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Construction of *n*-ary (H, G)-hypergroups

Said M. Anvariyeh, Bijan Davvaz and Saeed Mirvakili

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Abstract. In this paper, we shall define the concepts of completion and complete part with respect to n-ary (H, G)-hypergroups. Moreover, we present a way to obtain a new n-ary hypergroup, starting with other n-ary hypergroups. Finally, we introduce the fundamental relation of an n-ary hypergroup and prove some results. Examples in known classes of n-ary (H, G)-hypergroups are also investigated.

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1. Introduction and preliminaries

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, if H is a non-empty set and $\wp^*(H)$ is the set of all non-empty subsets of H, then we consider the maps of the following type:

$$f_i : H \times H \to \wp^*(H),$$

where $i \in \{1, 2, \dots, n\}$. The maps f_i are called (*binary*) *hyperoperations*. For all x, y of H, $f_i(x, y)$ is called the (*binary*) *hyperproduct* of x and y. The algebraic system $(H, f_1, \dots, f_n)$ is called a (*binary*) *hyperstructure*, where usually $n = 1$ or $n = 2$. Under certain conditions, imposed on the maps f_i, we obtain the so-called semihypergroups, hypergroups, hyperrings or hyperfields. Since 1934, when Marty [26] introduced for the first time the notion of a hypergroup, the hyperstructure theory had applications to several domains. Several books have been written on this topic, for example see [5], [6], [11], [32]. A recent book on hyperstructures [6] outlines applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs.

Let H be a non-empty set and let $\wp^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \to \wp^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*. If A and B are non-empty subsets of H, then we write

$$A \circ B = \bigcup \{a \circ b \mid a \in A, b \in B\}, \qquad x \circ A = \{x\} \circ A \quad \text{and} \quad A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, we have $x \circ H = H \circ x = H$. If a hypergroup H contains an element ε with the property that, for all x in H, one has $x \in x\varepsilon$ (resp. $x \in \varepsilon x$), we say that ε is a *right identity* (resp. *left identity*) of H. If $x\varepsilon = \{x\}$ (resp. $\varepsilon x = \{x\}$), for all x in H, then ε is a *right scalar identity* (resp. *left scalar identity*). The element ε is said to be an *identity* (resp. *scalar identity*) if it is both right and left identity (resp. right and left scalar identity). If H is a hypergroup with identity ε, then an element $x' \in H$ is called *inverse* of an element $x \in H$ if $\varepsilon \in xx' \cap x'x$. A hypergroup H is said to be *of type U on the right* if it fulfills the following conditions:

(U_1) H has a right scalar identity ε;

(U_2) for all $x, y \in H$, $x \in xy \Rightarrow y = \varepsilon$.

In [16], Fasino and Freni exploited the aforementioned properties in order to complete the classification of hypergroups of type U. The class of n^*-complete hypergroups is introduced by De Salvo and Lo Faro [14]. Several properties and examples are found and a geometric interpretation is given by means of hypergraphs. The class of γ_n^*-complete hypergroups is studied by Davvaz and Karimian [10]. In [7], Cristea and Ştefănescu found sufficient and necessary conditions for partial hypergroupoids associated with binary relations in order to be reduced hypergroups. They also determined when the cartesian product of two hypergroupoids associated with a binary relation is a reduced hypergroup. A convolution on a hypergroup, especially in the generalization called H_v-group is given by Vougiouklis [34]. In [20], Hošková and Chvalina presented transformation hyperstructures, namely semihypergroups and hypergroups, acting on tolerance spaces. One knows the construction of a hypergroup K having as core a fixed hypergroup H. In [13], the aforesaid construction is generalized to a large class of hypergroups obtained from a group and from a family of fixed sets, and its properties are analyzed especially in the finite case. Also see [2], [1], [4], [28], [30], [33].

As a generalization of the notion of a group, the notion of an n-ary group (n-group) was introduced by Dörnate in 1928 [15]. In 1940, Post published an extensive study of n-groups in which the well-known Post's Coset Theorem appeared [29]. Hosszú [21] and Gluskin [19] described n-groups for $= 3$ using one group, one automorphism of this group and a constant. Notice that if $(G, \cdot)$ is a group and $n > 2$, then we obtain an n-ary group (G, f), where $f(x_1, \ldots, x_n) = x_1 \ldots x_n$, but for every $n > 2$ there are n-groups which are not of this form. The reader will find in [31] a deep discussion of n-group theory.

Recently, research about n-ary hyperstructures has been initiated by Davvaz and Vougiouklis, who introduced these structures in [12]. They were studied by Anvariyeh, Davvaz, Dudek, Ghadiri, Leoreanu-Fotea Mirvakili, Vougiouklis, Zhan and others, see [3], [8], [9], [18], [24], [25], [23], [27], [35]. n-ary hypergroups are a generalization of hypergroups in the sense of Marty. Also, we can consider n-ary hypergroups as a nice generalization of n-ary groups.

In this paper, the notion of n-ary (H, G)-hypergroups, as a subclass of n-ary hypergroups and generalization of n-ary (H, G)-groups is defined. In fact, the n-ary (H, G)-hypergroup is an n-ary hypergroup in the general sense can be obtained using an n-ary group and a family of fixed sets. In addition, we introduce the completion of an n-ary hypergroup and prove some results in this respect. Finally, examples in known classes of n-ary (H, G)-hypergroups are also investigated.

Let H be a non-empty set. A mapping $h : H^n \to \wp^*(H)$ is called an *n-ary hyperoperation* and n is called the *arity* of hyperoperation.

Let h be an n-ary hyperoperation on H and $A_1, \ldots, A_n$ be non-empty subsets of H. We define

$$h(A_1, \ldots, A_n) = \bigcup \{h(x_1, \ldots, x_n) \mid x_i \in A_i, i = 1, \ldots, n\}.$$

We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. Also, for every $a \in H$, we write $h(\underbrace{a, \ldots, a}_{n}) = h(\overset{(n)}{a})$ and for $j < i$, x_i^j is the empty set. In this convention for $j < i$, x_i^j is the empty set and also

$$h(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, x_{j+1}, \ldots, x_n)$$

is written as $h(x_1^i, y_{i+1}^j, x_{j+1}^n)$.

A non-empty set H with an n-ary hyperoperation $h : H^n \to \wp^*(H)$ is called an *n-ary hypergroupoid* and is denoted by (H, h).

An n-ary hypergroupoid (H, h) is an *n-ary semihypergroup* if and only if the following associative axiom holds:

$$h\big(x_1^{i-1}, h(x_i^{n+i-1}), x_{n+i}^{2n-1}\big) = h\big(x_1^{j-1}, h(x_j^{n+j-1}), x_{n+j}^{2n-1}\big),$$

for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in H$.

An n-ary semihypergroup (H,h), in which the equation $b \in h(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b \in H$ and $1 \le i \le n$, is called an *n-ary hypergroup*. An n-ary hypergroup (H,h) is called an *n-ary group* if for every a_1^n, $h(a_1^n)$ is singleton.

Example 1. Let H be a set and $|H| > 3$. Suppose that $u, v \in H$ and $u \neq v$. Define an n-ary hyperoperation f as follows:

$$f(x_1^n) = \begin{cases} H - \{u\}, & \text{if } (x_1^n) = (u, \overset{(n-1)}{v}), \\ H, & \text{if } x_1^n \in H \text{ and } (x_1^n) \neq (u, \overset{(n-1)}{v}). \end{cases}$$

Then $f\big(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\big) = H$ for every $i, j \in \{1, \dots, n\}$ and $x_1^{2n-1} \in H$. Hence, f is associative and (H,f) is a non-commutative n-ary hypergroup.

Example 2 ([9]). Let $\mathbb{Z}_4$ be the additive group of order 4 and let $H = \mathbb{Z}_4 \cup \{\boldsymbol{\theta}\}$ and $x_1^n \in \mathbb{Z}_4$. Define the commutative n-ary hyperoperation f as follows:

if $(x_1 + x_2 + \cdots + x_n + 2) = 0 \pmod 4$, then $f(x_1^n) = \{0, \theta\}$,

if $(x_1 + x_2 + \cdots + x_n + 2) \neq 0 \pmod 4$, then

$f(x_1^n) = (x_1 + x_2 + \cdots + x_n + 2) \pmod 4$ and $f(\overset{(i)}{\theta}, x_{i+1}^n) = f(\overset{(i)}{0}, x_{i+1}^n)$.

Then (H,f) is a commutative n-ary hypergroup.

Example 3. Let $(G, \circ)$ be an abelian group. We define a ternary (3-ary) hyperoperation on G in the following way:

$$f(x,y,z) = x \circ y^{-1} \circ z, \qquad \text{for all } x, y, z \in G.$$

Then (G,f) is a ternary (3-ary) group.

Example 4. Let $G = (\mathbb{Z}_{16}, \cdot)$ and $H = 2\mathbb{Z}_{16}$. Now we define a ternary(3-ary) hyperoperation on H in the following way:

$$f(x,y,z) = x \cdot y \cdot z + 4, \qquad \text{for all } x, y, z \in H.$$

f is associative since for every $x_1^5 \in R$ we have

$$f\big(f(x_1^3), x_4^5\big) = f\big(x_1, f(x_2^4), x_5\big) = f\big(x_1^2, f(x_3^5)\big) = 4.$$

It is not difficult to see that (H,f) is a ternary group.

An element $e \in H$ is called a *neutral element* if $x = h(\overset{(i-1)}{e}, x, \overset{(n-i)}{e})$, for every $1 \le i \le n$ and for every $x \in H$.

Example 5. Let $(H, \leq, +)$ be a totally ordered group and $x_1^n \in H$. Set $k \in \{i \mid x_i = \max\{x_1, \dots, x_n\}\}$ and $c = \mathrm{card}\{i \mid x_i = \max\{x_1, \dots, x_n\}\}$. Now suppose that

$$f(x_1, \dots, x_n) = \begin{cases} \{t \in H \mid t \leq x_k\}, & \text{if } c > 1, \\ x_k, & \text{if } c = 1. \end{cases}$$

It follows that (H, f) is an n-ary hypergroup. If $e = \min\{x \mid x \in H\}$, then e is a neutral element of H.

An element $e \in H$ called a *weak neutral element* if $x \in h(\overset{(i-1)}{e}, x, \overset{(n-i)}{e})$, for every $1 \leq i \leq n$ and for every $x \in H$.

Example 6. Let (H, f) be the n-ary hypergroup in Example 2. For $n = 3$ it has two weak neutral elements 1 and 3, for $n = 4$ it has only one weak neutral element 2, for $n = 5$ it has no weak neutral elements. Also, it has not a weak neutral element for every n.

Let (A, f) and (B, g) be two n-ary hypergroups. A homomorphism from A to B is a mapping $\phi : A \to B$ such that $\phi\big(f(a_1^n)\big) = f\big(\phi(a_1), \dots, \phi(a_n)\big)$ holds for all $a_1^n \in A$.

According to [17], an n-ary polygroup is an n-ary hypergroup (P, f) such that the following axioms hold for all $1 \leq i, j \leq n$ and $x, x_1^n \in P$:

1. There exists a unique element $0 \in P$ such that $x = f(\overset{(i-1)}{0}, x, \overset{(n-i)}{0})$,
2. There exists an unitary operation $-$ on P such that $x \in f(x_1^n)$ implies $x_i \in f(-x_{i-1}, \dots, -x_1, x, -x_n, \dots, -x_{i+1})$.

It clear that every 2-ary polygroup is a polygroup. Also, every n-ary group with a scaler neutral element is an n-ary polygroup.

Also, Leoreanu in [22] defined a canonical n-ary hypergroup. A canonical n-ary hypergroup is a commutative n-ary polygroup.

Let (H, f) be a commutative n-ary hypergroup and $a, b_1^n \in H$, set $a/b_1^n = \{x \mid a \in f(x, b_1^n)\}$. H is said to an *n-join space* [25] if for any a, c, b_1^n, d_1^n of H the following implication holds:

$$a/b_1^n \cap c/d_1^n \neq \emptyset \implies f(a, d_1^n) \cap f(c, b_1^n) \neq \emptyset.$$

2. n-ary (H, G)-hypergroups

Definition 2.1. Let (H, f) be an n-ary hypergroup and (G, h) be an n-ary group with a neutral element e. Also, let $\{A_g\}_{g \in G}$ be a family of non-empty subsets indexed in G such that for all $x, y \in G$, $x \neq y$, $A_x \cap A_y = \emptyset$, and $A_e = H$. We set $K = \bigcup_{g \in G} A_g$ and we define the hyperoperation $\bar{f}$ in K in the following way:

$$\bar{f}(x_1^n) = f(x_1^n) \qquad \text{for all } (x_1^n) \in H^n,$$
$$\bar{f}(x_1^n) = A_{h(g_1^n)} \qquad \text{for all } (x_1^n) \in A_{g_1} \times \cdots \times A_{g_n} \neq H^n.$$

Theorem 2.2. *The structure $(K, \bar{f})$ is an n-ary hypergroup.*

Proof. Since (H, f) and (G, h) are associative, so $(K, \bar{f})$ has the associative property.

If $y \in K$ and $x_1^{i-1}, x_{i+1}^n \in H$, since (H, f) is an n-ary hypergroup, then the equation $y \in \bar{f}(x_1^{i-1}, x, x_{i+1}^n)$ has the solution $x = b \in H$.

Now let $y \in \bar{f}(x_1^{i-1}, x, x_{i+1}^n)$ and $x_j \notin H$ for $1 \le j \le n$. Hence, there exists $g_j \in A_j$, such that $x_j \in A_{g_j}$. Since (G, h) is an n-ary group, the equation $z \in h(g_1^{i-1}, t, g_{i+1}^n)$ has the solution $t = c \in G$ and so $y \in \bar{f}(x_1^{i-1}, x, x_{i+1}^n) = A_z = A_{h(g_1^{i-1}, c, g_{i+1}^n)}$. □

Definition 2.3. The n-ary hypergroup $(K, \bar{f})$ in Theorem 2.2, is called an *n-ary (H, G)-hypergroup* with *support $K = \bigcup_{g \in G} A_g$*.

Example 7. Let $G = \{0, 1, 2\}$ with a ternary operation $h(x_1, x_2, x_3) = x_1 + x_2 + x_3$ and $H = \{\bar{0}, \bar{1}\}$ with a commutative ternary hyperoperation f defined as follows:

$$f(\bar{0}, \bar{0}, \bar{0}) = \bar{0}, \qquad f(\bar{1}, \bar{0}, \bar{0}) = f(\bar{1}, 1, \bar{0}) = f(\bar{1}, \bar{1}, \bar{1}) = H.$$

It is easy to see that (G, h) is a ternary group and (H, f) is a ternary hypergroup. Let $A_0 = H$, $A_1 = \{a, b\}$ and $A_2 = \{c\}$. Now, $\bar{f}$ is a commutative ternary hyperoperation defined as follows:

$$\bar{f}(\bar{0}, \bar{0}, \bar{0}) = \bar{0}, \bar{f}(\bar{1}, \bar{0}, \bar{0}) = \bar{f}(\bar{1}, 1, \bar{0}) = \bar{f}(\bar{1}, \bar{1}, \bar{1}) = H,$$
$$\bar{f}(\bar{0}, \bar{0}, a) = \bar{f}(\bar{0}, \bar{0}, b) = \bar{f}(\bar{0}, \bar{1}, a) = \bar{f}(\bar{0}, \bar{1}, b) = \bar{f}(\bar{1}, \bar{1}, a) = \bar{f}(\bar{1}, \bar{1}, b) = A_1,$$
$$\bar{f}(\bar{0}, \bar{0}, c) = \bar{f}(\bar{0}, \bar{1}, c) = \bar{f}(\bar{1}, \bar{1}, c) = A_2,$$
$$\bar{f}(\bar{0}, a, a) = \bar{f}(\bar{0}, a, b) = \bar{f}(\bar{0}, b, b) = \bar{f}(\bar{1}, a, a) = \bar{f}(\bar{1}, a, b) = \bar{f}(\bar{1}, b, b) = A_2,$$
$$\bar{f}(\bar{0}, a, c) = \bar{f}(\bar{0}, b, c) = \bar{f}(\bar{1}, a, c) = \bar{f}(\bar{1}, b, c) = H,$$
$$\bar{f}(\bar{1}, c, c) = \bar{f}(\bar{0}, c, c) = A_1, \qquad \text{and} \qquad \bar{f}(c, c, c) = H,$$
$$\bar{f}(x_1, x_2, x_3) = H, \quad \bar{f}(x_1, x_2, c) = A_1, \quad \bar{f}(x_1, c, c) = A_2 \qquad \text{for all } x_1, x_2, x_3 \in A_2,$$

and $(K = \{\bar{0}, \bar{1}, a, b, c\}, \bar{f})$ is a ternary hypergroup.

In the paper, we consider card $G > 1$, because in the opposite case the n-ary hypergroup K is identified with H.

Lemma 2.4. *The n-ary hypergroup $(K, \bar{f})$ is weak commutative if and only if the n-ary hypergroup (H, f) is weak commutative and the n-ary group (G, h) is commutative.*

Proof. This is obvious. □

Lemma 2.5. *The n-ary hypergroup $(K, \bar{f})$ is commutative if and only if the n-ary hypergroup (H, f) and the n-ary group (G, h) are commutative.*

Proof. For all $x, y \in G$, if $x \neq y$, $A_x \cap A_y = \emptyset$, then by Lemma 2.4 the proof is obvious. □

Lemma 2.6.

(1) *If an n-ary hypergroup (H, f) has a weak neutral element, then the n-ary (H, G)-hypergroup $(K, \bar{f})$ has a weak neutral element.*

(2) *If $n = 2$, then the converse of (1) is true, too.*

Proof. (1) Let $\varepsilon \in H$ be a weak neutral element of (H, f) and $x \in K$. If $x \in A_e = H$, then we obtain

$$x \in f(\overset{(i-1)}{\varepsilon}, x, \overset{(n-i)}{\varepsilon}) = \bar{f}(\overset{(i-1)}{\varepsilon}, x, \overset{(n-i)}{\varepsilon}).$$

If $x \in A_g$ and $g \neq e$, then

$$x \in A_g = A_{h(\overset{(i-1)}{e}, g, \overset{(n-i)}{e})} = \bar{f}(\overset{(i-1)}{\varepsilon}, x, \overset{(n-i)}{\varepsilon}).$$

Therefore, $\varepsilon \in K$ is a weak neutral element of $(K, \bar{f})$.

(2) Let $\varepsilon \in K$ be a weak neutral element of $(K, \bar{f})$. If $\varepsilon \in H = A_e$, then ε is a weak neutral element of (H, f). Now let there exists $g \in G - \{e\}$ such that $\varepsilon \in A_g$. So, for every $x \in H$, $x \in \bar{f}(x, \varepsilon) = A_{h(e,g)} = A_g$, which contradicts $x \in H = A_e$. □

Example 8. The part (2) of Lemma 2.6 is not true for $n > 2$. Let H be a set such that $|H| \geq 3$ and $e, a \in H$. For every $x, y \in H$, we define a hyperoperation $+$ as follows:

$$x + y = \begin{cases} e & \text{if } (x, y) = (e, e) \text{ or } x \neq e \neq y, \\ H - \{e\} & \text{if } (x, y) \neq (e, e) \text{ and } x = e \text{ or } y = e. \end{cases}$$

Set $f(x, y, z) = a + x + y + z$. Then (H, f) has not any weak neutral element. In fact, for every $x \in H$, $f(e, x, x) = H - \{e\}$. If $(G = \mathbb{Z}_2, h)$ with $h(x, y, z) =$

$x + y + z$ is a 3-ary group, $A_0 = H$ and $A_1 = \{1\}$, then 1 is a weak neutral element of K, since for every $x \in H$, $\bar{f}(1,1,x) = A_{h(1,1,0)} = A_0$ and so, $x \in \bar{f}(1,1,x)$ and $\bar{f}(1,1,1) = A_{h(1,1,1)} = A_1 = \{1\}$.

Lemma 2.7. *The n-ary hypergroup $(K, \bar{f})$ has a neutral element if and only if for every $s \in G - \{e\}$, A_s is singleton and the n-ary hypergroup (H, f) has a neutral element.*

Proof. Let $\varepsilon \in K$ be a neutral element. Then we show that $|H| = 1$ or $\varepsilon \in A_e = H$. Also, we have $\varepsilon \in A_g$ and $g \neq e$. Then, for every $s \in G$ and $x \in A_s$, we have

$$\{x\} = \bar{f}(\overset{(n-1)}{\varepsilon}, x) = A_{h(\overset{(n-1)}{g}, s)}.$$

Now we have $x \in A_s$ which implies that

$$A_s = A_{h(\overset{(n-1)}{g}, 0)} = \{x\}$$

and so for every $s \in G$, A_s is singleton. In particular, $A_e = H$ is singleton and $x \in H$ is a neutral element of (H, f). Hence, if $|H| > 1$, then $\varepsilon \in H$ and ε is a neutral element of (H, f). If $\varepsilon \in H$, then for every $s \in G - \{e\}$ and $x \in A_s$ we get

$$\{x\} = \bar{f}(\overset{(n-1)}{\varepsilon}, x) = A_{h(\overset{(n-1)}{e}, s)} = A_s.$$

Hence, A_s is singleton for every $s \neq e$.

Conversely, let $\varepsilon \in H$ be a neutral element and A_g be singleton for every $g \neq e$ in G. For every $x \in K - H$ there exists $g \neq e$ such that $A_g = \{x\}$, so we have

$$\bar{f}(\overset{(i-1)}{\varepsilon}, x, \overset{(n-i)}{\varepsilon}) = A_{h(\overset{(i-1)}{e}, g, \overset{(n-i)}{e})} = A_g = \{x\}.$$

Therefore, ε is a neutral element of $(K, \bar{f})$. □

Corollary 2.8. *If $|H| > 1$, then every neutral element of an n-ary (H, G)-hypergroup is a neutral element of H.*

Remark 1. An n-ary (H, G)-hypergroup is called an *n-ary $(1, G)$-hypergroup* if H is singleton.

Remark 2. Let $H = \{\bar{0}\}$ and $(G, h) = (\mathbb{Z}_2, h)$ such that for every $x, y, z \in \mathbb{Z}_2$, $h(x, y, z) = x + y + z \pmod 2$. Let $A_0 = \{\bar{0}\}$ and $A_1 = \{\bar{1}\}$. It is not difficult to see that $(K, \bar{f}) \cong (G, h)$. Now, $(K, \bar{f})$ has two neutral elements and $\bar{1} \notin H$. This example shows that every neutral element of an n-ary $(1, G)$-hypergroup is not belong to H.

Theorem 2.9. *Let (G, h) have a unique neutral element. Then $(K, \bar{f})$ is an n-ary polygroup if and only if for every $s \in G - \{e\}$, A_s is singleton and (H, f) is an n-ary polygroup and (G, h) is an n-ary group with polygroup properties.*

Proof. $(\Rightarrow)$ Let $(K, \bar{f})$ be an n-ary polygroup and $\varepsilon \in K$ be a neutral element. If $|H| > 1$, then by Lemma 2.7 and Corollary 2.8, $\varepsilon \in H$ and A_g is singleton for every $g \in G - \{e\}$. So $(H, \bar{f})$ is an n-ary subpolygroup of $(K, \bar{f})$. Since $\bar{f}|_H = f$, hence (H, f) is an n-ary polygroup. Now let $g, g_1^n \in G$, $g = h(g_1^n)$ and $x \in A_g$ where $x_i \in A_{g_i}$. Then $x \in A_g = A_{h(g_1^n)} = \bar{f}(x_1^n)$. Since $(K, \bar{f})$ is an n-ary polygroup, we have

$$x_i \in \bar{f}(x_{i-1}^{-1}, \dots, x_1^{-1}, x, x_n^{-1}, \dots, x_{i+1}^{-1}) = A_{h(g_{i-1}^{-1}, \dots, g_1^{-1}, g, g_n^{-1}, \dots, g_{i+1}^{-1})}.$$

But, $x_i \in A_{g_i}$ and so

$$A_{g_i} = A_{h(g_{i-1}^{-1}, \dots, g_1^{-1}, g, g_n^{-1}, \dots, g_{i+1}^{-1})}$$

and $g_i = h(g_{i-1}^{-1}, \dots, g_1^{-1}, g, g_n^{-1}, \dots, g_{i+1}^{-1})$. Therefore, (G, h) is an n-ary group with polygroup properties. If $|H| = 1$, then $(K, \bar{f}) \cong (G, h)$ and the implication is immediate.

$(\Leftarrow)$ Let $x, x_1^n \in K$ and $x \in \bar{f}(x_1^n)$. If $x, x_1^n \in H$, then $x \in \bar{f}(x_1^n) = f(x_1^n)$ and so

$$x_i \in f(x_{i-1}^{-1}, \dots, x_1^{-1}, x, x_n^{-1}, \dots, x_{i+1}^{-1}) = \bar{f}(x_{i-1}^{-1}, \dots, x_1^{-1}, x, x_n^{-1}, \dots, x_{i+1}^{-1}).$$

Let $x \in A_g$ and $x_i \in A_{g_i}$. Then $x \in \bar{f}(x_1^n) = A_{h(g_1^n)}$ and $x \in A_g$ implies that $A_g = A_{h(g_1^n)}$ and $g = h(g_1^n)$. Hence, $g_i \in h(g_{i-1}^{-1}, \dots, g_1^{-1}, g, g_n^{-1}, \dots, g_{i+1}^{-1})$ and thus

$$x_i \in A_{g_i} = A_{h(g_{i-1}^{-1}, \dots, g_1^{-1}, g, g_n^{-1}, \dots, g_{i+1}^{-1})} = \bar{f}(x_{i-1}^{-1}, \dots, x_1^{-1}, x, x_n^{-1}, \dots, x_{i+1}^{-1}).$$

Moreover, Lemma 2.7, implies that the neutral element of H is a neutral element of K and $(K, \bar{f})$ is an n-ary polygroup. □

Corollary 2.10. *Let (G, h) have a unique neutral element. Then $(K, \bar{f})$ is a canonical n-ary hypergroup if and only if for every $s \in G - \{e\}$, A_s is singleton and (H, f) is a canonical n-ary hypergroup and (G, h) is a canonical n-ary group.*

Proof. By Theorem 2.9 and Lemma 2.5, the implication is immediate. □

Example 9. Let $H = \{\varepsilon, a\}$ with a commutative 3-ary hyperoperation f as follows:

$$f(\varepsilon, \varepsilon, \varepsilon) = \varepsilon, \qquad f(\varepsilon, \varepsilon, a) = a, \qquad f(\varepsilon, a, a) = f(a, a, a) = H.$$

Let $(G,h) = (\mathbb{Z}_2, h)$ such that for every $x, y, z \in \mathbb{Z}_2$, $h(x,y,z) = x + y + z \pmod 2$ and set $A_0 = H$ and $A_1 = \{\bar{1}\}$. Then $(K, \bar{f})$ is a 3-ary polygroup, but (G,h) is not. Indeed, (G,h) has two neutral elements and it is not a 3-ary polygroup.

Theorem 2.11. *The n-ary (H,G)-hypergroup $(K,\bar{f})$ is a join n-space if and only if H and G are join n-spaces.*

Proof. $(\Rightarrow)$ Let $x, x_1^{n-1}, y, y_1^{n-1} \in H$ and $(x/x_1^{n-1})_H \cap (y/y_1^{n-1})_H \neq \emptyset$. By definition of $\bar{f}$, we have $(x/x_1^{n-1})_K \cap (y/y_1^{n-1})_K \neq \emptyset$ which implies that $\bar{f}(x, y_1^{n-1}) \cap \bar{f}(y, x_1^{n-1}) \neq \emptyset$ and so $f(x, y_1^{n-1}) \cap f(y, x_1^{n-1}) \neq \emptyset$.

$(\Leftarrow)$ Let $x, x_1^{n-1}, y, y_1^{n-1} \in K$ and $(x/x_1^{n-1})_K \cap (y/y_1^{n-1})_K \neq \emptyset$. If $x, x_1^{n-1} \in H$, $y \in A_l$ and $y_i \in A_{l_i}$, where $i = 1, \dots, n-1$, then $x/x_1^{n-1} \subseteq A_e$ and so $y/y_1^{n-1} \cap A_e \neq \emptyset$. So there exists $u \in A_e$ such that $x \in \bar{f}(u, x_1^{n-1})$ and $y \in \bar{f}(u, y_1^{n-1})$. Then $A_l = A_{h(e, l_1^{n-1})}$. If $y_1^{n-1} \in H$, then $y \in H$ and implication is immediate. Suppose that $(y_1^{n-1}) \notin H^{n-1}$. Then $\bar{f}(x, y_1^{n-1}) = A_{h(e, l_1^{n-1})}$. If $y \in H = A_e$, then $\bar{f}(y, x_1^n) \subseteq A_e = A_l$ and if $y \in A_l$, $l \neq 0$, then $\bar{f}(y, x_1^n) = A_{h(l, \overset{(n-1)}{e})} = A_l$. Since $l = h(e, l_1^{n-1})$, we have $\bar{f}(y, x_1^n) \cap \bar{f}(x, y_1^n) \neq \emptyset$. One comes to the same conclusion when $y, y_1^{n-1} \in H$ and $x \in A_g$ and $x_i \in A_{g_i}$. In the same way the remaining cases are discussed. □

Given the n-ary semihypergroups (H,f) and (K,g). We say that (K,g) is an *enlargement* of (H,f) if

(1) $H \subseteq K$;

(2) $f(x_1^n) \subseteq g(x_1^n)$ for all $x_1^n \in H^n$.

Example 10. Let (K,g) be an n-ary hypergroup and (H,g) be a sub n-ary hypergroup. Then (K,g) is an enlargement (H,g). In particular, every n-ary (H,G)-hypergroup $(K,\bar{f})$ is an enlargement of (H,f).

Suppose that $\mathscr{H} = (H, f, e, ^{-1})$, and $\mathscr{G} = (G, g, e, ^{-1})$ are n-ary polygroups, whose elements have been renamed so that $H \cap G = \{e\}$, where e is the neutral element of both $\mathscr{H}$ and $\mathscr{G}$. A new system $\mathscr{H}[\mathscr{G}] = (M, k, e, ^{-1})$, called the *extension* of $\mathscr{H}$ by $\mathscr{G}$ (or extension of n-ary polygroups of $\mathscr{H}$ and $\mathscr{G}$), is formed in the following way. Set $M = H \cup G$ and let $e^{-1} = e$ and for all $x \in M$, we set $k(\overset{(i-1)}{e}, x, \overset{(n-1)}{e}) = x$. In fact, e is a neutral element of k. Now, for all $x_1^n \in M$, set

$$y_i = \begin{cases} e, & \text{if } x_i \in A, \\ x_i, & \text{if } x_i \in B, \end{cases}$$

and define

$$k(x_1^n) = \begin{cases} f(x_1^n), & x_1^n \in H, \\ g(y_1^n), & \text{if } \{x_1^n\} \not\subset H \text{ and } e \notin g(y_1^n), \\ g(y_1^n) \cup H, & \text{if } \{x_1^n\} \not\subset H \text{ and } e \in g(y_1^n). \end{cases}$$

Theorem 2.12. *$\mathscr{H}[\mathscr{G}] = (M, k, e, ^{-1})$ is an n-ary polygroup.*

Proof. The properties of $\mathscr{A}$ and $\mathscr{B}$ as n-ary polygroups, with tedious and rarely long computations guarantee that the extension $\mathscr{A}[\mathscr{B}] = (M, h, 0, -)$ has the n-ary polygroup properties. □

The above theorem is an extension of Comer's theorem in [4].

Theorem 2.13. *Let $\mathscr{H} = (H, f, e, ^{-1})$ be an n-ary polygroup and $\mathscr{G} = (G, g, e, ^{-1})$ be an n-ary group with polygroup properties whose elements have been renamed so that $H \cap G = \{e\}$, where e is the neutral element of both $\mathscr{H}$ and $\mathscr{G}$. Set $A_g = \{g\}$ for every $g \in G - \{e\}$. Then $\mathscr{H}[\mathscr{G}] = (M, k)$ is an enlargement n-ary (H, G)-hypergroup $(K, \bar{f})$.*

Proof. We have $K = \bigcup_{g \in G} A_g = H \cup G = M$. Also, suppose that $x_1^n \in K$. Then, for $x_1^n \in H$, we have $\bar{f}(x_1^n) = f(x_1^n) = k(x_1^n)$. By definition of $\bar{f}$ and k we have $k(x_1^n) = \bar{f}(x_1^n)$ or $k(x_1^n) = \bar{f}(x_1^n) \cup H$. Therefore, for every $x_1^n \in M = K$ we have $\bar{f}(x_1^n) \subseteq k(x_1^n)$. □

Example 11. Suppose that $A = \{0, a\}$ and $B = \{0, 1, 2\}$. Let $\mathscr{A} = (A, f, 0, -)$ and $\mathscr{B} = (B, g, 0, -)$ be two commutative 3-ary polygroups such that for all $x \in A \cup B$, $-x = x$, $f(0, a, a) = f(a, a, a) = \{0, a\}$ and g be a 3-ary hyperoperation as follows:

$$g(0,1,2) = g(1,1,1) = g(2,2,2) = \{1,2\}, \qquad g(0,1,1) = \{0,2\},$$
$$g(1,1,2) = g(0,2,2) = \{0,1\} \quad \text{and} \quad g(1,2,2) = \{0,1,2\}.$$

Then we have $M = A \cup B = \{0, a, 1, 2\}$ and a 3-ary polygroup $\mathscr{A}[\mathscr{B}] = (M, h, 0, -)$ such that $h(x_1^3) = f(x_1^3)$ for every $x_1^3 \in A$ and

$$h(0,1,1) = h(a,1,1) = \{0,a,2\}, \qquad h(1,1,2) = h(0,2,2) = h(a,2,2) = \{0,a,1\},$$
$$h(1,2,2) = \{0,a,1,2\}, \qquad h(0,1,2) = h(a,1,2) = h(1,1,1) = h(2,2,2) = \{1,2\},$$
$$h(0,a,x) = x, \ \forall x \in B, \qquad h(a,a,x) = \{0,a\}, \ \forall x \in B.$$

Also, $\mathscr{B}[\mathscr{A}] = (M, k, 0, -)$ is a 3-ary polygroup with the 3-ary hyperoperation k as follows:

$$k(x_1^3) = g(x_1^3) \quad \text{for all } x_1^3 \in B,$$
$$k(x, y, a) = a \quad \text{for all } x, y \in B,$$
$$k(x, a, a) = \{0, a, 1, 2\} \quad \text{for all } x \in M.$$

Definition 2.14. If (H, f) is an n-ary hypergroup, then the relation γ is defined

$$\gamma = \bigcup_{k \geq 1} \gamma_k,$$

where γ_1 is the diagonal relation, and for every integer $k > 1$, γ_k is the relation defined as follows:

If $m = k(n-1)+1$, then there exist $a_1^m \in H$ and $\sigma \in \mathbb{S}_n$ such that

$$x\gamma_k y \iff x \in f_{(k)}(a_1^m) \text{ and } y \in f_{(k)}(a_{\sigma(1)}^{\sigma(m)}).$$

In fact, if $u_k = f_{(k)}(a_1^m)$ and $u_k^\sigma = f_{(k)}(a_{\sigma(1)}^{\sigma(m)})$, then

$$x \in u_k \quad \text{and} \quad y \in u_k^\sigma \quad \text{if and only if } x\gamma_k y.$$

Remark 3. The relation γ_k in Definition 2.14 is called β_k relation if $\sigma = Id$ (identity map). In the other words, $x\beta_k y$ if and only if there exist $a_1^m \in H$ and an integer $k > 1$, such that $x, y \in f_{(k)}(a_1^m)$.

Theorem 2.15. *If $(K, \bar{f})$ is an n-ary (H, G)-hypergroup, with support $K = \bigcup_{g \in G} A_g$, then $K/\beta_K \cong G$.*

Proof. Define $\phi : K/\beta_K \to G$ by $\beta_K(a_g) \mapsto g$, where $a_g \in A_g$. It is clear that ϕ is an n-ary group epimorphism. Now let $\phi(\beta_K(a_g)) = \phi(\beta_K(a_{g'}))$. Then $g = g'$ and so $\beta_K(a_g) = \beta_K(a_{g'})$. Therefore, ϕ is an isomorphism. □

Theorem 2.16. *If $(K, \bar{f})$ is an n-ary (H, G)-hypergroup, with support $K = \bigcup_{g \in G} A_g$, then $K/\gamma_K \cong G/\gamma_G$.*

Proof. Define $\phi : K/\gamma_K \to G/\gamma_G$ by $\gamma_K(a_g) \mapsto \gamma_G(g)$, where $a_g \in A_g$. Then ϕ is a group epimorphism. Since $\ker \phi = \{\gamma_K(a_0)\}$, hence ϕ is one to one and therefore ϕ is an isomorphism. □

For $1 \leq i \leq n$, we denote S_i, the set of *i-scalars* of the n-ary (H, G)-hypergroup with respect to the n-ary hyperoperation $\bar{f}$ as follows:

$$S_i = \left\{x \in K \,\middle|\, \text{card}\left(\bar{f}(x_1^{i-1}, x, x_{i+1}^n)\right) = 1 \text{ for all } x_1^{i-1}, x_{i+1}^n \in K\right\}.$$

Theorem 2.17. *Let $(K, \bar{f})$ be an n-ary (H, G)-hypergroup. Then:*

(1) *If there exists $i \in \{1, \dots, n\}$ such that $S_i \cap (K - H) \neq \emptyset$, then $(K, \bar{f})$ is an n-ary group.*

(2) *If there exists $i \in \{1, \dots, n\}$ such that $S_i \neq \emptyset$ and $S_i \cap (K - H) = \emptyset$, then* $\operatorname{card} A_g = 1$ *for every $g \in G - \{e\}$.*

Proof. (1) Let $u \in S_i \cap (K - H)$. So there exists $g_i \in G$ such that $u \in A_{g_i} \neq H$. Let $a \in G$, so there exist $g_1^{i-1}, g_{i+1}^n \in G$ such that $a = h(g_1^{i-1}, g_i, g_{i+1}^n)$. If $u_k \in A_{g_k}$, $k \neq i$, then

$$\bar{f}(u_1^{k-1}, u, u_{k+1}^n) = A_{h(g_1^{i-1}, g_i, g_{i+1}^n)} = A_a.$$

But $u \in S_i$ and therefore A_a is singleton.

(2) By hypothesis, we have $S_i \subseteq H$. Moreover, if $u \in S_i$, then for all $a \in G - \{0\}$, there exist $g_1^{i-1}, g_{i+1}^n \in G$ such that $a = h(g_1^{i-1}, g_i, g_{i+1}^n)$ and so if $u_k \in A_{g_k}$, $k \neq i$, then $\bar{f}(u_1^{k-1}, u, u_{k+1}^n) = A_{h(g_1^{i-1}, g_i, g_{i+1}^n)} = A_a$ is singleton. Therefore, for every $g \neq 0$ we have $\operatorname{card} A_g = 1$. □

Let $(H, \circ)$ be a hypergroup. Define $f(x_1^n) = x_1 \circ \cdots \circ x_n$ for every $x_1^n \in H$. Hence, $\operatorname{der}_n(H) = (H, f)$ is an n-ary hypergroup. Now we have the following theorem:

Theorem 2.18. *Let $(H, \circ)$ be a hypergroup, $(G, \cdot)$ be a group and $(K, \diamond)$ be the (H, G)-hypergroup. Then the n-ary $(\operatorname{der}_n(H), \operatorname{der}_n(G))$-hypergroup $(K, \bar{f})$ coincides with the n-ary hypergroup $\operatorname{der}_n(K) = (K, h)$, such that $h(x_1^n) = x_1 \diamond \cdots \diamond x_n$ for every $x_1^n \in K$.*

Proof. By definition we have

$$x \diamond y = \begin{cases} x \circ y, & x, y \in H, \\ A_{i \cdot j}, & (x, y) \in A_i \times A_j \neq H^2, \end{cases}$$

$$\bar{f}(x_1^n) = \begin{cases} f(x_1^n), & \text{for all } (x_1^n) \in H^n, \\ A_{h(g_1^n)}, & \text{for all } (x_1^n) \in A_{g_1} \times \cdots \times A_{g_n} \neq H^n. \end{cases}$$

It is not difficult to see that $h(x_1^n) = \bar{f}(x_1^n)$ for every $x_1^n \in K$. □

Let A be a subset of an n-ary hypergroupoid (H, f). Then, for every $m = l(n-1) + 1$ where $l \in \mathbb{N}$, the m-ary hyperoperation given by

$$f_{(l)}(x_1^{l(n-1)+1}) = f\big(f\big(\dots, f\big(f(x_1^n), x_{n+1}^{2n-1}\big), \dots\big), x_{(l-1)(n-1)+2}^{l(n-1)+1}\big),$$

is denoted by $f_{(l)}$.

A is called a *complete part* if the following implication is valid:

$$f_{(l)}(x_1^{l(n-1)+1}) \cap A \neq \emptyset \Rightarrow f_{(l)}(x_1^{l(n-1)+1}) \subseteq A.$$

Let A be a non-empty subset of H. The intersection of the subsets of H which are complete parts and contain A is called the *closure* of A in H and it is denoted by $C(A)$ (or $C_H(A)$).

Lemma 2.19. *The relation $xKy \Leftrightarrow x \in C(\{y\})$ is an equivalence relation.*

Proof. This is straightforward. □

Let (G, g) be an n-ary group, e be a neutral element of G and φ be a homomorphism of n-ary hypergroup (H, f) in (G, g). Set $\ker \varphi = \{x \in H \mid k(x) = e\}$. If $\phi_H : H \to H/K$ is the canonical projection, then we denote the $\ker \phi_H$ by ω_H.

Definition 2.20. Let (H, f) be an n-ary hypergroup. We define for every $a_1^n \in H^n$, $k(a_1, \ldots, a_n) = C_H\big(f(a_1, \ldots, a_n)\big)$. Then (H, k) is an n-ary hypergroup which is called the *completion* of (H, f) and denoted by $\Delta(H)$.

3. Enlargement of an *n*-ary hypergroup

Definition 3.1. Let (H, f) be an n-ary hypergroupoid and $\{A(x)\}_{x \in H}$ be a family of non-empty sets such that for every $(x, y) \in H^2$, $x \neq y$ implies $A(x) \cap A(y) = \emptyset$. Set $K_H = \bigcup_{x \in H} A(x)$ and define

$$g(a) = x \iff a \in A(x) \qquad \text{for all } a \in K_H.$$

Now we define the following n-ary hyperoperation in K_H

$$h(a_1^n) = \bigcup_{z \in f(g(a_1), \ldots, g(a_n))} \big(A(z)\big) \qquad \text{for all } a_1^n \in K_H.$$

Remark 4. For every $x \in H$, if $x \in A(x)$, then (K_H, h) is an enlargement of (H, f).

Theorem 3.2. (1) *(H, f) is an n-ary semihypergroup if and only if (K_H, h) is an n-ary semihypergroup.*

(2) *(H, f) is an n-ary hypergroup if and only if (K_H, h) is an n-ary hypergroup.*

Proof. (1) Let (H, f) be an n-ary semihypergroup. Then, for every $a_1^{2n-1} \in K_H$ such that $g(a_i) = x_i$, we have

$$h\big(a_1^{i-1}, h(a_i^{n+i-1}), a_{n+i}^{2n-1}\big) = \bigcup_{z \in f_{(2)}(x_1^{2n-1})} \big(A(z)\big).$$

Therefore, (K_H, h) is an n-ary semihypergroup.

Conversely, let (K_H, h) be an n-ary semihypergroup and $u \in f\big(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\big)$. Then there exists $v \in f(x_i^{n+i-1})$ such that $u \in f(x_1^{i-1}, v, x_{n+i}^{2n-1})$. So,

$$A(v) \subseteq \bigcup_{w \in f(x_i^{n+i-1})} \big(A(w)\big) \quad \text{and} \quad A(u) \subseteq \bigcup_{z \in f(x_1^{i-1}, v, x_{n+i}^{2n-1})} \big(A(z)\big).$$

Now let $a_i \in A(x_i)$, $b \in A(v)$ and $c \in A(u)$. Then

$$h(a_i^{n+i-1}) = \bigcup_{w \in f(x_i^{n+i-1})} \big(A(w)\big)$$

and

$$h(a_1^{i-1}, b, a_{n+i}^{2n-1}) = \bigcup_{z \in f(x_1^{i-1}, v, x_{n+i}^{2n-1})} \big(A(z)\big).$$

Therefore, $c \in h\big(a_1^{i-1}, h(a_i^{n+i-1}), a_{n+i}^{2n-1}\big) = h\big(a_1^{j-1}, h(a_j^{n+j-1}), a_{n+j}^{2n-1}\big)$. Hence, there exists $d \in h(a_j^{n+j-1})$ such that $c \in h(a_1^{j-1}, d, a_{n+j}^{2n-1})$. But

$$h(a_j^{n+j-1}) = \bigcup_{t \in f(x_j^{n+j-1})} \big(A(t)\big),$$

so there exists $r \in f(x_j^{n+j-1})$ such that $d \in A(r)$. Then

$$h(a_1^{j-1}, d, a_{n+j}^{2n-1}) = \bigcup_{s \in f(x_1^{j-1}, r, x_{n+j}^{2n-1})} \big(A(s)\big),$$

from which $q \in f(x_1^{j-1}, r, x_{n+j}^{2n-1})$ exists such that $c \in A(q)$. But $c \in A(u) \cap A(q)$ implies that $u = q$. For this reason $u \in f\big(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}\big)$. Analogously one proves

$$f\big(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}\big) \subseteq f\big(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\big).$$

(2) We first prove that the implication $\Rightarrow$. Let $a_k \in A(x_k)$, where $k = 1, \ldots, n$. Let $1 \leq i \leq n$. Hence, there exists $x \in H$ such that $x_i \in f(x_1^{i-1}, x, x_{i+1}^n)$. If

$a \in A(x)$, then

$$h(a_1^{i-1}, a, A_{i+1}^n) = \bigcup_{w \in f(x_1^{i-1}, x, x_{i+1}^n)} \big(A(w)\big),$$

therefore $a_i \in h(a_1^{i-1}, a, A_{i+1}^n)$.

Conversely, let $x_1^n \in H$ and $a_j \in A(x_j)$. Then, for every $1 \le i \le n$, $a \in K_H$, exists such that $a_i \in h(a_1^{i-1}, a, A_{i+1}^n)$. Suppose that $a \in A(x)$. Then

$$h(a_1^{i-1}, a, A_{i+1}^n) = \bigcup_{w \in f(x_1^{i-1}, x, x_{i+1}^n)} \big(A(w)\big).$$

From $a_i \in h(a_1^{i-1}, a, A_{i+1}^n)$ it follows that $y \in f(x_1^{i-1}, x, x_{i+1}^n)$ exists such that $a_i \in A(y)$. Therefore, $a_i \in A(x_i) \cap A(y)$ implies that $x_i = y$ and $x_i \in f(x_1^{i-1}, x, x_{i+1}^n)$. □

We set $K(P) := \bigcup_{x \in P} A(x)$, where $\emptyset \neq P \subseteq H$. It is clear that $K(P) = g^{-1}(P)$ and if $a_1^n \in K_H$, then

$$h(a_1^n) = K\big(f\big(g(a_1), \dots, g(a_n)\big)\big) = g^{-1}\big(f\big(g(a_1), \dots, g(a_n)\big)\big) = g^{-1}g\big(h(a_1^n)\big)$$

since $f\big(g(a_1), \dots, g(a_n)\big) = g\big(h(a_1^n)\big)$. Hence g is an epimorphism.

Let (H, f) be an n-ary hypergroup, we say that H is *regular* if it has at least one neutral element. The element a' is inverse of a if $f(a, a', \overset{(n-2)}{e}) = e$. If (H, f) is regular, we denoted by $E_H = E(H)$ the set of the identities, and for every $a \in H$, by $i(a)$ the set of the inverses of a.

Theorem 3.3. *Let (H, f) be an n-ary hypergroup. Then*

(1) $E(K_H) = K(E_H)$;

(2) $i(a) = K\big(i\big(g(a)\big)\big) = g^{-1}\big(i\big(g(a)\big)\big)$ *for every* $a \in K_H$.

Proof. (1) It is obvious.

(2) Indeed, for $a_1^n \in K_H$, one has

$$c \in h(a_1^n) \iff c \in g^{-1}g\big(h(a_1^n)\big) \iff g(c) \in g\big(h(a_1^n)\big) = h\big(g(a_1), \dots, g(a_n)\big). \quad \square$$

Theorem 3.4. *Let (H, f) be an n-ary hypergroup. If P is a complete part of H, then $K(P)$ is a complete part of (K_H, h).*

Proof. Let $x_1^t \in K_H$, where $t = l(n-1) + 1$ and $u \in h_{(l)}(x_1^t) \cap K(P) \neq \emptyset$. Then there exist $z_1^t \in H$ such that $x_i \in A(z_i)$. Hence,

$$u \in \bigcup_{z_i \in f_{(l)}(z_1^t)} A(z_i),$$

so $z_r \in f_l(z_1^t)$ exists such that $u \in A(z_r)$. Since $u \in K(P)$, there exists $z_k \in P$ such that $u \in A(z_k)$, for this reason $z_r = z_k \in f_l(z_1^t) \cap P$. This follows for the completeness of P, $f_{(l)}(z_1^t) \subseteq P$.

Now suppose that

$$v \in h_{(l)}(x_1^t) = \bigcup_{y \in f_{(l)}(z_1^t)} A_y.$$

Then there exists $w \in f_{(l)}(z_1^t)$ such that $v \in A(w)$. Since $f_{(l)}(z_1^t) \subseteq P$, we have $A(w) \subseteq K(P)$. Therefore, $v \in K(P)$. □

Theorem 3.5. *Let P be a non-empty subset of an n-ary semihypergroup H. Then P is a sub n-ary hypergroup of H if and only if $K(P)$ is a sub n-ary hypergroup of K_H.*

Proof. Let (P, f) be a sub n-ary hypergroup of H. It is immediate that $K(P)$ is a sub n-ary semihypergroup. Let $b \in h(a_1^{i-1}, x_i, a_{i+1}^n)$ and there exist $c, c_1^{i-1}, c_{i+1}^n, y \in P$ such that $b \in A(c)$, $a_j \in A(c_j)$ $(j = 1, \dots, i-1, i+1, \dots, n)$ and $x_i \in A(y_i)$. The hypothesis implies the equation $c \in f(c_1^{i-1}, y_i, c_{i+1}^n)$ has a solution $z \in H$. If $q \in A(z)$, we have

$$h(a_1^{i-1}, q, a_{i+1}^n) = \bigcup_{t \in f(c_1^{i-1}, z, c_{i+1}^n)} A(t).$$

Therefore, $A(c) \subseteq h(a_1^{i-1}, q, a_{i+1}^n)$ and for this reason $b \in A(c) \subseteq h(a_1^{i-1}, q, a_{i+1}^n)$.

Conversely, let $K(P)$ be a sub n-ary hypergroup of K_H. If $x, x_1^n \in P$, then $A(x_i) \subseteq K(P)$ $(1 \le i \le n)$. Let $u \in f(x_1^{i-1}, x, x_{i+1}^n)$, $a_m \in A(x_m)$ $(i = 1, \dots, i-1, i+1, \dots, n)$ and $a \in A(x)$. Then

$$h(a_1^{i-1}, a, a_{i+1}^n) = \bigcup_{t \in f(x_1^{i-1}, x, x_{i+1}^n)} A(t) \subseteq K(P).$$

Since $A(u) \subseteq K(P)$, there exists $p \in P$ such that $A(u) = A(p)$. Therefore, $u = p$ and P is a sub n-ary hypergroup of H.

Let $b \in f(x_1^{i-1}, x, x_{i+1}^n)$. Since $A(x), A(x_i) \subseteq K(P)$, for every $a_i \in A(x_i)$, $y \in A(x)$ and $a \in A(b)$, there exists $z \in K(P)$ such that the equation $a \in h(a_1^{i-1}, y, a_{i+1}^n)$ has a solution $z \in K(P)$. Hence, $c \in P$ exists such that $z \in A(c)$. For this reason

$$h(a_1^{i-1}, z, a_{i+1}^n) = \bigcup_{s \in f(x_1^{i-1}, c, x_{i+1}^n)} A(s)$$

and so there exists $t \in f(x_1^{i-1}, c, x_{i+1}^n)$ and $a \in A(t)$. This implies that $b = t \in f(x_1^{i-1}, c, x_{i+1}^n)$. □

Theorem 3.6. *Let* $(x, y) \in H^2$ *and* $(u, v) \in A(x) \times A(y)$. *If* $u\beta_{K_H} v$, *then* $x\beta_H y$.

Proof. Let $u\beta_{K_H} v$. Then $a_1^t \in K_H$, with $t = l(n-1)+1$ exist such that $a_k \in A(x_k)$. Then, we have

$$\{u, v\} \subseteq \bigcup_{w \in f_{(l)}(x_1^t)} A(t).$$

So $\{w_1, w_2\} \subseteq f_{(l)}(x_1^t)$ exists such that $u \in A(w_1)$, $v \in A(w_2)$. Clearly, $w_1\beta_H w_2$ and so $w_1 = x$, $w_2 = y$. Therefore, $x\beta_H y$. □

Theorem 3.7. *Let* $(x, y) \in H^2$ *and* $(u, v) \in A(x) \times A(y)$. *If* $u\beta_{K_H} v$, *then* $A(x)\bar{\bar{\beta}}_{K_n} A(y)$.

Proof. The hypothesis implies that $m \in \mathbb{N}$ and $a_1^t \in H$ exists such that $\{x, y\} \subseteq f_l(x_1^t)$. For every $(b_1, \dots, b_t) \in A(a_1) \times \cdots \times A(a_m)$, we have

$$h_l(b_1^t) = \bigcup_{v \in f_{(l)}(a_1^t)} A(v).$$

So, $A(x) \cup A(y) \subseteq h_{(l)}(b_1^t)$ and thus $A(x)\bar{\bar{\beta}}_{K_n} A(y)$. The inverse implication follows from Theorem 2.15. □

Theorem 3.8. *For every* $a \in A(x)$, *we have* $C_{K_H}(a) = \bigcup_{w \in C_H(x)} A(w)$.

Proof. Let $u \in C_{K_H}(a)$. Then $u\beta_{K_n} a$. Now let $u \in A(y)$, since $u \in A(x)$ and $a \in A(x)$ by Theorem 3.7, $y \in C_H(x)$ and

$$A(y) \subseteq \bigcup_{w \in C_H(x)} A(w),$$

so

$$C_{K_H}(a) \subseteq \bigcup_{w \in C_H(x)} A(w).$$

Conversely, suppose that

$$u \in \bigcup_{w \in C_H(x)} A(w).$$

Then $v \in C_H(x)$ exists such that $u \in A(v)$. By Theorem 3.7, this follows that $A(v)\overline{\overline{\beta^*}}_{k+1} A(x)$. Thus, $u\beta_k^* a$. □

Theorem 3.9. *We have* $\Delta(K_H) = K_{\Delta(H)}$.

Proof. Let k and h be two n-ary hyperoperations in $\Delta(K_H)$ and $K_{\Delta(H)}$, respectively. Then, for every $a_1^n \in K_H$ with $a_i \in A(x_i)$, we have

$$k(a_1^n) = C_{K_H}\big(h(a_1^n)\big) = C_{K_H}\Big(e \bigcup_{w \in f(x_1^n)} A(w)\Big)$$

and

$$h(a_1^n) = \bigcup_{z \in C_H(f(x_1^n))} A(z).$$

Let $u \in k(a_1^n)$. Then there exists

$$v \in \bigcup_{w \in f(x_1^n)} A(w)$$

such that $u \in C_{K_H}(v)$, and there exists $z \in f(x_1^n)$ such that $v \in A(z)$. By Theorem 3.8, we obtain

$$C_{K_H}(v) = \bigcup_{w \in C_H(z)} A(w).$$

So there exists $w \in C_H(z)$ such that $u \in A(w)$, since $C_H(z) = C_H\big(f(x_1^n)\big)$, hence $u \in h(a_1^n)$.

Conversely, let $u \in h(a_1^n)$. Then there exists $w \in C_H\big(f(x_1^n)\big)$ such that $u \in A(w)$. So there exists $z \in f(x_1^n)$ with $w \in C_H(z)$. Now let $v \in A(z)$. By Theorem 3.7, $A(w) \subseteq C_{K_H}(v)$ and therefore $u \in C_{K_H}(v)$. On the other hand,

$$C_{K_H}(v) \subseteq C_{K_H}\big(A(z)\big) \subseteq C_{K_H}\Big(\bigcup_{t \in f(x_1^n)} A(t)\Big) = k(a_1^n).$$

Thus, $u \in k(a_1^n)$. □

Theorem 3.10. *Let* (H, f) *be an n-ary hypergroup. Then:*

(1) $\omega_{K_H} = \omega_{\Delta(K_H)} = \omega_{K_{\Delta H}}$.

(2) $\omega_{K_H} = K(\omega_H)$.

Proof. (1) This follows from Definition 2.20 and Theorem 3.9.

(2) Let $x \in \omega_H$ and $a \in A(x)$. Then

$$K(\omega_H) = K\big(C_H(x)\big) = \bigcup_{t \in C_H(x)} A(t).$$

By Theorem 3.8, we have

$$C_{K_H}(a) = \bigcup_{t \in C_H(x)} A(t) = K(\omega_H).$$

By Theorems 3.4 and 3.5, we have $K(\omega_H)$ is a complete part sub n-ary hypergroup of K_H and so $\omega_{K_H} \subseteq K(\omega_H)$. Therefore, $\omega_{K_H} \subseteq C_{K_H}(a)$ which implies that $K(\omega_H) = C_{K_H}(a) = \omega_{K_H}$. □

Definition 3.11. Let (G, f) be an n-ary hypergroup. We say K is a *closed* sub n-ary hypergroup if for every $y, x_1^{i-1}, x_{i+1}^n \in K$ and $y \in f(x_1^{i-1}, x, x_{i+1}^n)$ follows $x \in K$.

If (G, f) is an n-ary hypergroup and $A \subseteq G$, we set

$$\langle A \rangle = \bigcup_{a_1^m \in A} f_{(l)}(a_1^m),$$

where $m = l(n-1)+1$ and $f_{(0)}(a_1) = \{a_1\}$. Let A be a subset of an n-ary hypergroup (G, f). Then A is said to be free or independent if either $A = \emptyset$ or for every $x \in A$, $x \notin \langle A - \{x\} \rangle$. A non-free subset is also called dependent. A subset A of G generates H if $\langle A \rangle = H$. In such a case, A is called a set of generators of H. A free set of generator is called a base. We shall denote V_G the set $\langle \emptyset \rangle$.

An n-ary hypergroup (G, f) is called *cambist* if for every $x, y \in G$ and $A \in \wp(G)$, the following implication is satisfied $x \in \langle A \cup \{y\} \rangle$, $x \notin \langle A \rangle \Rightarrow y \in \langle A \cup \{x\} \rangle$.

Theorem 3.12. *Let (G, f) be an n-ary group, H be an n-ary hypergroup without closed proper sub n-ary hypergroups and K be the n-ary (H, G)-hypergroup. Then:*

(1) $V_K = H$,

(2) *for every $A \in \wp(K)$, a subset B of G exists such that $\langle A \rangle = K(\langle B \rangle)$. Moreover, if $\langle A \rangle \neq K$, then for all $z \in K - \langle A \rangle$, there exists $j \in G - \langle B \rangle$ such that $\langle \langle A \rangle \cup \{z\} \rangle = K(\langle B \cup \{j\} \rangle)$.*

(3) *K is a cambist n-ary hypergroup if and only if G is a cambist n-ary group.*

Proof. (1) Since H has not closed proper sub n-ary hypergroup, by Theorem 2.15, H is closed in K and we have $V_K = H$.

(2) By Theorems 2.15 and 3.5, a sub n-ary hypergroup G' of G exists such that $\langle A \rangle = K(G')$. Let $\langle A \rangle \neq K$ and $z \in K - \langle A \rangle$. Let B the subset of $G = \{i \in G \mid A_i \cap A \neq \emptyset\}$. Now we show that if $i \in B$, then $A_i \cap \langle A \rangle \neq \emptyset$. But $\langle A \rangle = K(G')$ is a complete part by Theorem 3.4, and $A_i = f(x_1^{i-1}, y, x_{i+1}^n)$ for any $x_1^{i-1}, x_{i+1}^n \in H$ and $y \in A_i$. Then $A_i \subseteq \langle A \rangle$ from which $i \in G'$ and hence $B \subseteq G'$. Therefore, $\langle B \rangle \subseteq G'$, whence $K(\langle B \rangle) \subseteq K(G')$ and consequently, since

$K(\langle B\rangle)$ is closed, so $\langle A\rangle = K(\langle B\rangle)$. Since $z \notin \langle A\rangle$, it follows that $j \in G$ exists such that $z \in A_i$ and $i \notin \langle B\rangle$ (if $j \in \langle B\rangle$, thus $z \in K(\langle B\rangle) = \langle A\rangle$). On the other hand, it is clear that $\langle A \cap \{z\}\rangle = K(\langle B \cup \{j\}\rangle)$ and (2) is proved.

(3) Let G be cambist and $x, y \in K$ such that $x \in \langle A \cup \{y\}\rangle$, $x \notin \langle A\rangle$. By (2), subset B of G exists such that $\langle A\rangle = K(\langle B\rangle)$. Since $x \notin \langle A\rangle$, we have $y \notin \langle A\rangle$. Let $y \in A_i$, so $j \notin \langle B\rangle$ and $\langle A \cup \{y\}\rangle = K(\langle B \cup \{j\}\rangle)$. It is clear that $s \in \langle B \cup \{j\}\rangle$ exists such that $x \in A_i$, where $s \notin \langle B\rangle$ (since $x \notin \langle A\rangle$). Since G is cambist, $j \in \langle B \cup \{s\}\rangle$. Finally, $y \in A_j \subseteq K(\langle B \cup \{s\}\rangle) = \langle A \cup \{x\}\rangle$ and consequently K is cambist. □

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S. M. Anvariyeh, Department of Mathematics, Yazd University, Yazd, Iran
E-mail: anvariyeh@yazd.ac.ir

B. Davvaz, Department of Mathematics, Yazd University, Yazd, Iran
E-mail: davvaz@yazduni.ac.ir, bdavvaz@yahoo.com

S. Mirvakili, Department of Mathematics, Payame Noor University, 19395-4697 Tehran, Iran
E-mail: saeed_mirvakili@pnu.ac.ir