha}$-domains G, a vanishing angle of contact. The theory for convex domains and a constant nonvanishing angle of contact κ was introduced in [Ura73]; The results were extended in [Ura75] to variable $\kappa = \kappa(\bar{x})$ and nonconvex $\mathscr{C}^3$-domains, but only for the case $\sigma = |q|$ (mean curvature equation). In these papers, it is assumed that $\Phi = \Phi(x)$. Other approaches to the results of [Ura75] for the mean curvature equation were discussed in the papers [SS76], Th. 3 or in [Ger79], that states the gradient estimate for (nonconvex) $\mathscr{C}^4$ domains. The boundedness result for gradient of solutions to the general quasilinear mean curvature equation with contact-angle $\kappa = \kappa(\bar{x}, x_{N+1})$ was proved in [Ura82]. In the latest paper σ is allowed to depend on the x_{N+1}-variable, but only in a very particular way.

The arguments on existence, uniqueness and *a priori* estimates for the problem (5), (6) are spread in the literature (mostly in papers by Uraltseva). Indeed the paper [Ura82], where the general quasilinear case is treated, only deals with the gradient estimate. In the present contribution, we aim at a complete overview on the classical solvability of the problem (5), (6) in smooth settings. We also propose two generalizations: A growth condition for the x_{N+1}-dependence of the function σ is formulated, and shown to yield well-posedness; The regularity assumptions for Φ are weakened.

2. Notations and statement of the main results

Let $N \geq 2$ denote the space dimension, and $G \subset \mathbb{R}^N$ be a bounded domain of class $\mathscr{C}^{2,\alpha}$, $\alpha > 0$, $\Omega := G \times \mathbb{R}$. Throughout the paper, the function σ is assumed to satisfy

$$\sigma \in C^3\big(\overline{G} \times \mathbb{R} \times (\mathbb{R}^{N+1} \backslash \{0\})\big). \tag{7}$$

We assume that there exist positive constants λ_j $(j = 0, 2)$ and μ_i $(i = 0, \ldots, 4)$ such that for all $(x, q) \in \Omega \times \mathbb{R}^{N+1}$,

$$\lambda_0 |q| \leq \sigma(x, q) \leq \mu_0 |q|, \tag{8a}$$

$$|\sigma_q(x, q)| \leq \mu_1, \tag{8b}$$

$$\frac{\lambda_2}{|q|}|\xi|^2 \leq \sum_{i,j=1}^{N+1} \sigma_{q_i,q_j}(x,q)\xi_i\xi_j \leq \frac{\mu_2}{|q|}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^{N+1} \text{ such that } \xi \cdot q = 0, \tag{8c}$$

$$\sum_{j=1}^{n+1} \sigma_{q_i,q_j}(x,q)q_j = 0 \qquad \text{for } i = 1,\ldots,N+1, \tag{8d}$$

$$|\sigma_{q,x}(x,q)| \leq \mu_3, \qquad |\sigma_{q,x,x}(x,q)| \leq \mu_4. \tag{8e}$$

The hypotheses (8a), (8b), (8c) and (8d) are well-knwon, and in particular satisfied if σ is convex and positively homogeneous of degree one in the q variable (cf. [LU70], [Ura71] for a proof). We need special assumptions on the x_{N+1}-derivatives of the function σ. We assume at $(x,q) \in \bar{\Omega} \times \mathbb{R}^{N+1}\backslash\{0\}$ that

$$|\sigma_{x_{N+1}}| + |\sigma_{x_{N+1},x_{N+1}}| \leq \mu_5 \frac{|q_{N+1}|^2}{|q|}, \tag{9a}$$

$$|\sigma_{\bar{x},x_{N+1}}| + |\sigma_{x_{N+1},q}| + |\sigma_{\bar{x},x_{N+1},q}| + |\sigma_{x_{N+1},x_{N+1},q}| \leq \mu_6 \frac{|q_{N+1}|}{|q|}. \tag{9b}$$

One purpose of the paper is also to relax the requirement of continuous differentiability of the right-hand side. We shall require that $\Phi \in V \subset W^{1,\infty}(\Omega \times \mathbb{R}^{N+1})$, where V is any closed linear subspace of $W^{1,\infty}$ that allows for $\nabla\Phi$ to have bounded traces on both sides of smooth submanifolds (for instance, $\nabla\Phi \in C_{pw}$ or even $\nabla\Phi \in W^{1,1}_{pw}$). We assume that

$$\Phi \in V(\Omega \times \mathbb{R}^{N+1}), \qquad \kappa \in C^{1,\alpha}(\partial G \times \mathbb{R}) \qquad (\alpha > 0). \tag{10}$$

Special assumptions are needed in connection with the x_{N+1}-derivatives of these functions:

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}^{N+1}} \Phi_{x_{N+1}} \leq -\gamma_0 < 0, \qquad \kappa_{x_{N+1}} \geq 0. \tag{11}$$

Choosing λ_0 as in (8a), there is a compatibility condition between the functions κ and σ:

$$\sup_{\partial G\times\mathbb{R}} |\kappa| < \lambda_0, \qquad \gamma_1 := \lambda_0 - \|\kappa\|_{L^\infty(\partial G\times\mathbb{R})} > 0. \tag{12}$$

For the existence and uniqueness of the solution, we have to assume that the parameters γ_0, λ_2 and μ_5, μ_6 in the conditions (11), (8d) and (9) satisfy

$$\gamma_0 > \frac{(\mu_5 + \mu_6 + \|\Phi_q\|_{L^\infty(\Omega\times\mathbb{R}^{N+1})})^2}{4\lambda_2}. \tag{13}$$

The main result on existence, uniqueness and regularity for the problem (5), (6) is formulated in the following theorem.

Theorem 2.1. *Let all the assumptions of this section be satisfied for the domain G and the functions σ, Φ and κ. Then, the problem* (5), (6) *possesses a unique solution $\psi \in C^{2,\alpha}(\bar{G})$. Denoting S the graph of the function ψ, there is a constant c depending on all the data in their respective norm, such that $\|D^2\psi\|_{C^\alpha(\bar{G})} \le c(\|\Phi\|_{C^\alpha(\bar{S})} + \|\kappa\|_{C^{1,\alpha}(\partial S)})$.*

A second result of the paper concerns the gradient estimate for solutions to (5), (6), which is the most essential step of the proof. In comparison to the result of [Ura82], we allow for a x_{N+1} dependence of σ, and we formulate the assumptions for the function Φ as integrability conditions.

Proposition 2.2. *Assumptions of Theorem* 2.1 (*the inequality* (13) *being not needed*). *Assume that $\psi \in C^2(\bar{G})$ is a solution to* (5), (6). *Let p and s be real numbers such that $p > N/2$ and $s > \max\{p, \frac{2Np}{2p-N}\}$. Then, there is a continuous* (*polynomial*) *function c such that*

$$\sup_G \sqrt{1 + |\nabla\psi|^2} \le c(X, \|\Phi\|_{L^s(S)}, \|\Phi_x\|_{L^s(S)}, \|\Phi_q\|_{L^{2s}(S)}),$$

where X depends on all the data in their respective norm, but not on Φ.

Remark 2.3. We will give an elementary proof of Proposition 2.2 as stated. Using the global Sobolev embedding on the manifold S, one can show that the statement holds true for $p > N/2$ and $s = p$.

Preliminary propositions. We terminate this section by stating explicitely a few elementary consequences of the hypotheses (see [LU70] or [Ura73] for similar considerations). Due to (8a) and the Taylor formula, there is for all $(x, q) \in \Omega \times \mathbb{R}^{N+1}\backslash\{0\}$ a $\lambda \in]0, 1[$ such that[1]

$$0 = \sigma(x, 0) = \sigma(x, q) - \sigma_q(x, q)q + \frac{1}{2}\sigma_{q_i,q_j}(x, \lambda q)q_i q_j.$$

The properties (8d) and (8a) therefore implies for all $q \in \mathbb{R}^{N+1}\backslash\{0\}$ that

$$\sigma(x, q) = \sigma_q(x, q) \cdot q, \qquad \sigma_q(x, q) \cdot q \ge \lambda_0|q|. \tag{14}$$

For $p \in \mathbb{R}^N$, $q := (-p, 1)$, it follows from (14) and the definition (3) that

$$\bar{\sigma}_p(x, p) \cdot p = \sigma_q(x, q) \cdot q - \sigma_{q_{N+1}}(x, q) = \sigma(x, q) - \sigma_{q_{N+1}}(x, q).$$

[1] Whenever confusion is impossible, we use the convention that repeated indices imply summation.

Using (8a) and (8b), one therefore obtains from the previous assumptions on the growth of σ that

$$\bar{\sigma}_p(x,p)\cdot p \geq \lambda_0\sqrt{1+|p|^2}-\mu_1 \qquad \text{for all } (x,p)\in\Omega\times\mathbb{R}^N. \tag{15}$$

Since $\sigma_q(x,q)\cdot q=\sigma(x,q)$, the assumption (8e) also implies that

$$|\sigma_x(x,q)|\leq\mu_3|q| \qquad \text{for all } q\in\mathbb{R}^{N+1}. \tag{16}$$

For $\xi,p\in\mathbb{R}^N$, the relation (8c) elementarily implies that

$$\frac{\lambda_2|\xi^T|^2}{\sqrt{1+|p|^2}}\leq\bar{\sigma}_{p_i,p_j}(x,p)\xi_i\xi_j\leq\frac{\mu_2|\xi^T|^2}{\sqrt{1+|p|^2}}. \tag{17}$$

Here, $\xi^T=\xi^T(p):=\hat{\xi}-\frac{q}{|q|}\big(\hat{\xi}\cdot\frac{q}{|q|}\big)\in\mathbb{R}^{N+1}$, with $\hat{\xi}:=(\xi_1,\dots,\xi_N,0)$ and $q:=(-p,1)$.

We also need extensions into G of the data n and κ given on $\partial G\times\mathbb{R}$.

Remark 2.4 (Data extension). Since G has a $\mathscr{C}^{2,\alpha}$ boundary, the unit normal has an extension $n:=\nabla\,\mathrm{dist}(\cdot,\partial G)$ into G such that $n\in[C^{1,\alpha}(\overline{G})]^N$. Setting $n_{N+1}=0$ and extending $n(\bar{x})$ by a constant in the $N+1$-direction, we obtain that $n\in[C^{1,\alpha}(\overline{G}\times\mathbb{R})]^{N+1}$. Under the assumption (10), it is possible to assume that $\kappa\in C^{1,\alpha}(\overline{G}\times\mathbb{R})$. We can ensure that the inequality (12) is preserved.

Finally, we recall some notations associated with the surface S. For $\psi\in C^2(\overline{G})$, the graph $S\subset\mathbb{R}^{N+1}$ of ψ is the set $S:=\{(\bar{x},x_{N+1})\in\overline{G}\times\mathbb{R}:x_{N+1}=\psi(\bar{x})\}$. A unit normal on the surface S is given by $\nu\big(\bar{x},\psi(\bar{x})\big):=\nu\big(\nabla\psi(\bar{x})\big)$ with $\nu(p)$ like in (4). The natural surface measure on the surface S is given by $dH_N:=\sqrt{1+|\nabla\psi|^2}\,d\lambda_N$. For $f\in C^1(\mathbb{R}^{N+1})$, the differential operator

$$\delta f:=\nabla f-(\nabla f\cdot\nu)\nu, \tag{18}$$

is identical on S with the surface gradient. Throughout the paper, we denote $\partial S:=\{(\bar{x},x_{N+1})\in\partial G\times\mathbb{R}:x_{N+1}=\psi(\bar{x})\}$. The tangential gradient of ψ on ∂G given by $\psi_t:=\nabla\psi-(\nabla\psi\cdot n)n$ on ∂G. If α denotes the angle of contact between S and $\partial G\times\mathbb{R}$ (that is, $\cos\alpha:=-\nabla\psi\cdot n/\sqrt{1+|\nabla\psi|^2}$ on ∂G), then

$$\sin\alpha=\left(\frac{1+|\psi_t|^2}{1+|\nabla\psi|^2}\right)^{1/2} \qquad \text{on } \partial G. \tag{19}$$

Denote dH_{N-1} the standard surface measure on ∂G. Then, a natural surface measure on ∂S is defined by

$$ds = \sqrt{1 + |\psi_t|^2}\, dH_{N-1} = \sin\alpha \sqrt{1 + |\nabla\psi|^2}\, dH_{N-1}. \tag{20}$$

3. Global L^∞-estimate on $\nabla\psi$

In this section, we are concerned with *a priori* estimates satisfied by $\nabla\psi$ in $L^\infty(G)$ for a function $\psi \in C^2(\overline{G})$ satisfying (5), (6). The assumptions considered for the data are those of Section 2. For local gradient estimates, we refer to the publications mentioned in the introduction. A gradient estimate up to the boundary of S was first proved in [Ura71], [Ura73] for convex domains $G \subset \mathbb{R}^N$ of class $\mathscr{C}^{2,\alpha}$, $\sigma = \sigma(q)$, $\Phi = \Phi(x)$, and $\kappa = \text{const}$. The proof was extended in [Ura75] for $\sigma(q) = |q|$ to nonconvex $\mathscr{C}^3$ domains, $\kappa = \kappa(\bar{x})$. For the later case, results are also to find in [SS76], [Ger79]. A worth-noticing difference is the following: thanks to the Sobolev embedding theorem up to the boundary of S, Uraltseva allows for the limiting case $\gamma_0 = 0$ (cf. the condition (11)), while the proof in the last two papers can be carried out from more elementary considerations. Finally, Uraltseva extended her methods in the paper [Ura82] to general quasilinear mean curvature equations, $\kappa = \kappa(\bar{x}, x_{N+1})$, and G of class $\mathscr{C}^2$.

In this section, we present a proof of the gradient estimate *using Uraltseva's methods*. We slightly extend the result of [Ura82] allowing for a general x_{N+1} dependence of σ via the conditions (9), and tracking the dependence on the right-hand side in the gradient bound in terms of integrability conditions.

Throughout the section, $S \subset \mathbb{R}^{N+1}$ denotes a N-dimensional submanifold that satisfies (1), (2). We abbreviate $\sigma = \sigma(x, v)$ and $\Phi = \Phi(x, v)$ on S. We start with a method to estimate integrals over ∂S which was the new ingredient for the advances in [Ura82] with respect to the former contributions [Ura71], [Ura73].[2] In the following two lemmas, we recall the proof of this fundamental statement.

Lemma 3.1. *Let $S \subset \mathbb{R}^{N+1}$ denote a N-dimensional manifold that satisfies* (1), (2). *Taking into account the assumptions* (8b) *and* (8e) *and the Remark* 2.4, *introduce the function $a_0 := |\Phi| + 2\mu_1|\nabla n| + \mu_3$. Then, for every nonnegative $f \in C^1(\mathbb{R}^{N+1})$*

$$\int_{\partial S} f \frac{ds}{\sin\alpha} \le \gamma_1^{-1}\Big(\mu_1 \int_S |\delta f|\, dH_N + \int_S a_0 f\, dH_N\Big),$$

where δ is defined by (18). *The function* $\sin\alpha$ *and the measure ds are defined in* (19) *and* (20).

[2]Some references on the original idea are also to find in [Ura82].

Proof. On the surface S, define a vector field $T := -(n \cdot \nu)\sigma_q + (\sigma_q \cdot \nu)n$. Note that T is tangent on S. Denote moreover $\nu' := \sin\alpha^{-1}\big(n - (\nu \cdot n)\nu\big)$ the conormal on ∂S. We use the identity $\int_S T \cdot \delta f \, dH_N + \int_S \operatorname{div}_S Tf \, dH_N = \int_{\partial S}(T \cdot \nu')f \, ds$. One easily verifies that

$$T \cdot \nu' = \sin\alpha^{-1}(\sigma_q - \kappa n) \cdot \nu \geq (\lambda_0 - \|\kappa\|_{L^\infty(\partial S)}) \sin\alpha^{-1} \qquad \text{on } \partial S. \tag{21}$$

We compute

$$\begin{aligned} \operatorname{div}_S T &= \delta_i\big((\sigma_q \cdot \nu)n_i - (\nu \cdot n)\sigma_{q_i}\big) \\ &= n_i\delta_i\sigma_q \cdot \nu + \sigma_q \cdot \nu \operatorname{div}_S n - \nu \cdot \delta_i n \sigma_{q_i} - (\nu \cdot n) \operatorname{div}_S \sigma_q \\ &\quad + n_i \delta_i \nu \cdot \sigma_q - n \cdot \delta_i \nu \sigma_{q_i}. \end{aligned} \tag{22}$$

Using the equation (1), it follows that $\operatorname{div}_S \sigma_q = \Phi - \sigma_x \cdot \nu$. Using the symmetry of the matrix $\{\delta_i \nu_j\}$, we show that $n_i\delta_i\nu \cdot \sigma_q - n \cdot \delta_i \nu \sigma_{q_i} = 0$. For $i \in \{1, \ldots, n\}$, the property (8d) and the identity (14) yield $\delta_i \sigma_q \cdot \nu = \sigma_{q_j,\delta_i}\nu_j + \sigma_{q_j,q_l}\delta_j \nu_l \nu_j = \sigma_{q_j,\delta_i}\nu_j = \sigma_{\delta_i}$, where $\sigma_{\delta_i} = \sigma_{x_i} - (\nu \cdot \sigma_x)\nu_i$. Thus

$$\operatorname{div}_S T = n_i \sigma_{\delta_i} + \sigma_q \cdot \nu \operatorname{div}_S n - \nu \cdot \delta_i n \sigma_{q_i} - (\nu \cdot n)(\Phi - \sigma_x \cdot \nu), \tag{23}$$

and the estimate $|\operatorname{div}_S T| \leq a_0$ is an easy consequence of the contitions (8). The claim follows combining $|T| \leq \mu_1$, (21) and (23). □

Note the following elementary precision concerning Lemma 3.1.

Lemma 3.2. *Assumptions of Lemma* 3.1. *Then* $\sin\alpha > \gamma_1/\mu_1$ *on* ∂S.

Proof. Denote $n' = \sin\alpha^{-1}(\nu - \cos\alpha n)$. It is easy to verify that $|n'| = 1$ on ∂S. From the conditions (8), it follows that $\mu_1 \geq \sigma_q \cdot n' = \sin\alpha^{-1}\big(\sigma - (\nu \cdot n)\kappa\big) \geq \sin\alpha^{-1}(\lambda_0 - \|\kappa\|_{L^\infty(\partial S)})$. □

We now turn to the core of the proof of the gradient estimate. It was noticed for the first time in [Ura73] that under the condition (12), it is both convenient and sufficient to estimate the quantity

$$v(x) := \nu_{N+1}^{-1}\big(\sigma(x, \nu) - \kappa(x)\big(\nu \cdot n(x)\big)\big), \qquad x \in S \tag{24}$$

since the conditions (8a) and (12) imply the inequalities

$$\gamma_1\sqrt{1 + |\nabla\psi|^2} \leq v \leq \gamma_2\sqrt{1 + |\nabla\psi|^2} \quad \text{on } S, \qquad \gamma_2 := \mu_0 + \|\kappa\|_{L^\infty(\partial G \times \mathbb{R})}. \tag{25}$$

The following Lemma provides the corner stone for the gradient estimate. We perform the computations for continuously differentiable Φ. In the case that $\Phi \in V$ (cf. (10)), the same is valid usign either the right or the left trace of $\nabla\Phi$ on S.

Lemma 3.3. *Let S be a N-dimensional hypersurface that satisfies* (1), (2), *such that $\nu_{N+1} > 0$ on S. Let v be defined by* (24) *on S. Then, there are functions a_1, a_2 and $b_1, \dots, b_{N+1}$ such that for all $\eta \in C^1(\overline{S})$, the relation*

$$\int_S \nu_{N+1}^2 \sigma_{q_i,q_j} \delta_j v \delta_i \eta - \int_S \Phi_{x_{N+1}} \nu_{N+1} v\eta + \int_{\partial S} \kappa_{x_{N+1}} \nu_{N+1} v\eta \frac{ds}{\sin\alpha}$$
$$= \int_S \nu_{N+1}\{a_1\eta + b\cdot\delta\eta\} + \int_{\partial S} \nu_{N+1} a_2 \eta \frac{ds}{\sin\alpha} \qquad (26)$$

is valid. There are constants c_i, $i = 1, \dots, 4$ depending only on the constants in the conditions (8), (9), *on $\|\kappa_x\|_{L^\infty(S)}$ and on the domain G, such that $a_2 \le c_3$, $|b| \le c_4$ and*

$$a_1 \le \frac{-\lambda_2^2}{2}|\delta v|^2 + c_1(1 + |\Phi_x| + |\Phi_q|) + c_2(1 + |\Phi_q|)\nu_{N+1}|\delta v|. \qquad (27)$$

Proof. Throughout the proof, $\sigma_q = \sigma_q(x, \nu)$ on S. Due to the assumption $\nu_{N+1} > 0$, S is the graph of a function $\psi \in C^2(\overline{G})$. For $k = 1, \dots, N$, we denote d_k the tangential differential operator $d_k := \partial_{x_k} + \psi_{x_k}\partial_{x_{N+1}}$ on S. For $\eta \in C^1(\mathbb{R}^{N+1})$, we denote $\eta_{d_k} := \eta_x \cdot u^k$ with the tangent vector field $u_i^k := \delta_i^k$ for $i = 1, \dots, N$, $u_{N+1}^k := \psi_{x_k}$.

For $k = 1, \dots, N+1$, we introduce $\xi_k := \sigma_{q_k} - \kappa n_k$, and $\zeta_k := \nu_{N+1}^{-1}\nu_k$, that is, $\zeta_k = -\psi_{x_k}$ for $k = 1, \dots, N$, and $\zeta_{N+1} = 1$. The identity (14) yields

$$v = \sum_{k=1}^{N+1} (\sigma_{q_k} - \kappa n_k)\zeta_k = \xi\cdot\zeta \qquad \text{on } S. \qquad (28)$$

For $k \in \{1, \dots, N\}$, we can differentiate the equation (5), multiply the result with $\eta \circ \psi = \eta(x, \psi)$ ($\eta \in C^1(\overline{S})$ arbitrary), and use integration by parts to obtain that

$$\int_G \frac{d\bar\sigma_{p_i}}{dx_k}\frac{d}{dx_i}\eta\circ\psi = \int_G \frac{d\overline{\Phi}}{dx_k}\eta\circ\psi + \int_{\partial G} n_i \frac{d\bar\sigma_{p_i}}{dx_k}\eta\circ\psi\, dH_{N-1},$$

which is nothing else but the identity

$$-\int_S \nu_{N+1} d_k \sigma_{q_i} d_i \eta = \int_S \nu_{N+1} d_k \Phi \eta - \int_{\partial S} \nu_{N+1} d_k \sigma_{q_i} n_i \eta \frac{ds}{\sin\alpha}, \qquad (29)$$

with summation over $i = 1, \ldots, N$. Choosing $\xi_k \eta$ as test function in (29), it follows that

$$-\int_S v_{N+1} \xi_k d_k \sigma_{q_i} d_i \eta = \int_S v_{N+1} (d_k \sigma_{q_i} d_i \xi_k + \xi_k d_k \Phi) \eta - \int_{\partial S} v_{N+1} \xi_k d_k \sigma_{q_i} n_i \eta \frac{ds}{\sin \alpha}, \tag{30}$$

with summation over $i, k = 1, \ldots, N$. Using the symmetry of the matrix $\{\delta_j v_l\}$ and the fact that $\psi_{x_k} v_{N+1} = -v_k$, one verifies that

$$d_k v_j = \delta_k v_j + \psi_{x_k} \delta_{N+1} v_j = -v_{N+1} \delta_j \psi_{x_k}, \qquad j = 1, \ldots, N+1. \tag{31}$$

For $i \in \{1, \ldots, N\}$, the latest yields

$$\begin{aligned} \xi_k d_k \sigma_{q_i} &= \xi_k \sigma_{q_i, d_k} + \xi_k \sigma_{q_i, q_j} (\delta_k v_j + \psi_{x_k} \delta_{N+1} v_j) \\ &= \xi_k \sigma_{q_i, d_k} - v_{N+1} \sigma_{q_i, q_j} \delta_j \psi_{x_k} \xi_k \end{aligned} \tag{32}$$

For $j \in 1, \ldots, N+1$, using that $\zeta_{N+1} = 1$ on S, we see that

$$\sum_{k=1}^{N} \delta_j \psi_{x_k} \xi_k = -\sum_{k=1}^{N+1} \delta_j \zeta_k \xi_k = -\delta_j (\zeta \cdot \xi) + \sum_{k=1}^{N+1} \zeta_k \delta_j \xi_k. \tag{33}$$

Using (8d), we compute that

$$\begin{aligned} v_{N+1} \zeta_k \delta_j \xi_k &= \sigma_{q_k, q_l} v_k \delta_j v_l + v_k \big(\sigma_{q_k, \delta_j} - \delta_j (n_k \kappa) \big) \\ &= v_k \big(\sigma_{q_k, \delta_j} - \delta_j (n_k \kappa) \big), \end{aligned} \tag{34}$$

with summation over $k = 1, \ldots, N+1$. Using (32), (33) and (34), we obtain for $i \in \{1, \ldots, N\}$ the identity

$$\begin{aligned} \sum_{k=1}^{N} d_k \sigma_{q_i} \xi_k &= \sum_{k=1}^{N} \xi_k \sigma_{q_i, d_k} - v_{N+1} \sigma_{q_i, q_j} \Big(-\delta_j v + \sum_{k=1}^{N+1} \zeta_k \big(\sigma_{q_k, \delta_j} - \delta_j (n_k \kappa) \big) \Big) \\ &= \sigma_{q_i, q_j} (v_{N+1} \delta_j v + \tilde{b}_j) + \sum_{k=1}^{N} \xi_k \sigma_{q_i, d_k}, \\ \tilde{b}_j &:= -\sum_{k=1}^{N+1} v_k \big(\sigma_{q_k, \delta_j} - \delta_j (n_k \kappa) \big). \end{aligned} \tag{35}$$

Due to (8d), we easily see that $\sum_{i=1}^{N} \sigma_{q_i, q_j} d_i \eta = \sum_{i=1}^{N+1} \sigma_{q_i, q_j} \delta_i \eta$. Thus, we obtain that

$$\int_S \nu_{N+1}\xi_k d_k \sigma_{q_i} d_i \eta = \int_S \nu_{N+1}\sigma_{q_i,q_j}(\nu_{N+1}\delta_j v + \tilde{b}_j)\delta_i \eta + \int_S \nu_{N+1}\sum_{k=1}^{N}\xi_k \sigma_{q_i,d_k} d_i \eta. \tag{36}$$

We easily verify that

$$\int_S \nu_{N+1}\xi_k \sigma_{q_i,d_k} d_i \eta = -\int_S \nu_{N+1} d_i(\xi_k \sigma_{q_i,d_k})\eta + \int_{\partial S} \nu_{N+1}\xi_k \sigma_{q_i,d_k} n_i \frac{ds}{\sin\alpha}.$$

with summation over $k, i = 1, \dots, N$, and we can rewrite (30) as

$$\begin{aligned}\int_S \nu_{N+1}\sigma_{q_i,q_j}(\nu_{N+1}\delta_j v + \tilde{b}_j)\delta_i \eta &= \int_S \nu_{N+1}\big(d_i(\xi_k \sigma_{q_i,d_k}) - d_k \sigma_{q_i} d_i \xi_k - \xi_k d_k \Phi\big)\eta \\ &\quad + \int_{\partial S} \nu_{N+1}(\xi_k d_k \sigma_{q_i} - \xi_k \sigma_{q_i,x_k}) n_i \eta \frac{ds}{\sin\alpha}.\end{aligned} \tag{37}$$

We consider in the first integral on the right-hand side the term $d_i(\xi_k \sigma_{q_i,d_k}) = d_i \xi_k \sigma_{q_i,d_k} + \xi_k d_i \sigma_{q_i,d_k}$. We compute

$$\begin{aligned}\xi_k d_i \sigma_{q_i,d_k} = \{&\sigma_{q_i,x_k,x_i} + \psi_{x_i}\sigma_{q_i,x_k,x_{N+1}} + \sigma_{x_k,q_i,q_j} d_i \nu_j \\ &+ \sigma_{q_i,x_{N+1}}(-\delta_i \zeta_k + \zeta_i \delta_{N+1}\zeta_k) \\ &- \zeta_k(\sigma_{x_i,x_{N+1},q_i} - \zeta_i \sigma_{q_i,x_{N+1},x_{N+1}} + \sigma_{x_{N+1},q_i,q_j} d_i \nu_j)\}\xi_k,\end{aligned} \tag{38}$$

with summation over $i, k = 1, \dots, N$. In (38), we use that $-\xi_k \zeta_k = -v + \sigma_{q_{N+1}}$, and the fact that $\sum_{i=1}^{N} \sigma_{x_{N+1},q_i,q_j} d_i \nu_j = \sum_{i=1}^{N+1} \sigma_{x_{N+1},q_i,q_j} \delta_i \nu_j$ (cp. (8d)) in order to reexpress

$$\begin{aligned}-\zeta_k \xi_k \sigma_{x_{N+1},q_i,q_j} d_i \nu_j &= (-v + \sigma_{q_{N+1}})\sigma_{x_{N+1},q_i,q_j}\delta_i \nu_j \\ &= -v(\delta_i \sigma_{x_{N+1},q_i} - \sigma_{x_{N+1},q_i,\delta_i}) + \sigma_{q_{N+1}}\sigma_{x_{N+1},q_i,q_j}\delta_i \nu_j.\end{aligned} \tag{39}$$

Using (38), (39), we obtain the identity $\xi_k d_i \sigma_{q_i,d_k} = -v\delta_i \sigma_{x_{N+1},q_i} + \tilde{A}_1$

$$\begin{aligned}\tilde{A}_1 := \{&\sigma_{q_i,x_k,x_i} + \psi_{x_i}\sigma_{q_i,x_k,x_{N+1}} + \sigma_{x_k,q_i,q_j} d_i \nu_j \\ &+ \sigma_{q_i,x_{N+1}}(-\delta_i \zeta_k + \zeta_i \delta_{N+1}\zeta_k) - \zeta_k(\sigma_{x_i,x_{N+1},q_i} - \zeta_i \sigma_{q_i,x_{N+1},x_{N+1}})\}\xi_k \\ &+ v\sigma_{x_{N+1},q_i,\delta_i} + \sigma_{q_{N+1}}\sigma_{x_{N+1},q_i,q_j}\delta_i \nu_j.\end{aligned} \tag{40}$$

Therefore, using the preceding identities and the Gauss theorem, it follows that

$$\begin{aligned}\int_S \nu_{N+1}\xi_k d_i\sigma_{q_i,d_k}\eta &= \int_S \nu_{N+1}\tilde{A}_1\eta - \int_S \nu_{N+1}v\delta_i\sigma_{x_{N+1},q_i}\eta\\ &= \int_S \nu_{N+1}A_1\eta + \int_S \nu_{N+1}v\sigma_{x_{N+1},q_i}\delta_i\eta\\ &\quad - \int_{\partial S}\nu_{N+1}v\sigma_{x_{N+1},q_i}\nu_i'\eta\,ds, \end{aligned}\tag{41}$$

where we have set

$$A_1 := \tilde{A}_1 + \nu_{N+1}^{-1}\sigma_{x_{N+1},q_i}\big(\delta_i(\nu_{N+1}v) + \nu_i\, v\, \nu_{N+1}\operatorname{div}_S \nu\big). \tag{42}$$

Using (41), we can now express the relation (37) in the form

$$\begin{aligned}&\int_S \nu_{N+1}^2\sigma_{q_i,q_j}\delta_j v\delta_i\eta + \nu_{N+1}(\sigma_{q_i,q_j}\tilde{b}_j - v\sigma_{x_{N+1},q_i})\delta_i\eta\\ &\quad = \int_S \nu_{N+1}(A_1 + d_i\xi_k\sigma_{q_i,d_k} - d_k\sigma_{q_i}d_i\xi_k - \xi_k d_k\Phi)\eta\\ &\qquad + \int_{\partial S}\nu_{N+1}(\xi_k d_k\sigma_{q_i} - \xi_k\sigma_{q_i,x_k} - v\sigma_{x_{N+1},q_i})n_i\eta\frac{ds}{\sin\alpha}. \end{aligned}\tag{43}$$

On the other hand, using the relation (31) $d_k\sigma_{q_i}d_i\xi_k = \sigma_{q_i,d_k}d_i\xi_k - A_2$

$$A_2 := \nu_{N+1}\sigma_{q_i,q_j}\delta_j\psi_{x_k}\big(\sigma_{q_k,d_i} - d_i(\kappa n_k) - \nu_{N+1}\sigma_{q_k,q_j}\delta_j\psi_{x_i}\big). \tag{44}$$

Due again to (31), we also have

$$\xi_k d_k\Phi = \xi_k(\Phi_{d_k} + \Phi_{q_j}d_k v_j) = \xi_k(\Phi_{x_k} + \psi_{x_k}\Phi_{x_{N+1}}) + \nu_{N+1}\Phi_{q_j}\xi_k\delta_j\zeta_k,$$

with summation over $k = 1,\ldots,N$ and $j = 1,\ldots,N+1$. Since $\xi_k\psi_{x_k} = -v + \sigma_{q_{N+1}}$, it follows that $\xi_k d_k\Phi = -v\Phi_{x_{N+1}} + A_3$,

$$A_3 := \sum_{k=1}^{N}\xi_k\Phi_{x_k} + \sigma_{q_{N+1}}\Phi_{x_{N+1}} + \nu_{N+1}\Phi_{q_j}\delta_j v - \Phi_{q_j}\sum_{k=1}^{N+1}\nu_k\big(\sigma_{q_k,\delta_j} - \delta_j(\kappa n_k)\big). \tag{45}$$

Finally, observe that $\xi_k d_k\sigma_{q_i}n_i = \xi_k d_k(\sigma_q\cdot n) - \xi_k d_k n\cdot\sigma_q$. Using the fact that $\xi_k d_k$ is a tangential differential operator on ∂S, it follows from the boundary condition (2) that

$$\xi_k d_k\sigma_{q_i}n_i = \xi_k d_k\kappa - \xi_k d_k n\cdot\sigma_q = \xi_k\kappa_{x_k} + (\sigma_{q_{N+1}} - v)\kappa_{x_{N+1}} - \xi_k d_k n\cdot\sigma_q, \tag{46}$$

with summation over $k = 1, \ldots, N$. Define

$$a_2 := \xi_k \kappa_{x_k} + \sigma_{q_{N+1}} \kappa_{x_{N+1}} - \xi_k d_k n \cdot \sigma_q - (\xi_k \sigma_{q_i, x_k} + v \sigma_{q_i, x_{N+1}}) n_i. \tag{47}$$

we have $(\xi_k d_k \sigma_{q_i} - \xi_k \sigma_{q_i, x_k} - v \sigma_{x_{N+1}, q_i}) n_i = -v \kappa_{x_{N+1}} + a_2$. For $i = 1, \ldots, n$, we define $b_i := \sigma_{q_i, q_j} \tilde{b}_j - v \sigma_{x_{N+1}, q_i}$, and we set $a_1 := A_1 + A_2 + A_3$. Then the representation (26) follows from (43). It remains to find the constants $c_1 \ldots c_4$. Using the definition (35) of $\tilde{b}$ and the assumptions (8), (10) and (14), we easily prove that $|b| \le \mu_2 \big(|\sigma_{q,x}| + |\delta(\kappa n)| \big) + \gamma_2 \mu_6$. In the definition (40) of $\tilde{A}_1$, we use the formula (cp. (14))

$$\sum_{i=1}^{n} \sigma_{q_i, x_j} \psi_{x_i} = -\sum_{i=1}^{N+1} \sigma_{q_i, x_j} \zeta_i + \sigma_{q_{N+1}, x_j} = -v_{N+1}^{-1} \sigma_{x_j} + \sigma_{q_{N+1}, x_j},$$

and analogously, that $\sum_{i=1}^{N} \sigma_{q_i, x_j, x_l} \psi_{x_i} = -v_{N+1}^{-1} \sigma_{x_j, x_l} + \sigma_{q_{N+1}, x_j, x_l}$, to prove with the help of (31) that

$$\begin{aligned} |\tilde{A}_1| \le{}& |\xi| (|\sigma_{q, \bar{x}, \bar{x}}| + |\sigma_{\bar{x}, x_{N+1}}| v_{N+1}^{-1} + |\sigma_{q_{N+1}, \bar{x}}|) + v_{N+1} |\sigma_{\bar{x}, q, q}| \, |\delta\zeta| \\ &+ |\sigma_{q, x_{N+1}}| \, |\delta\zeta| + (|\sigma_{q_{N+1}, x_{N+1}}| + |\sigma_{x_{N+1}}| v_{N+1}^{-1}) |\delta_{n+1} \zeta| \\ &+ |\zeta| (\sigma_{\bar{x}, q, x_{N+1}} - v_{N+1}^{-1} \sigma_{x_{N+1}, x_{N+1}} + \sigma_{q_{N+1}, x_{N+1}, x_{N+1}}). \end{aligned}$$

Due to the assumptions (8) and (9), we therefore have $|\tilde{A}_1| \le c_1 + c_2 v_{N+1} |\delta\zeta|$. To estimate A_1 (cf. (42)), we also use the facts

$$\begin{aligned} |v_{N+1}^{-1} \sigma_{x_{N+1}, q_i} | \delta_i (v_{N+1} v) | &\le \mu_6 (v_{N+1} |\delta v| + v_{N+1}^2 v |\delta\zeta|) \\ |v_{N+1}^{-1} \sigma_{x_{N+1}, q_i} v_i \operatorname{div}_S v v v_{N+1}| &= v |\sigma_{x_{N+1}}| \, |\operatorname{div}_S v| \le \gamma_2 \mu_5 v_{N+1}^2 |\delta\zeta|. \end{aligned}$$

This finally proves that $|A_1| \le c_1 + c_2 v_{N+1} (|\delta\zeta| + |\delta v|)$.

Using that $\sigma_{q_i, q_j} d_i = \sigma_{q_i, q_j} \delta_i$, we readily see that

$$\begin{aligned} A_2 &= v_{N+1} \sigma_{q_i, q_j} \delta_j \psi_{x_k} \big(\sigma_{q_k, x_i} - \delta_i (\kappa n_k) - v_{N+1} \sigma_{q_k, q_j} \delta_j \psi_{x_i} \big) \\ &\le -v_{N+1}^2 \sigma_{q_i, q_j} \sigma_{q_k, q_j} \delta_j \zeta_i \delta_j \zeta_k + v_{N+1} \big(|\sigma_{q, \bar{x}}| + |\delta(\kappa n)| \big) |\delta\zeta|. \end{aligned} \tag{48}$$

The condition (8c) implies that $\sigma_{q_i, q_j} \delta_j \zeta_k \sigma_{q_k, q_l} \delta_l \zeta_i \ge \lambda_2^2 |\delta\zeta|^2$. This yields the inequality $A_2 \le -\lambda_2^2 v_{N+1}^2 |\delta\zeta|^2 + c v_{N+1} |\delta\zeta|$. Using the conditions (8), we easily see that

$$|A_3| \le \mu_1 |\Phi_x| + |\Phi_q| \big(|\sigma_{\bar{x}, q}| + |\delta(\kappa n)| \big) + v_{N+1} |\Phi_q| \, |\delta v|.$$

Thus, for the function a_1 we have the following estimate:

$$a_1 \le -\lambda_2^2 |\delta\zeta|^2 v_{N+1}^2 + C v_{N+1} |\delta\zeta| + (1 + |\Phi_q|) v_{N+1} |\delta v| + C(1 + |\Phi_x| + |\Phi_q|),$$

where the constants depend on G, $\|\kappa_x\|_{L^\infty(S)}$ and the constants of the conditions (8), (9). It follows from Youngs inequality that

$$a_1 \leq -\frac{\lambda_2^2}{2}|\delta\zeta|^2 v_{N+1}^2 + C(1 + |\Phi_x| + |\Phi_q|) + C(1 + |\Phi_q|)v_{N+1}|\delta v|.$$

Finally, we use the assumptions (9) to show that $|a_2| \leq \mu_1(|\kappa_x| + |n_x| + |\sigma_{q,\bar{x}}|) + \gamma_2\mu_6 \leq C$. This concludes the proof. □

From Lemma 3.3, there are several ways to finish the proof. Uraltseva's technique in [LU70], [Ura71], [Ura82] is based on estimating $w := \log v$. Assume that $\Phi_{x_{N+1}} \leq 0$ and $\kappa_{x_{N+1}} \geq 0$. Choosing ηv with η nonnegative as a test function in (26), and using (25), one easily deduces that

$$\int_S \sigma_{q_i,q_j}\delta_j w \delta_i \eta + \int_S v_{N+1}^2 \sigma_{q_i,q_j}\delta_j v \delta_i v \eta$$
$$\leq C_1 \int_S \{(1 + |\delta w|)\eta + |\delta\eta|\} + C_2 \int_{\partial S} \eta \frac{ds}{\sin\alpha}.$$

Thus, Lemma 3.1 now yields $\int_S \sigma_{q_i,q_j}\delta_j w \delta_i \eta \leq C\int_S\{(1 + |\delta w|)\eta + |\delta\eta|\}$. It is possible to derive the boundedness of w like in the standard Stampacchia proof for second order elliptic equation with L^∞ coefficients, provided that a Sobolev embedding theorem is globally available on the manifold S (cf. [LU70], [MS73] for local embedding results, [Ura71], [Ura82] for the extension to global embedding). Here we rather show an elementary manner to finish the proof in the case that $\gamma_0 > 0$ in the condition (11). Under this strong monotonicity condition, the estimate on $\nabla\psi$ is only polynomial in the norm of the data.

Lemma 3.4. *Same assumptions as in Lemma* 3.3. *Then, there is a constant K depending on the constants in* (8), (9), *and on $\|\kappa_x\|_{L^\infty(S)}$ and G such that for all* $1 \leq q < \infty$

$$\int_G v^{q-4}|\nabla v|^2 - \frac{1}{q}\int_G \Phi_{x_{N+1}} v^{q+1} \leq K\int_G \tilde{a} v^q, \qquad \tilde{a} \leq 1 + |\Phi| + |\Phi_x| + |\Phi_q|^2. \quad (49)$$

Proof. Choose in Lemma 3.3 $\eta = v^q$. We obtain that

$$\int_S q v_{N+1}^2 v^{q-1}\sigma_{q_i,q_j}\delta_j v \delta_i v - \int_S \Phi_{x_{N+1}} v_{N+1} v^{q+1} + \int_{\partial S} \kappa_{x_{N+1}} v_{N+1} v^{q+1} \frac{ds}{\sin\alpha}$$
$$= \int_S v_{N+1}\{a_1 v^q + qb \cdot \delta v v^{q-1}\} + \int_{\partial S} v_{N+1} a_2 v^q \frac{ds}{\sin\alpha}.$$

Due to (8b), $\sigma_{q_i,q_j}\delta_j v\delta_i v \geq \lambda_2|\delta v|^2$. Applying Young's inequality, and using the bounds derived in Lemma 3.3 for the functions a_i, $|b|$ we can estimate

$$\begin{aligned} \nu_{N+1}a_1 v^q &\leq c_1(1+|\Phi_x|+|\Phi_q|)\nu_{N+1}v^q + \frac{\lambda_2 q}{6}\nu_{N+1}^2|\delta v|^2 v^{q-1} \\ &\quad + \frac{3c_2^2|\Phi_q|^2}{2\lambda_2 q}v^{q-1} q\nu_{N+1}b\cdot\delta v v^{q-1} \\ &\leq \frac{\lambda_2 q}{6}\nu_{N+1}^2|\delta v|^2 v^{q-1} + \frac{3c_4^2 q}{2\lambda_2}v^{q-1}. \end{aligned}$$

Using the fact that $\kappa_{x_{N+1}} \geq 0$, and (25), we prove that

$$\begin{aligned} &\frac{2q\lambda_2}{3}\int_S \nu_{N+1}^2 v^{q-1}|\delta v|^2 - \int_S \Phi_{x_{N+1}}\nu_{N+1}v^{q+1} \\ &\quad\leq \int_S\left(\frac{3c_2^2|\Phi_q|^2}{2\lambda_2 q} + \frac{3c_4^2 q}{2\lambda_2} + c_1\gamma_2(1+|\nabla\Phi|)\right)v^{q-1} + \int_{\partial S}\nu_{N+1}|a_2|v^q\frac{ds}{\sin\alpha}. \end{aligned}$$

We apply the estimates (25) and (27) and the Lemma 3.1 to estimate

$$\begin{aligned} \int_{\partial S}\nu_{N+1}|a_2|v^q\frac{ds}{\sin\alpha} &\leq c_3\gamma_2\int_{\partial S}v^{q-1}\frac{ds}{\sin\alpha} \\ &\leq c_3\gamma_2\gamma_1^{-1}\left((q-1)\int_S v^{q-2}|\delta v| + \int_S|a_0|v^{q-1}\right) \\ &\leq \frac{\lambda_2 q}{6}\int_S \nu_{N+1}^2|\delta v|^2 v^{q-1} \\ &\quad + \int_S\left\{\frac{3c_3^2\gamma_2^2(q-1)^2}{2\lambda_2\gamma_1^2 q}\nu_{N+1}^{-2}v^{q-3} + c_3\gamma_2\gamma_1^{-1}|a_0|v^{q-1}\right\}. \end{aligned}$$

Using (25) again and the bound derived in Lemma 3.1 for the function a_0, we derive the estimate

$$\begin{aligned} q\int_S \nu_{N+1}v^{q-2}|\delta v|^2 - \int_S\Phi_{x_{N+1}}v^q &\leq Cq\int_S\tilde{a}v^{q-1} \\ \tilde{a} &:= \frac{1}{q}(q+|\Phi_x|+1/q|\Phi_q|^2+|\Phi|), \end{aligned} \tag{50}$$

where C depends on all the data but not on Φ. Note that

$$\int_S \nu_{N+1}v^{q-2}|\delta v|^2 = \int_G v^{q-2}|\delta v|^2 \geq \int_G \nu_{N+1}^2 v^{q-2}|\nabla v|^2.$$

The claim follows using again (25). □

Proposition 3.5. *Same assumptions as in Lemma* 3.4. *Let* $p > N/2$ *and* $\alpha_0 > \frac{2Np}{2p-N}$ *arbitrary. Then, there exist a constant* C *depending on* K, G, α_0 *and* p, *and functions* ζ_0, ζ_1 *of* α_0 *and* p *such that* $\max_S v \le C(1 + \|\Phi\|_{L^p(S)} + \|\Phi_x\|_{L^p(S)} + \|\Phi_q\|^2_{L^{2p}(S)})^{\zeta_1}\|v\|^{\zeta_0}_{L^{\alpha_0}(G)}$.

Proof. Due to the condition (11), Lemma 3.4 implies that

$$\int_G |\nabla v^{(q-2)/2}|^2 \le \frac{K(q-2)^2}{4}\int_G \tilde{a} v^q. \tag{51}$$

We add $\|v^{(q-2)/2}\|^2_{L^2(G)}$ on both sides of (51). Thanks to Hölder's inequality, it follows that

$$\int_G \{|\nabla v^{(q-2)/2}|^2 + |v|^{q-2}\} \le \frac{K(q-2)^2}{4}\int_G \tilde{a} v^q + \text{meas}(G)^{2/q}\|v\|^{q-2}_{L^q(G)}. \tag{52}$$

Define $q_0 := \alpha_0/p'$, $p' = p/(p-1)$. The choice of α_0 garanties that $q_0 > N$. Define $\chi := \frac{q_0-2}{N-2}\frac{N}{q_0}$ if $N > 2$, and $\chi \in]p', +\infty[$ arbitrary if $N = 2$. The choice of α_0 implies that $\chi > p'$. We can also verify that $\frac{2\chi q}{q-2} \le \frac{2N}{N-2}$ for $N \ge 3$, $\frac{2\chi q}{q-2} < \infty$ for $N = 2$, for all $q_0 \le q < \infty$. It follows that the embedding $W^{1,2}(G) \hookrightarrow L^r(G)$ for $r := 2\chi q/(q-2)$ is continuous, and that the embedding constants are uniformly bounded. The relation (52) implies that

$$\begin{aligned}
\|v\|^{q-2}_{L^{\chi q}(G)} &= \|v^{(q-2)/2}\|^2_{L^r(G)} \\
&\le c\big((q-2)^2 K\|\tilde{a}\|_{L^p(G)}\|v\|^q_{L^{p'q}(G)} + \text{meas}(G)^{2/q+(q-2)/qp}\|v\|^{q-2}_{L^{p'q}(G)}\big) \\
&\le c\max\{(q-2)^2 K\|\tilde{a}\|_{L^p(G)}, \text{meas}(G)^{2/q+(q-2)/qp}\} \\
&\quad \times \max\{\|v\|^q_{L^{p'q}(G)}, \|v\|^{q-2}_{L^{p'q}(G)}\}.
\end{aligned} \tag{53}$$

For $m \in \mathbb{N}$, set $\alpha_m := \frac{\chi}{p'}\alpha_{m-1}$, $A_m := \|v\|_{L^{\alpha_m}(G)}$. As a consequence of (53) with $q_m = \alpha_m/p'$, one finds the recursive inequalities $A_{m+1} \le c_m^{1/(q_m-2)} A_m^{\xi_m}$ that imply

$$\begin{aligned}
A_{m+1} &\le c_m^{1/(q_m-2)}\Big\{\prod_{i=0}^{m-1}[c_i]^{\xi_{i+1}/(q_i-2)}\Big\}A_0^{\prod_{i=0}^m \xi_i}, \\
\xi_m &:= \begin{cases}\lambda_m := \frac{q_m}{q_m-2} & \text{if } A_m \ge 1, \\ 1 & \text{otherwise,}\end{cases} \\
c_m &:= c[(q_m-2)^2 K\|\tilde{a}\|_{L^p(G)} + \text{meas}(G)^{2/q_m+(q_m-2)/q_m p}].
\end{aligned} \tag{54}$$

We now provide rough bounds for the products appearing in (54). We abbreviate $\tilde{\chi} := \chi/p' > 1$. Note first that

$$\log\Big(\prod_{i=0}^{m} \xi_i\Big) \le \sum_{i=0}^{m} \log\big(q_i/(q_i-2)\big) \le 2\sum_{i=0}^{m} \frac{1}{q_i-2} \le \frac{2}{q_0-2}\sum_{i=0}^{\infty} \tilde{\chi}^{-i},$$

Thus $\zeta_0 := \prod_{i=0}^{\infty} \xi_i$ satisfies the estimate $\zeta_0 \le \exp\big(2\tilde{\chi}/(q_0-2)(\tilde{\chi}-1)\big)$. Observe also that

$$\log\Big(\prod_{i=0}^{m-1} [c_i]^{\xi_{i+1}/(q_i-2)}\Big) = \sum_{i=0}^{m-1} \frac{\xi_{i+1}}{q_i-2}\log(c_i),$$

$$\log(c_i) \le \log c + \log(K\|\tilde{a}\|_{L^p(G)}) + 2\log(q_i-2) + \left(\frac{2}{q_i} + \frac{q_i-2}{q_i p}\right)\log \operatorname{meas}(G). \tag{55}$$

Using the estimate $\xi_{i+1} \le q_0/(q_0-2)$ for $i \in \mathbb{N}$ we can bound

$$\sum_{i=0}^{m-1} \frac{\xi_{i+1}}{q_i-2}\log(q_i-2) \le \frac{q_0}{q_0-2}\sum_{i=0}^{m-1} \frac{i\log\tilde{\chi} + \log q_0}{\tilde{\chi}^i q_0 - 2} \le \frac{q_0 \log q_0}{(q_0-2)^2}\Big(\sum_{i=0}^{m-1} \frac{i+1}{\tilde{\chi}^i}\Big),$$

and $\zeta_1 := \sum_{i=0}^{\infty} \frac{\xi_{i+1}}{q_i-2}$, $\zeta_2 := \sum_{i=0}^{\infty} \frac{\xi_{i+1}}{(q_i-2)}\big(\frac{2}{q_i} + \frac{q_i-2}{q_i}\big)$ are obviously finite. Therefore, (55) implies that $\prod_{i=0}^{m-1} [c_i]^{\xi_{i+1}/(q_i-2)} \le c_1(q_0)\big((K\|\tilde{a}\|_{L^p(G)})^{\zeta_1} + \operatorname{meas}(G)^{\zeta_2}\big)$, and the claim follows from (54). □

Everything is therefore reduced to estimating the L^{q_0}-norm of v for a $\alpha_0 > \frac{2Np}{2p-N}$. We directly obtain this bound, if we require the strong monotonicity condition (11). It trivially follows from (50) that for all $2 < t < \infty$

$$\|v\|_{L^t(G)} \le \frac{Ct}{\gamma_0}(1 + \|\Phi\|_{L^t(S)} + \|\Phi_x\|_{L^t(S)} + \|\Phi_q\|_{L^{2t}(S)}). \tag{56}$$

This achieves the proof of Theorem 2.2.

4. Higher-order estimates

The gradient bound is the corner stone in the problem (5), (6). Higher-order estimates can be derived whenever a L^∞-bound on the derivatives of ψ has been proved, since the equation (5) is then a *uniformly elliptic* equation of quasilinear type, due to (cp. (17))

$$\sum_{i,j=1}^{N} \bar{\sigma}_{p_i,p_j}\xi_i\xi_j \geq \frac{\lambda_2|\xi^T|^2}{\sqrt{1+|\nabla\psi|^2}} \geq \frac{\lambda_2|\xi|^2}{(1+|\nabla\psi|^2)^{3/2}} \qquad \text{for all } \xi \in \mathbb{R}^N.$$

Define $c_6 := \sup_G \sqrt{1+|\nabla\psi|^2}$. The following Lemma states the Hölder continuity estimate.

Lemma 4.1. *Assume that G is a domain of class* $\mathscr{C}^2$. *Let* $\psi \in \mathscr{C}^2(\bar{G})$ *be a solution to* (5), (6). *Then, for all* $\beta \in [0,1[$, *there is* $c = c(G, c_6, \beta)$ *such that*

$$\|\nabla\psi\|_{C^{0,\beta}(\bar{G})} \leq c(1 + \|\nabla(n\kappa)\|_{L^\infty(S)} + \|\Phi\|_{L^\infty(S)}).$$

Proof. Due to Remark 2.4 and Gauss's divergence theorem, ψ satisfies

$$\int_G (\bar{\sigma}_p + \kappa n)\cdot\nabla\xi = \int_G \big(\bar{\Phi} - \operatorname{div}(\kappa n)\big)\xi \qquad \forall \xi \in W^{1,1}(G). \tag{57}$$

Here and throughout the proof, the functions $\bar{\sigma}$ and $\bar{\Phi}$ are evaluated at $(\bar{x}, \psi, \nabla\psi)$. In order to simplify the discussion, we prove the regularity in a smooth open domain $G_0 \subset G$, assuming that $\Gamma_0 := \partial G \cap \overline{G_0}$ is flat and such that the $N-1$ first basis vectors are tangent on Γ_0 and $n = e_N$ on Γ_0. In the general case, it is possible to use the definition of a domain of class $\mathscr{C}^2$ to locally map a neighbourhood of $\bar{x} \in \partial G$ onto the model configuration.

For $l = 1, \ldots, N-1$, we insert the test function $\partial_{x_l}\xi$ for $\xi \in C_c^1(G_0 \cup \Gamma_0)$ in (57). Using integration by parts, it follows that

$$-\int_G \{\bar{\sigma}_{p_i,p_j}\partial^2_{x_j,x_l}\psi + \bar{\sigma}_{p_i,x_l} + \bar{\sigma}_{p_i,x_{N+1}}\partial_{x_l}\psi + \partial_{x_l}(\kappa n_i)\}\partial_{x_i}\xi$$
$$+ \int_{\partial G} (\bar{\sigma}_{p_i} + \kappa n_i)\partial_{x_i}\xi n_l = \int_G \big(\bar{\Phi} - \operatorname{div}(\kappa n)\big)\partial_{x_l}\xi. \tag{58}$$

Since $n_l = 0$ on Γ_0, the choice of ξ yields vanishing of the surface integral. Equivalently

$$\int_G \bar{\sigma}_{p_i,p_j}\partial_{x_j}\psi_{x_l}\partial_{x_i}\xi = \int_G V\cdot\nabla\xi, \tag{59}$$

$$V_i := -\bar{\sigma}_{p_i,x_l} - \bar{\sigma}_{p_i,x_{N+1}}\partial_{x_l}\psi - \partial_{x_l}(n_i\kappa) - \big(\bar{\Phi} - \operatorname{div}(\kappa n)\big)\delta^i_l \qquad \text{for } i = 1,\ldots,N. \tag{60}$$

Using in particular the growth assumptions (8), and (9b), it follows that $|V| \leq \mu_2 + \mu_1 + \mu_6 + \|\nabla(\kappa n)\|_{L^\infty(S)} + \|\Phi\|_{L^\infty(S)}$. According to classical linear regularity theory (cf. for instance the Theorem 3.16 in [Tro87]), there is for $0 \le \beta < 1$ arbitrary a constant c depending only on β, G_0, the ellipticity constant of the matrix $\{\bar{\sigma}_{p_i,p_j}\}$ and its norm in L^∞ such that

$$\|\psi_{x_l}\|_{C^{0,\beta}_{\text{loc}}(G_0\cup\Gamma_0)} \le c\|V\|_{[L^\infty(G_0)]^n}. \tag{61}$$

It follows that $\psi_t := \nabla\psi - (n\cdot\nabla\psi)n \in C^{0,\beta}_{\text{loc}}(G_0\cup\Gamma_0)$. Using an open covering of ∂G, and applying the reasoning locally to each section, we obtain that $\psi_t \in C^{0,\beta}(\partial G)$ with corresponding norm estimate. We show that also $\psi_n := n\cdot\nabla\psi$ satisfies a Hölder condition on ∂G. For $\bar{x} \in \bar{G}$, $y \in \mathbb{R}$, define

$$H(\bar{x}, y) := \bar{\sigma}_p\big(\bar{x}, \psi(\bar{x}), \psi_t(\bar{x}) + n(\bar{x})y\big)\cdot n(\bar{x}) + \kappa\big(\bar{x}, \psi(\bar{x})\big).$$

Using the growth condition (8b), $|H(\bar{x}, y)| \le \mu_1 + \|\kappa\|_{L^\infty(S)}$ for all $(\bar{x}, y) \in \bar{G}\times\mathbb{R}$. Moreover, for $\bar{x}_1, \bar{x}_2 \in \bar{G}$, $y \in \mathbb{R}$

$$\begin{aligned}|H(\bar{x}_1, y) - H(\bar{x}_2, y)| \le{}& \|\bar{\sigma}_{p,\bar{x}}\|_{L^\infty}|\bar{x}_1 - \bar{x}_2| + \|\bar{\sigma}_{p,x_{N+1}}\|_{L^\infty}|\psi(\bar{x}_1) - \psi(\bar{x}_2)|\\ &+ \|\bar{\sigma}_{p,p}\|\big(|\psi_t(\bar{x}_1) - \psi_t(\bar{x}_2)| + |y|\,|n(\bar{x}_1) - n(\bar{x}_2)|\big)\\ &+ \|\kappa_{\bar{x}}\|_{L^\infty}|\bar{x}_1 - \bar{x}_2| + \|\kappa_{x_{N+1}}\|_{L^\infty}|\psi(\bar{x}_1) - \psi(\bar{x}_2)|,\end{aligned}$$

so that the following estimate holds:

$$\frac{|H(\bar{x}_1, y) - H(\bar{x}_2, y)|}{|\bar{x}_1 - \bar{x}_2|^\beta} \le c(1 + \|\psi_t\|_{C^{0,\beta}(\partial G)} + |y|\,\|n\|_{C^{0,\beta}(\bar{G})}). \tag{62}$$

By virtue of the condition (8c), note that

$$\partial_y H(\bar{x}, y) = \bar{\sigma}_{p_i,p_j} n_j(\bar{x}) n_i(\bar{x}) \ge \lambda_2(1 + |\nabla\psi|^2)^{-1/2}\big(1 - (v\cdot n)^2\big) \ge \lambda_2 c_6^{-3}. \tag{63}$$

On the other hand, the boundary condition (6) implies that $H\big(x, \psi_n(x)\big) = 0$ on ∂G. For $\bar{x}, \bar{x}' \in \partial G$ arbitrary, it follows that

$$\begin{aligned}\lambda_2 c_6^{-3}\big(\psi_n(\bar{x}) - \psi_n(\bar{x}')\big) &\le \int_{\psi_n(\bar{x})}^{\psi_n(\bar{x}')} \partial_y H(\bar{x}, s)\,ds\\ &= H\big(\bar{x}, \psi_n(\bar{x}')\big) - H\big(\bar{x}, \psi_n(\bar{x})\big)\\ &= H\big(\bar{x}, \psi_n(\bar{x}')\big) - H\big(\bar{x}', \psi_n(\bar{x}')\big).\end{aligned}$$

The latest yields

$$\frac{|\psi_n(\bar{x}) - \psi_n(\bar{x}')|}{|\bar{x} - \bar{x}'|^\beta} \le c\,\frac{\big|H\big(\bar{x}, \psi_n(\bar{x}')\big) - H\big(\bar{x}', \psi_n(\bar{x}')\big)\big|}{|\bar{x} - \bar{x}'|^\beta}. \tag{64}$$

Therefore, taking (62) into account

$$\|\psi_n\|_{C^{0,\beta}(\partial G)} \le c(1 + \|\psi_t\|_{C^{0,\beta}(\bar{G})} + c_6\|n\|_{C^{0,\beta}(\bar{G})}) \tag{65}$$

which finally implies that $\nabla\psi \in C^{0,\beta}(\partial G)$. Return to (59) for $\xi \in C_c^1(G_0)$. With the help of regularity results for linear equations (cf. for instance the Theorem 3.16 in [Tro87]), it now follow that $\partial_{x_l}\psi \in C_{\text{loc}}^{0,\beta}(G_0 \cup \Gamma_0)$ for $l = 1, \ldots N-1$ with corresponding norm estimate. Since the same relation is valid for $l = N$ if the test function ξ vanishes on ∂G (note: the operator $(\bar{\sigma}_p + \kappa n)\cdot\nabla$ is tangent on ∂G), we can argue the same for ψ_n in view of (65). □

The estimate in $C^{2,\alpha}$ is obtained with similar ideas.

Lemma 4.2. *Same assumptions as in Lemma* 4.1. *Then* $\|D^2\psi\|_{C^\alpha(\bar{G})} \le C(\|\Phi\|_{C^\alpha(\bar{S})} + \|\kappa\|_{C^{1,\alpha}(\partial S)})$, *where C depends on the constants in the conditions* (8), (9), (12) *and on* c_6.

Proof. Consider the relation (59). Lemma 4.1 implies that $\bar{\sigma}_{p_i,p_j} \in C^{0,\beta}(\overline{G})$ for all $\beta \in [0,1[$. Analogously, $\overline{\Phi} \in C^{0,\beta}(\overline{G})$ for all $\beta \in [0,1[$ (cf. (4) and (7)).

The definition (60) together with Lemma 4.1 now implies that $V \in [C^{0,\alpha}(\overline{G})]^N$ (cp. (60)). Thus, the linear regularity theory (cf. Theorem 3.17 in [Tro87]) now yields for $l = 1, \ldots, N-1$

$$\|\psi_{x_l}\|_{[C^{1,\alpha}(\bar{G})]^N} \le c\|V\|_{[C^{0,\alpha}(\bar{G})]^n}.$$

We are now allowed to differentiate the relation $H(\bar{x}, \psi_n(\bar{x})) = 0$ in any tangential direction τ over ∂G, which yields $\partial_y H(\bar{x}, \psi_n(\bar{x}))(\tau\cdot\nabla\psi_n) = \tau \cdot H_{\bar{x}}(\bar{x}, \psi_n(\bar{x}))$ for $\bar{x} \in \partial G$. Due to commutation rules, the mixed-derivatives $\psi_{t,n}$ belongs to $C^{0,\alpha}(\overline{G})$, with corresponding continuity estimates. In order to show that also $\psi_{n,n} \in C^{0,\alpha}(\overline{G})$, we use the previous results in connection with equation (5) yielding on Γ_0

$$\begin{aligned}\overline{\Phi} - \sum_{i,j=1}^{N-1} \bar{\sigma}_{p_i,p_j}\psi_{x_i,x_j} - 2\sum_{i=1}^{N-1} \bar{\sigma}_{p_i,p_N}\psi_{x_i,x_N} - \bar{\sigma}_{p_i,x_i} - \bar{\sigma}_{p_i,x_{N+1}}\psi_{x_i} \\ = \bar{\sigma}_{p_N,p_N}\psi_{n,n} \in C^{0,\alpha}(\overline{G}).\end{aligned}$$

Since $n_i\bar{\sigma}_{p_i,p_j}n_j \ge \lambda_2 c_6^{-3}$, the function $(n_i\bar{\sigma}_{p_i,p_j}n_j)^{-1}$ belongs also to $C^{0,\alpha}(\partial G)$. We finally can conclude that $\psi_{n,n} \in C^{0,\alpha}(\Gamma_0)$, and that $\psi_{n,n} \in C^{0,\alpha}(\partial G)$ due to localization arguments. Thus $D^2\psi \in C^{0,\alpha}(\partial G)$, and the claim follows (Theorem 3.17 in [Tro87]). □

5. *A priori* estimates on ψ in L^∞

The natural $W^{1,1}$ estimate, and the global boundedness of weak solutions to (5), (6) have been discussed in different papers. In the case that σ and κ do not depend on the x_{N+1} variable, and that $\Phi = \Phi(\bar{x}, x_{N+1})$ the inequality $\|\kappa\|_{L^\infty(\partial G)} < \lambda_0$ and the condition (11) is known to be sufficient to obtain these bounds. The arguments easily carry over to the general case.

Lemma 5.1. *Assume that* $\psi \in W^{1,1}(G)$ *is a weak solution to* (5), (6). *Assume that* (11) *is valid. Assume that* $p > 2N$. *Then, there is a constant depending on* $2N - p$, *on* G, *on the constants* μ_1, γ_1 *and* γ_0, *and on* $\|\kappa_{\bar{x}}(\cdot, 0)\|_{L^\infty(G)}$ *such that*

$$\|\psi\|_{L^\infty(G)} \le c\big(1 + \|\bar{\Phi}(\bar{x}, 0, \nabla\psi)\|_{L^p(G)}^2\big).$$

Proof. Multiply the equation with $\xi \in W^{1,1}(G)$ and integrate by parts. We add the zero $\int_G \operatorname{div}\big(\kappa(\bar{x}, 0)n\xi\big) - \int_{\partial G} \kappa(\bar{x}, 0)\xi$, to obtain the identity

$$\begin{aligned}\int_G \big(\bar{\sigma}_q(\bar{x}, \psi, \nabla\psi) + \kappa(\bar{x}, 0)n\big) \cdot \nabla\xi + \int_{\partial G} \big(\kappa(\bar{x}, \psi) - \kappa(\bar{x}, 0)\big)\xi \\ = \int_G \big(\bar{\Phi}(\bar{x}, \psi, \nabla\psi) - \bar{\Phi}(\bar{x}, 0, \nabla\psi)\big)\xi + \int_G \big(\bar{\Phi}(\bar{x}, 0, \nabla\psi) - \operatorname{div}\big(\kappa(\bar{x}, 0)n\big)\big)\xi.\end{aligned}$$

Choose $\xi = (\psi - k)^+$, $k \in \mathbb{R}^+$. Due to (11), $\big(\kappa(\bar{x}, \psi) - \kappa(\bar{x}, 0)\big)(\psi - k)^+ \ge 0$, and $\big(\bar{\Phi}(\bar{x}, \psi, \nabla\psi) - \bar{\Phi}(\bar{x}, 0, \nabla\psi)\big)(\psi - k)^+ \le -\gamma_0\psi(\psi - k)^+$. Using (15), and the constant γ_1 from (12), we can prove that

$$\begin{aligned}\gamma_1 \int_G |\nabla(\psi - k)^+| - \mu_1 \operatorname{meas}(A_k) + \gamma_0 \int_G \psi(\psi - k)^+ \\ \le \int_G \big(|\bar{\Phi}(\bar{x}, 0, \nabla\psi)| + \big|\operatorname{div}\big(\kappa(\bar{x}, 0)n\big)\big|\big)(\psi - k)^+,\end{aligned}$$

where $A_k := \operatorname{supp}(\psi - k)^+$. Using Young's and Hölder's inequalities, we can prove that

$$\begin{aligned}&\gamma_1 \|(\psi - k)^+\|_{W^{1,1}(G)} \\ &\quad \le \left(\mu_1 + \frac{\gamma_1^2}{2\gamma_0}\right) \operatorname{meas}(A_k) + \frac{1}{2\gamma_0} \int_{A_k} \big(|\bar{\Phi}(\bar{x}, 0, \nabla\psi)| + \big|\operatorname{div}\big(\kappa(\bar{x}, 0)n\big)\big|\big)^2 \\ &\quad \le \left(\mu_1 + \frac{\gamma_1^2}{2\gamma_0}\right) \operatorname{meas}(A_k) \\ &\qquad + \frac{1}{2\gamma_0} \big(\|\bar{\Phi}(\bar{x}, 0, \nabla\psi)\|_{L^p(G)}^2 + \big\|\operatorname{div}\big(\kappa(\bar{x}, 0)n\big)\big\|_{L^p(G)}^2\big) \operatorname{meas}(A_k)^{(N-1)/N+\varepsilon},\end{aligned}$$

with $\varepsilon := 1/N - 2/p$. It follows that $\psi \leq c(1 + \|\overline{\Phi}(\bar{x}, 0, \nabla\psi)\|^2_{L^p(G)})$ (Stampacchia's Lemma, cf. [Tro87], Lemma 2.9). We prove analogously a lower bound, and the claim follows. □

6. Existence

It was shown for the first time in [Ura71] that *a priori* estimates on the gradient of C^2 solutions to (5), (6) joined to the Hölder estimate of Lemma 4.1 leads to an existence theorem via continuation methods in Banach-spaces exposed in [LU68], Ch. 10. Here, existence is obtained via the implicit function theorem.[3] Note that we need somewhat weaker hypotheses on Φ than usually in the literature. Moreover, the condition (13) seems not to be yet known in the present context. At first, we formulate a simple continuation Lemma.

Proposition 6.1. *Let X, Y, Z be Banach spaces such that $Y \hookrightarrow X$ with compact embedding. For $a, b \in \mathbb{R}$, $a < b$, let $\mathscr{G} : X \times]a, b[\to Z$ be a Fréchet differentiable mapping, such that the derivative $\partial_x \mathscr{G}(x^*, \lambda^*) \in \mathscr{L}(X, Z)$ is an isomorphism for all $(x^*, \lambda^*) \in X \times]a, b[$. Assume that there is $K > 0$ such that for all $\lambda \in]a, b[$, all solutions $x \in X$ to the equation $\mathscr{G}(x, \lambda) = 0$ belong to $B_K(0; Y)$. If there is $(x_0, \lambda_0) \in X \times]a, b[$ such that $\mathscr{G}(x_0, \lambda_0) = 0$, then the equation $\mathscr{G}(x, \lambda) = 0$ has a unique solution in $B_K(0; Y)$ for all $\lambda \in [a, b]$.*

Proof. Define $M := \{\lambda \in [a, b] : \exists x \in X, \mathscr{G}(x, \lambda) = 0\}$. The set M is nonvoid since $\mathscr{G}(x_0, \lambda_0) = 0$. Moreover $\lambda^* := \sup M$ belongs to M. To see this, choose $\{\lambda_k\}_{k \in \mathbb{N}} \subseteq M$, $\lambda_k \to \lambda^*$. By definition, there is $x_k \in X$ such that $\mathscr{G}(x_k, \lambda_k) = 0$. By assumption $x_k \in B_K(0; Y)$ for all $k \in \mathbb{N}$, and therefore, there is a subsequence x_{k_j} that strongly converges in X to some x^*. Obviously, $\mathscr{G}(x^*, \lambda^*) = 0$, implying $\lambda^* \in M$.

Seeking a contradiction, assume that $\lambda^* < b$. Then, due to the implicit function theorem (see [GT01], Th. 17.6), there is an open neighborhood $]\lambda^* - \varepsilon, \lambda^* + \varepsilon[$ in $]a, b[$ such that the equation $\mathscr{G}(x, \lambda) = 0$ defines a unique implicit vector-valued function $\lambda \mapsto x(\lambda) \in X$. Therefore $\lambda^* \neq \sup M$, the contradiction. Analogously, one shows that $\inf M = a$. This proves the existence.

If $x_1, x_2 \in X$ both solve $\mathscr{G}(x, \lambda) = 0$, then $\partial_x \mathscr{G}(x^*, \lambda)(x_1 - x_2) = 0$ for some $x^* \in [x_1, x_2]$. Due to the assumption that $\partial_x \mathscr{G}$ is an isomorphism, the uniqueness follows. □

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[3] We thank the referree for the indication that a similar simplification of the existence proof was already achieved in the second edition (1972) of the book [LU68], which unfortunately has not been translated into English.

Theorem 6.2. *Assumptions of the Theorem* 2.1. *Then there is a unique* $\psi \in C^{2,\alpha}(\bar{G})$ *that solves* (5), (6).

Proof. In the first step, we prove the existence claim assuming that $\Phi \in C^{1,\alpha}(\bar{G} \times \mathbb{R} \times \mathbb{R}^{N+1})$. Let $0 < \beta < \alpha$. Define two mappings $\mathscr{G}_1 : C^{2,\beta}(\bar{G}) \times]-1,1[\to C^{\beta}(\bar{G})$ and $\mathscr{G}_2(w,\lambda) : C^{2,\beta}(\bar{G}) \times]-1,1[\to C^{1,\beta}(\partial G)$ via

$$\begin{aligned}\mathscr{G}_1(w,\lambda) &:= -\frac{d}{dx_i}\bar{\sigma}_{p_i}(\bar{x}, w, \nabla w) - \bar{\Phi}(\bar{x}, w, \nabla w)\\ &\quad + (1-\lambda)\big(\partial_{x_i}\bar{\sigma}_{p_i}(\bar{x},0,0) + \bar{\Phi}(\bar{x},0,0)\big),\\ \mathscr{G}_2(w,\lambda) &:= -\bar{\sigma}_{p_i}(\bar{x}, w, \nabla w)n_i(\bar{x}) - \kappa(\bar{x}, w)\\ &\quad + (1-\lambda)\big(\bar{\sigma}_{p_i}(\bar{x},0,0)n_i(\bar{x}) + \kappa(\bar{x},0)\big).\end{aligned}$$

We define a mapping $\mathscr{G}(w,\lambda) := \big(\mathscr{G}_1(w,\lambda), \mathscr{G}_2(w,\lambda)\big)$. Obviously, $\mathscr{G}(0,0) = 0$. Moreover, due to the regularity assumptions on σ, Φ and κ, the mapping $\mathscr{G}$ is clearly Fréchet-differentiable. The derivative $\partial_w\mathscr{G}(w^*,\lambda^*)$ at an arbitrary point $(w^*,\lambda^*) \in C^{2,\beta}(\bar{G}) \times]-1,1[$ in the direction $w \in C^{2,\beta}(\bar{G})$ has the expression

$$\partial_w\mathscr{G}(w^*,\lambda^*)w = \begin{cases} -\frac{d}{dx_i}(\bar{\sigma}^*_{p_i,p_j}\partial_{x_j}w + \bar{\sigma}^*_{p_i,x_{N+1}}w) - \bar{\Phi}^*_{x_{N+1}}w - \bar{\Phi}^*_{p_i}w_{x_i},\\ -(\bar{\sigma}^*_{p_i,p_j}\partial_{x_j}w + \bar{\sigma}^*_{p_i,x_{N+1}}w)n_i(\bar{x}) - \kappa^*_{x_{N+1}}w,\end{cases}$$

where the indice $*$ means that the value is taken at $\big(\bar{x}, w^*(\bar{x}), \nabla w^*(\bar{x})\big)$. In the Lemma 6.3 below, we show that for every $f \in C^{\beta}(\bar{G}) \times C^{1,\beta}(\partial G)$, the equation $\partial_w\mathscr{G}(w^*,\lambda^*)w = f$ has a unique solution in $w \in C^{2,\beta}(\bar{G})$, that is nothing else but the invertibility of the Fréchet derivative $\partial_w\mathscr{G}(w^*,\lambda^*)$.

Moreover, any function $w \in C^{2,\beta}(\bar{G}) := X$ satisfying $\mathscr{G}(w,\lambda) = 0$ solves the problem (1), (2) with right-hand $\tilde{\Phi}(x,q) := \Phi(x,q) + (1-\lambda)\big(\sigma(\bar{x},0,0) - \Phi(\bar{x},0,0)\big)$, and with contact-angle $\tilde{\kappa}(x) := \kappa(x) + (1-\lambda)\big(\sigma_q(\bar{x},0,0)\cdot n(\bar{x}) - \kappa(\bar{x},0)\big)$. Due to the results of the preceding sections 3, 4 and 5, all solutions to the equation $\mathscr{G}(w,\lambda) = 0$ lay therefore in a bounded set of $C^{2,\alpha}(\bar{G}) =: Y$.

The assumptions of the Lemma 6.1 are satisfied, and we obtain in particular the existence of a unique $\psi \in C^{2,\alpha}(\bar{G})$ such that $\mathscr{G}(w,1) = 0$, that is the claim.

In order to obtain the Fréchet differentiability of $\mathscr{G}$, we had to assume in the first step of the proof that $\Phi \in C^{1,\alpha}$. In the second step, we have to show that this assumption can be removed. Let $\Phi \in V$ (cf. (10)). At first, we apply the Sobolev extension operator outside of G in the $\bar{x}$-variable, to obtain for arbitrary fixed $q \in \mathbb{R}^{N+1}$ that $\Phi(\cdot,q)$ is in $W^{1,\infty}(\mathbb{R}^{N+1})$. Obviously, $\sup_{\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}} \Phi_{x_{N+1}} = \sup_{\Omega\times\mathbb{R}^{N+1}} \Phi_{x_{N+1}} \le -\gamma_0$. We choose $\Phi_\varepsilon(x,q) := \int_{\mathbb{R}^{N+1}} w_\varepsilon(x-y)\Phi(y,q)\,dy$, where w_ε is a smooth nonnegative mollifier. Then, the sequence $\{\Phi_\varepsilon(q)\} \subset C^{\infty}(\mathbb{R}^{N+1})$ is uniformly bounded in $W^{1,\infty}(\mathbb{R}^{N+1})$, and $\Phi_{\varepsilon,x_{N+1}} \le -\gamma_0$. Moreover $\Phi_\varepsilon(q) \to \Phi(q)$ in $W^{1,p}(\Omega)$ for all $1 \le p < \infty$.

For $\varepsilon > 0$, let $\psi_\varepsilon \in C^{2,\alpha}(\overline{G})$ denote the unique solution to (5), (6) with right-hand Φ_ε. This solution exists according to the first step. Moreover, the sequence $\{\psi_\varepsilon\}$ is uniformly bounded in $C^{2,\alpha}(\overline{G})$, since the bounds obtained in Sections 3, 4 and 5 only depend on the $W^{1,\infty}$ norm of Φ_ε and on γ_0. The claim follows letting $\varepsilon \to 0$. □

Lemma 6.3. *Assumptions of Theorem 6.2. For every $w^* \in C^{2,\beta}(\overline{G})$ and $f \in C^{\beta}(\overline{G}) \times C^{1,\beta}(\partial G)$, the equation $\partial_w \mathscr{G}(w^*, \lambda^*)w = f$ has a unique solution in $w \in C^{2,\beta}(\overline{G})$.*

Proof. Existence is clear and follows from standard linear theory (cf. for instance the Theorem 3.28 in [Tro87]). For the uniqueness, we assume that $w_i \in C^{2,\beta}(\overline{G})$ is a solution for $i = 1, 2$. Then, the difference $\tilde{w}$ satisfies $\partial_w \mathscr{G}(w^*, \lambda^*)\tilde{w} = 0$. We abbreviate $\xi := \nabla \tilde{w}$. We moreover define $q^* := (\nabla w^*, -1) \in \mathbb{R}^{N+1}$, $\hat{\xi} = (\xi, 0) \in \mathbb{R}^{N+1}$, and the orthogonal part to q^* via $\xi^T := \hat{\xi} - \left(\hat{\xi} \cdot \frac{q^*}{|q^*|}\right)\frac{q^*}{|q^*|}$. Using also (14), we obtain that

$$\begin{aligned}\bar{\sigma}^*_{p,x_{N+1}} \cdot \xi &= -\sigma_{q,x_{N+1}}(\bar{x}, w^*, q^*) \cdot \hat{\xi} \\ &= -\sigma^*_{q,x_{N+1}} \cdot \xi^T - \sigma^*_{q,x_{N+1}} \cdot \frac{q^*}{|q^*|}\frac{q^*}{|q^*|} \cdot \hat{\xi} \\ &= -\sigma^*_{q,x_{N+1}} \cdot \xi^T + [q^*_{N+1}]^{-1} \sigma^*_{x_{N+1}} \xi^T_{N+1}. \end{aligned} \tag{66}$$

where $\sigma^* =$ value at $(\bar{x}, w^*, q^*)$. Using the assumptions (9), it follows that $|\bar{\sigma}^*_{p,x_{N+1}} \cdot \xi| \le (\mu_5 + \mu_6)|\xi^T|/|q^*|$. On the other hand, it follows from (8d) that $\bar{\sigma}^*_{p_i,p_j}\xi_i\xi_j \ge \lambda_2 |\xi^T|^2/|q^*|$. Thus, employing Young's inequality, we obtain the inequality

$$\bar{\sigma}^*_{p_i,p_j}\xi_i\xi_j + \bar{\sigma}^*_{p_i,x_{N+1}}\xi_i\tilde{w} \ge (1-\delta_1)\lambda_2 \frac{|\xi^T|^2}{\sqrt{1+|\nabla w^*|^2}} - \frac{(\mu_5+\mu_6)^2}{4\delta_1\lambda_2}\tilde{w}^2, \tag{67}$$

with $\delta_1 \in]0,1[$ arbitrary. On the other hand, using the definition (4) of $v_j(p)$ for $j = 1, \dots, N+1$, we compute for $i = 1, \dots, N$ the derivative

$$\partial_{p_i} v_j(p) = \begin{cases} \frac{-1}{\sqrt{1+|p|^2}}\left(\delta_i^j - \frac{p_i p_j}{1+|p|^2}\right) & \text{for } j \in \{1, \dots, N\} \\ \frac{-p_i}{(\sqrt{1+|p|^2})^{3/2}} & \text{if } j = N+1. \end{cases}$$

We have for $k = 1, \dots, N$ that $\partial_{p_j} v_k(\nabla w^*)\xi_j = \xi^T_k/|q^*|$. Since $\overline{\Phi}_{p_j} = \Phi_{q_k}\partial_{p_j} v_k(p)$, we easily see that $|\overline{\Phi}^*_{p_j}\xi_j| \le |\Phi_q|\,|\xi^T|/|q^*|$. It follows for $\delta_2 \in]0,1[$ arbitrary that

$$|\overline{\Phi}^*_{q_j}\xi_j\tilde{w}| \le \delta_2\lambda_2 \frac{|\xi^T|^2}{\sqrt{1+|\nabla w^*|^2}} + \frac{|\Phi_q|^2}{4\delta_2\lambda_2}\tilde{w}^2. \tag{68}$$

Summarizing, (67) and (68) imply for $\delta_1 + \delta_2 \leq 1$ the inequality

$$\bar{\sigma}^*_{p_i,p_j}\xi_i\xi_j + \bar{\sigma}^*_{p_i,x_{N+1}}\xi_i\tilde{w} - \Phi^*_{x_{N+1}}\tilde{w}^2 - \overline{\Phi}^*_{p_j}\xi_j\tilde{w} \geq \left(\gamma_0 - \frac{(\mu_5+\mu_6)^2}{4\delta_1\lambda_2} + \frac{|\Phi_q|^2}{4\delta_2\lambda_2}\right)\tilde{w}^2. \quad (69)$$

Due to the equation $\partial_w \mathscr{G}(w^*, \lambda^*)\tilde{w} = 0$, we have the identity

$$\int_G \{\bar{\sigma}^*_{p_i,p_j}\partial_i\tilde{w}\partial_j\tilde{w} + \bar{\sigma}^*_{p_i,x_{N+1}}\partial_i\tilde{w}\tilde{w} - \Phi^*_{x_{N+1}}\tilde{w}^2 - \overline{\Phi}^*_{p_j}\partial_j\tilde{w}\tilde{w}\} + \int_{\partial G}\kappa^*_{x_{N+1}}\tilde{w}^2 = 0.$$

Since $\kappa_{x_{N+1}} \geq 0$, we can use (69) and the assumption to show that $\tilde{w} = 0$. □

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References

[BDM69] E. Bombieri, E. De Giorgi, and M. Miranda, Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche. *Arch. Rational Mech. Anal.* **32** (1969), 255–267. Zbl 0184.32803 MR 0248647

[DDEN08] W. Dreyer, F. Duderstadt, S. Eichler, and M. Naldzhieva, On unwanted nucleation phenomena at the wall of a VGF chamber. Preprint 1312 of the Weierstass Institute for Applied Analysis and Stochastics (WIAS), Berlin 2008. http://www.wias-berlin.de/preprint/1312.

[Fin65] R. Finn, Remarks relevant to minimal surfaces, and to surfaces of prescribed mean curvature. *J. Analyse Math.* **14** (1965), 139–160. Zbl 0163.34604 MR 0188909

[Ger74] C. Gerhardt, Existence, regularity, and boundary behaviour of generalized surfaces of prescribed mean curvature. *Math. Z.* **139** (1974), 173–198. Zbl 0316.49005 MR 0437925

[Ger79] C. Gerhardt, Boundary value problems for surfaces of prescribed mean curvature. *J. Math. Pures Appl.* (9) **58** (1979), 75–109. Zbl 0413.35024 MR 533236

[Gia74] M. Giaquinta, On the Dirichlet problem for surfaces of prescribed mean curvature. *Manuscripta Math.* **12** (1974), 73–86. Zbl 0276.35038 MR 0336532

[GT01] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin 2001. Zbl 1042.35002 MR 1814364

[Giu76] E. Giusti, Boundary value problems for non-parametric surfaces of prescribed mean curvature. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **3** (1976), 501–548. Zbl 0344.35036 MR 0482506

[LU68] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and quasilinear elliptic equations*. Academic Press, New York 1968. Zbl 0164.13002 MR 0244627

[LU70] O. A. Ladyzhenskaya and N. N. Ural'tseva, Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations. *Comm. Pure Appl. Math.* **23** (1970), 677–703. Zbl 0193.07202 MR 0265745

[MS73] J. H. Michael and L. M. Simon, Sobolev and mean-value inequalities on generalized submanifolds of R^n. *Comm. Pure Appl. Math.* **26** (1973), 361–379. Zbl 0256.53006 MR 0344978

[Mir67] M. Miranda, Diseguaglianze di Sobolev sulle ipersuperfici minimali. *Rend. Sem. Mat. Univ. Padova* **38** (1967), 69–79. Zbl 0175.11802 MR 0221350

[Ser69] J. Serrin, The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Philos. Trans. Roy. Soc. London Ser. A* **264** (1969), 413–496. Zbl 0181.38003 MR 0282058

[SS76] L. Simon and J. Spruck, Existence and regularity of a capillary surface with prescribed contact angle. *Arch. Rational Mech. Anal.* **61** (1976), 19–34. Zbl 0361.35014 MR 0487724

[Tro87] G. M. Troianiello, *Elliptic differential equations and obstacle problems*. Plenum Press, New York 1987. Zbl 0655.35002 MR 1094820

[Tru73] N. S. Trudinger, Gradient estimates and mean curvature. *Math. Z.* **131** (1973), 165–175. Zbl 0253.53003 MR 0324187

[Ura71] N. N. Ural'tseva, Nonlinear boundary value problems for equations of minimal-surface type. *Trudy Mat. Inst. Steklov.* **116** (1971), 217–226; English transl. *Proc. Steklov Inst. Math.* **116** (1971), 227–237. Zbl 0269.35039 MR 0364860

[Ura73] N. N. Ural'tseva, The solvability of the capillarity problem. *Vestnik Leningrad. Univ.* **19** (1973), 54–64; English transl. *Vestn. Leningr. Univ. Math.* **6** (1979), 363–375. Zbl 276.35045 MR 0638359

[Ura75] N. N. Ural'tseva, The solvability of the capillarity problem. II. *Vestnik Leningrad. Univ.* **1** (1975), 143–149; English transl. *Vestn. Leningr. Univ. Math.* **8** (1980), 151–158. Zbl 303.35026 MR 0638360

[Ura82] N. N. Ural'tseva, Estimates for the maxima of the moduli of the gradients for solutions of capillary problems. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **115** (1982), 274–284; English transl. *J. Math. Sci.* **28** (1985), 806–813. Zbl 0557.49028 MR 660089

[Vis96] A. Visintin, *Models of phase transitions*. Progr. Nonlinear Differential Equations Appl. 28, Birkhäuser, Boston 1996. Zbl 0882.35004 MR 1423808

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