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Class $wA(s,t)$ operators and quasisimilarity

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Abstract. In this paper it is shown that the normal parts of quasisimilar $wA(s,t)$ operators with $s+t=1$ are unitarily equivalent. Also, we establish the orthogonality of the range and the kernel of a nonnormal derivation with respect to the unitarily invariant norms associated with norm ideals of operators. Moreover, we obtain that the range of the generalized derivation induced by an pair satisfies Fuglede–Putnam property is orthogonal to its kernel.

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1. Introduction

Let $\mathscr{H}$ be an infinite dimensional complex Hilbert and $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on $\mathscr{H}$. Every operator T can be decomposed into $T=U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $\ker(U)=\ker(|T|)$, then this decomposition is called the *polar decomposition*, which is one of the most important results in operator theory ([15], [19], [24] and [27]). In this paper, $T=U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker(U)=\ker(|T|)$.

Given two operators $T,S\in\mathscr{L}(\mathscr{H})$, define $\delta_{T,S}:\mathscr{L}(\mathscr{H})\to\mathscr{L}(\mathscr{H})$ by $\delta_{T,S}(X)=TX-XS$ for all $X\in\mathscr{L}(\mathscr{H})$. The classical Fuglede–Putnam Theorem says if T and S^* are normal operators, then $\ker(\delta_{T,S})=\ker(\delta_{T^*,S^*})$.

A number of generalizations of the Putnam–Fuglede Theorem can be found in the extant literature, amongst them generalizations where the normal operators T and S are replaced by larger classes than the normal operators. The particular classes which have received a lot of attention are those consisting of either subnormal or hyponormal or M-hyponormal or dominant or k-quasi-hyponormal operators as well as p-hyponormal operators.

It is well known that $\ker(\delta_{T,S}) \subseteq \ker(\delta_{T^*,S^*})$ for T and S^* belonging to many a pair of these classes ([7], [12], [13], [14], [28], [32], [31], [30], [35] and some of the references therein) except for when both T and S^* are dominant (see [12], [13], [14]).

Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is *positive*, $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathscr{H}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators ([2], [9], [11], [17], [18] and [22]). An operator T is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for $p \in (0,1]$ and an operator T is said to be log-*hyponormal* if T is invertible and $\log|T| \geq \log|T^*|$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *paranormal* if it satisfies the norm inequality $\|T^2\|\,\|x\| \geq \|Tx\|^2$ for all $x \in \mathscr{H}$. Ando [5] proved that every log-hyponormal operators is paranormal. According to [25], an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be (p,k)-*quasihyponormal* operator if $T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \geq 0$, where $k \in \mathbb{N}$ and $p \in (0,1]$.

2. Complementary results

In this section, we shall show some properties on class $wA(s,t)$ operators.

Definition 2.1. Let $s > 0$ and $t > 0$ and $T = U|T|$ be the polar decomposition of T.

(i) T belongs to class $A(s,t) \Leftrightarrow (|T^*|^t|T|^{2s}|T^*|^t)^{t/(t+s)} \geq |T^*|^{2t}$ [16].

(ii) T belongs to class $wA(s,t)$

$$\iff (|T^*|^t|T|^{2s}|T^*|^t)^{t/(t+s)} \geq |T^*|^{2t} \quad \text{and} \quad |T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{s/(s+t)}.$$

$$\iff |\tilde{T}_{s,t}|^{2t/(s+t)} \geq |T|^{2t} \quad \text{and} \quad |T|^{2s} \geq |\tilde{T}^*_{s,t}|^{2s/(s+t)},$$

where $\tilde{T}_{s,t} = |T|^s U |T|^t$ generalized Aluthge transformation [20].

(iii) T belongs to class $A \Leftrightarrow |T^2| \geq |T|^2$, that is, $T \in$ class $A(1,1)$ [18].

(iv) T is w-hyponormal $\Leftrightarrow |\tilde{T}| \geq |T| \geq |\tilde{T}^*|$, that is, T belongs to class $wA\left(\frac{1}{2},\frac{1}{2}\right)$, where $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ [3].

We remark that Aluthge transformation has many interesting properties, and many authors study this transformation, for instance [1], [18], [20], [33]. These classes are included in normaloid (i.e., $\|T\| = r(T)$, the spectral radius of T). It has been known that for each $s > 0$ and $t > 0$, class $A(s,t)$ includes class $wA(s,t)$ by parts (i) and (ii) of Definition 2.1.

Lemma 2.2 ([20]). *Let $T \in \mathscr{L}(\mathscr{H})$. For each $0 < p_1 \le p_2$ and $0 < q_1 \le q_2$. If T belongs to class $wA(p_1, q_1)$, then T belongs to class $wA(p_2, q_2)$.*

Here in general, we can obtain that class $A(s,t)$ coincides with class $wA(s,t)$ by Lemma 2.2 as follows:

Theorem 2.3. *For each $s > 0$ and $t > 0$. The class $A(s,t)$ coincides with class $wA(s,t)$.*

Theorem 2.4. *Let $T \in \mathscr{L}(\mathscr{H})$. If T belongs to class $wA(s,t)$ operators for $s > 0$ and $t > 0$ and polar decomposition $T = U|T|$. Then T belongs to class $wA(p,p)$, where $p = \max\{s,t\}$. Moreover, the Aluthge transformation $\tilde{T} = |T|^p U |T|^p$ is semi-hyponormal operator and $\tilde{\tilde{T}} = |\tilde{T}|^q \tilde{U} |\tilde{T}|^q$ is hyponormal operator with $0 < q \le \frac{1}{2}$.*

Proof. Let $p = \max\{s,t\}$. Then it follows from Lemma 2.2 that T belongs to class $wA(p,p)$. Now,

$$\begin{aligned}
|\tilde{T}| &= (|T|^p U^* |T|^{2p} U |T|^p)^{1/2} \\
&= (U^* U |T|^p U^* |T|^{2p} U |T|^p U^* U)^{1/2} \\
&= U^* (U |T|^p U^* |T|^{2p} U |T|^p U^*)^{1/2} U \\
&= U^* (|T^*|^p |T|^{2p} |T^*|^p)^{1/2} U \\
&\ge U^* |T^*|^{2p} U \qquad (\text{since } T \in wA(p,p)) \\
&= |T|^{2p}.
\end{aligned}$$

Also

$$|\tilde{T}^*| = (|T|^p U |T|^{2p} U^* |T|^p)^{1/2} = (|T|^p |T^*|^{2p} |T|^p)^{1/2} \le |T|^{2p}.$$

Therefore, we have $|\tilde{T}| \ge |T|^{2p} \ge |\tilde{T}^*|$. That is, $\tilde{T}$ is semi-hyponormal.

Since $\tilde{T}$ is semi-hyponormal then $\tilde{T}$ is q-hyponormal such that $0 < q \le \frac{1}{2}$. Hence we have

$$\tilde{U}^* |\tilde{T}|^{2q} \tilde{U} \ge |\tilde{T}|^{2q} \ge \tilde{U} |\tilde{T}|^{2q} \tilde{U}^*.$$

Now

$$\tilde{\tilde{T}}^* \tilde{\tilde{T}} - \tilde{\tilde{T}} \tilde{\tilde{T}}^* = |\tilde{T}|^q (\tilde{U}^* |\tilde{T}|^{2q} \tilde{U} - \tilde{U} |\tilde{T}|^{2q} \tilde{U}^*) |\tilde{T}|^q \ge 0,$$

and hence $\tilde{\tilde{T}}$ is hyponormal. □

A pair (T, S) is said to have the Fuglede–Putnam property if $T^*X = XS^*$ whenever $TX = XS$ for every $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$.

Lemma 2.5 ([36]). *Let $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$. Then the following assertions equivalent:*

(a) *The pair (T, S) satisfies Fuglede–Putnam theorem.*

(b) *If $TX = XS$ for some $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, then $\overline{\mathrm{Re}(X)}$ reduces T, $\ker(X)^{\perp}$ reduces S and $T|_{\overline{\mathrm{Re}(X)}}$ and $S|_{\ker(X)^{\perp}}$ are normal operators.*

Lemma 2.6 ([23]). *Let $T \in \mathscr{L}(\mathscr{H})$ and $S^* \in \mathscr{L}(\mathscr{K})$ be either* log-*hyponormal or p-hyponormal operators. Then the pair (T, S) has the Fuglede–Putnam property.*

Since class $wA(s, t)$ operators coincide with class $A(s, t)$ for each $s > 0$ and $t > 0$ by [20], the following three results follow immediately from [29], Corollary 2.2, [29], Lemma 4.3 and [29], Lemma 4.7, respectively.

Lemma 2.7. *Let $T \in \mathscr{L}(\mathscr{H})$ be a class $wA(s, t)$ operator with $s > 0$ and $t > 0$. If $\tilde{T}(s, t) = |T|^s U |T|^t$ is normal, then T is also normal.*

Lemma 2.8. *Let $T \in \mathscr{L}(\mathscr{H})$ be class a $wA(s, t)$ operator for some $s, t \in (0, 1]$ and $\mathscr{M}$ be an invariant subspace of T, then the restriction $T|_{\mathscr{M}}$ of T onto $\mathscr{M}$ is also a class $wA(s, t)$ operator.*

Lemma 2.9. *Let $T = U|T| \in \mathscr{L}(\mathscr{H})$ be a class $wA(s, t)$ operator with $s + t = 1$ and $\ker(T) \subset \ker(T^*)$. Let $\tilde{T}(s, t) = |T|^s U |T|^t$. Suppose $\tilde{T}(s, t)$ be of the form $N \oplus T'$ on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$, where N is a normal on $\mathscr{M}$. Then $T = N \oplus T_1$ and $U = U_{11} \oplus U_{22}$, where T_1 is a class $wA(s, t)$ operator with $\ker(T_1) \subset \ker(T_1^*)$ and $N = U_{11}|N|$ is the polar decomposition of N.*

Theorem 2.10. *Let $T \in \mathscr{L}(\mathscr{H})$ be a class $wA(s, t)$ for some $t > 0$ and $s > 0$. If $\mathrm{meas}(\sigma(T)) = 0$, then T is normal.*

Proof. Let $T = U|T|$ be the polar decomposition of T. It is known from Lemma 2.2 that if T belongs to class $wA(s, t)$, then $\tilde{T}(s, t)$ is p-hyponormal where $p = \frac{\min\{s, t\}}{s+t}$ and $\sigma(\tilde{T}) = \{r^{s+t} e^{i\theta} : re^{i\theta} \in \sigma(T)\}$ by [37]. Hence $\mathrm{meas}(\sigma(\tilde{T})) = 0$. So it follows from Putnam inequality of p-hyponormal operators [8] that $\tilde{T}(s, t)$ is normal. Therefore, T is normal by Lemma 2.7. $\square$

Theorem 2.11. *Let $p_1 > 0$, $p_2 > 0$, $q_1 > 0$ and $q_2 > 0$. If T belongs to class $wA(p_1, q_1)$ and T^* belongs to class $wA(p_2, q_2)$, then T is normal.*

To prove this Theorem, we need the following lemma from [40].

Lemma 2.12. *Let $A \geq 0$ and $B \geq 0$. If $B^{1/2}AB^{1/2} \geq B^2$ and $A^{1/2}BA^{1/2} \geq A^2$. Then $A = B$.*

Proof of Theorem 2.11. Let $t = \max\{p_1, q_1, p_2, q_2\}$. If T belongs to class $wA(p_1, q_1)$, then T belongs to class $wA(t,t)$ by Theorem 2.4. Hence we have

$$(|T^*|^t|T|^{2t}|T^*|^t)^{1/2} \geq |T^*|^{2t} \quad \text{and} \quad |T|^{2t} \geq (|T|^t|T^*|^{2t}|T|^t)^{1/2}. \tag{1}$$

Also, if T^* belongs to class $wA(p_2, q_2)$, then by Theorem 2.4 T^* belongs to class $wA(t,t)$. Hence we have

$$(|T|^t|T^*|^{2t}|T|^t)^{1/2} \geq |T|^{2t} \quad \text{and} \quad |T^*|^{2t} \geq (|T^*|^t|T|^{2t}|T^*|^t)^{1/2}. \tag{2}$$

Therefore

$$|T|^t|T^*|^{2t}|T|^t = |T|^{4t} \quad \text{and} \quad |T^*|^{4t} = |T^*|^t|T|^{2t}|T^*|^t$$

holds by (1) and (2), and hence it follows from Lemma 2.12 that $|T| = |T^*|$. □

3. Class $wA(s,t)$ operators and quasi-similarity

An operator $X \in \mathscr{L}(\mathscr{H})$ is called *quasiaffinity* if X is both injective and has a dense range. Two operators T and S are said to quasi-similar if there exist quasiaffinities X and Y such that $X \in \ker(\delta_{T,S})$ and $Y \in \ker(\delta_{S,T})$.

The operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *pure* if there exists no non-trivial reducing subspace $\mathscr{M}$ of $\mathscr{H}$ such that the restriction of T to $\mathscr{M}$ is normal and is completely hyponormal if it is pure.

Recall that every operator $T \in \mathscr{L}(\mathscr{H})$ has a direct sum decomposition $T = T_1 \oplus T_2$, where T_1 and T_2 are normal and pure parts, respectively. Of course in the sum decomposition, either T_1 or T_2 may be absent.

The following Lemma is due to Williams [39], Lemma 1.1.

Lemma 3.1. *Let T and S be normal operators. If there exist injective operators such that $X \in \ker(\delta_{T,S})$ and $Y \in \ker(\delta_{S,T})$, then T and S are unitarily equivalent.*

Lemma 3.2. *Let T be a class $wA(s,t)$ operator with $s + t = 1$ such that $\ker T \subset \ker T^*$ and S be normal operator. If there exist an operator X with dense range such that $TX = XS$, then T is normal.*

Proof. Decompose T into normal and pure parts by $T = T_1 \oplus T_2$ with respect to the decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Letting T_2 have the polar decomposition $T_2 = U_2|T_2|$, we consider its generalized Aluthge transform $\tilde{T}_2(s,t) = |T_2|^s U_2 |T_2|^t$. Let

$\tilde{T}_2(s,t) = V_2|\tilde{T}_2(s,t)|$, and define $\hat{T}_2(s,t) = |\tilde{T}_2(s,t)|^s V_2 |\tilde{T}_2(s,t)|^t$. Letting $W = |\tilde{T}_2(s,t)|^s |T_2|^s$, by the kernel condition we see that W is a quasiaffinity such that $\hat{T}_2(s,t)W = WT_2$. Now let $\hat{T}(s,t) = T_1 \oplus \hat{T}_2(s,t)$ and $Y = I_{\mathscr{H}_1} \oplus W$. Then $\hat{T}(s,t)$ is p-hyponormal, where $p = \max\{s,t\}$ and Y is a quasiaffinity such that $\hat{T}(s,t)Y = YT$. So we have that $\hat{T}(s,t)(YX) = (YX)S$ and YX has dense range. Thus by Lemma 3 of [21] $\hat{T}(s,t)$ is normal and so by Lemma 2.7, T is normal. □

Theorem 3.3. *Let T and S^* be class $wA(s,t)$ operators with $s+t=1$. If there exist a quasiaffinity X such that $X \in \ker(\delta_{T,S})$, then T and S are unitarily equivalent normal operators.*

Proof. First decompose T and S^* into their normal and pure parts by $T = T_1 \oplus T_2$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and $S^* = S_1^* \oplus S_2^*$ on $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2$, where T_1, S_1 are normal and T_2, S_2^* are pure. Let $X = [X_{i,j}]_{i,j=1}^2$. Then $TX = XS$ implies that $T_2X_{21} = X_{21}S_2$ and $T_2X_{22} = X_{22}S_2$. Let $T_2 = U_2|T_2|$, $S_2^* = V_2^*|S_2^*|$ be the polar decompositions of T_2 and S_2^*, respectively and

$$\tilde{T}_2(s,t) = |T_2|^s U_2 |T_2|^t, \qquad \tilde{S}_2^*(s,t) = |S_2^*|^s V_2^* |S_2^*|^t, \qquad W = |T_2|^s X_{22} |S_2^*|^s.$$

Then

$$\tilde{T}_2(s,t)W = |T_2|^s T_2 X_{22} |S_2^*|^s = |T_2|^s X_{22} S_2 |S_2^*|^s = W\big(\tilde{S}_2^*(s,t)\big)^*.$$

Since $\tilde{T}_2(s,t)$, $\tilde{S}_2^*(s,t)$ are class $wA(s,t)$ operators, then $\tilde{T}_2(s,t)$, $\tilde{S}_2^*(s,t)$ are $\min\{s,t\}$-hyponormal and W is quasiaffinity. Now by Lemma 2.6, we have $\tilde{T}_2^*(s,t)W = W\tilde{S}_2^*(s,t)$ and $\overline{\mathrm{Re}(W)}$ reduces $\tilde{T}_2(s,t)$ and $\ker(W)^\perp$ reduces $\tilde{S}_2^*(s,t)$ and $\tilde{T}_2(s,t)|_{\overline{\mathrm{Re}(W)}}$ and $\tilde{S}_2^*(s,t)|_{\ker(W)^\perp}$ are unitarily equivalent normal operators. Since W is quasiaffinity, we have $\overline{\mathrm{Re}(W)} = \mathscr{H}$ and $\ker(W)^\perp = \mathscr{H}$ and $\tilde{T}_2(s,t)$ and $\tilde{S}_2(s,t)$ are unitarily equivalent normal operators. In particular, $\tilde{T}_2(s,t)$ and $\tilde{S}_2(s,t)$ are normal operators and so the result follows now by Theorem 2.7 and Lemma 3.1. □

From Theorem 3.3, it is easy to deduce that a pure class $wA(s,t)$ operator is normal.

Corollary 3.4. *A pure class $wA(s,t)$ operator with $s+t=1$ such that* $\ker T \subset \ker T^*$ *is normal.*

Conway [10] proved that the normal parts of quasisimilar subnormal operators are unitarily equivalent and gave an example showing that the pure parts of quasisimilar subnormal operators need not be quasisimilar. This result was generalized to classes of p-hyponormal operators in [21] and log-hyponormal operators in [23], respectively. We prove that these results hold for class $wA(s,t)$ operators with $s+t=1$.

Theorem 3.5. *Let $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$ be class $wA(s,t)$ operators with $s+t=1$ such that $\ker T \subset \ker T^*$ and $\ker S \subset \ker S^*$ and let $T = N_1 \oplus R_1$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and $S = N_2 \oplus R_2$ on $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2$, where N_i, R_i $(i=1,2)$ are the normal and the pure parts of T and S, respectively. If T and S are quasisimilar, then N_1 and N_2 are unitarily equivalent and there exist $X_* \in \mathscr{L}(\mathscr{K}_2, \mathscr{H}_2)$, $Y_* \in \mathscr{L}(\mathscr{H}_2, \mathscr{K}_2)$ having dense ranges such that $R_1 X_* = X_* R_2$ and $Y_* R_1 = R_2 Y_*$.*

Proof. By hypotheses there exist quasiaffinities $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ and $Y \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ such that $TX = XS$ and $YT = SY$. Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$$

with respect to $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2$ and $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, respectively. A simple matrix calculation shows that

$$R_1 X_3 = X_3 N_2 \quad \text{and} \quad R_2 Y_3 = Y_3 N_1.$$

We claim that $X_3 = Y_3 = 0$. To prove this, let $\mathscr{M} = \overline{\mathrm{Re}(X_3)}$ and assume that $X_3 \neq 0$. Then $\mathscr{M}$ is an non-trivial invariant subspace of R_1. If R_1' is the restriction of R_1 to $\mathscr{M}$, then R_1' is class $wA(s,t)$ by Lemma 2.8. If we define an operator $X_3' : \mathscr{K}_1 \to \mathscr{M}$ by $X_3' x = X_3 x$ for each $x \in \mathscr{K}_1$, then we can see that X_3' has dense range and satisfies that $R_1' X_3' = X_3' N_2$. By Lemma 3.2, R_1' is normal. But which contradicts the hypothesis R_1 is pure. This forces $X_3 = 0$. Similarly, $Y_3 = 0$. Thus it follows that X_1 and Y_1 are injective. Since $N_1 X_1 = X_1 N_2$ and $Y_1 N_1 = N_2 Y_1$, by Lemma 3.1 we have that N_1 and N_2 are unitarily equivalent. Also, we can notice X_4 and Y_4 have dense ranges and

$$R_1 X_4 = X_4 R_2 \quad \text{and} \quad Y_4 R_1 = R_2 Y_4.$$

Hence the proof is complete. □

From Theorem 3.5 we easily obtain the following corollaries, and so we omit their proofs.

Corollary 3.6. *Let T and S be quasisimilar class $wA(s,t)$ with $s+t=1$ such that $\ker T \subset \ker T^*$ and $\ker S \subset \ker S^*$. If T is pure, then S is also pure.*

Corollary 3.7. *Let $T \in \mathscr{L}(\mathscr{H})$ be a class $wA(s,t)$ with $s+t=1$ such that* $\ker T \subset \ker T^*$ *and let $S \in \mathscr{L}(\mathscr{K})$ be normal. If T and S are quasisimilar, then T and S are unitarily equivalent normal operators.*

The *numerical range* of an operator T, denoted by $W(T)$, is the set defined by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

In general, the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$ do not imply that T is normal. If $T = SB$, where S is positive and invertible, B is self-adjoint, and S and B do not commute, then $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, but T is not normal. Therefore the following question arises naturally.

Question. Which operator T satisfying the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$ is normal?

In 1966, Sheth [34] showed that if T is a hyponormal operator and $S^{-1}TS = T^*$ for any operator S, where $0 \notin \overline{W(S)}$, then T is self-adjoint. We extend the result of Sheth to the class $wA(s,t)$, $s,t > 0$ operators as follows.

Theorem 3.8. *Let $T \in \mathscr{L}(\mathscr{K})$. If T or T^* belongs to class $wA(s,t)$ for some $s > 0$ and $t > 0$ and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

To prove Theorem 3.8 we need the following Lemma from [38].

Lemma 3.9. *If $T \in \mathscr{L}(\mathscr{H})$ is any operator such that $S^{-1}TS = T^*$, where $0 \notin \overline{W(S)}$, then $\sigma(T) \subseteq \mathbb{R}$.*

Proof of Theorem 3.8. Suppose first that T is a class $wA(s,t)$ operator. Since $\sigma(T) \subseteq \overline{W(S)}$, S is invertible and hence $ST = T^*S$ becomes $S^{-1}T^*S = T = (T^*)^*$. Apply Lemma 3.9 to T to get $\sigma(T) \subseteq \mathbb{R}$. Thus $\text{meas}(\sigma(T)) = 0$ for the planer Lebesgue measure $\text{meas}(\cdot)$. It follows from Theorem 2.10 that T is normal. Since $\sigma(T) \subseteq \mathbb{R}$. Therefore, T is self-adjoint.

Now assume that T^* is a class $wA(s,t)$ operator. Since $\sigma(T) \subseteq \overline{W(S)}$, S is invertible and hence $ST = T^*S$ becomes $S^{-1}T^*S = T = (T^*)^*$. Apply Lemma 3.9 to T^* to get $\sigma(T^*) \subseteq \mathbb{R}$. Then $\sigma(T) = \overline{\sigma(T^*)} = \sigma(T^*) \subseteq \mathbb{R}$. Thus $\text{meas}(\sigma(T)) = \text{meas}(\sigma(T^*)) = 0$ for the planer Lebesgue measure $\text{meas}(\cdot)$. It follows from Theorem 2.10 that T^* is normal. Since $\sigma(T) = \sigma(T^*) \subseteq \mathbb{R}$. Therefore, T is self-adjoint. □

Theorem 3.10. *Let $T, S \in \mathscr{L}(\mathscr{H})$ be such that T is an injective (p,k)-quasihyponormal and S^* is an injective class $wA(s,t)$ operator with $s+t=1$. If $X \in \ker(\delta_{T,S})$ for some $X \in \mathscr{L}(\mathscr{H})$, then $X \in \ker(\delta_{T^*,S^*})$.*

Proof. Since $\overline{\mathrm{Re}(X)}$ is invariant under T and $\ker(X)^{\perp}$ is invariant under S^*, we can consider the following decompositions $\mathscr{H} = \overline{\mathrm{Re}(X)} \oplus \overline{\mathrm{Re}(X)}^{\perp} = \ker(X)^{\perp} \oplus \ker(X)$ and we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & 0 \\ S_2 & S_3 \end{pmatrix},$$
$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : \ker(X)^{\perp} \oplus \ker(X) \to \overline{\mathrm{Re}(X)} \oplus \overline{\mathrm{Re}(X)}^{\perp}.$$

From $TX = XS$, we obtain

$$T_1X_1 = X_1S_1. \tag{3}$$

Let $S_1^* = U^*|S_1^*|$ be the polar decomposition of S_1^*. Let $\tilde{S}_1^* = |S_1^*|^s U^* |S_1^*|^t$ be the generalized Aluthge transform of S_1. From Equation (3), we have

$$T_1X_1 = X_1|S_1^*|U \tag{4}$$

Let $W = X_1|S_1^*|^s$. Then

$$\begin{aligned} T_1W = T_1(X_1|S_1^*|^s) &= X_1|S_1^*|U|S_1^*|^s \\ &= (X_1|S_1^*|^s)|S_1^*|^t U|S_1^*|^s \\ &= W\big(\tilde{S}_1^*(s,t)\big)^*. \end{aligned}$$

Since $\tilde{S}_1^*(s,t)$ is $\min\{s,t\}$-hyponormal by [20]. Thus it follows from Theorem 11 of [25] that the pair $\big(T_1, \tilde{S}_1^*(s,t)\big)$ satisfies the Fuglede–Putnam theorem. Therefore, $T_1|_{\overline{\mathrm{Re}(W)}}$ and $\tilde{S}_1^*(s,t)|_{\ker(W)^{\perp}}$ are normal operators. Since X_1 is injective with dense range and $|S_1^*|^s$ is an injective, we have

$$\overline{\mathrm{Re}(W)} = \overline{\mathrm{Re}(X_1)} = \overline{\mathrm{Re}(X)}$$

and

$$\ker(W) = \ker(X_1) = \ker(X).$$

It follows that T_1 and $\tilde{S}_1^*(s,t)$ are normal, and hence it follows from Lemma 2.7 that S_1 is also normal. Since T is an injective (p,k)-quasi-hyponormal and its restriction T_1 is normal, then $\overline{\mathrm{Re}(X)}$ reduces T by [25, Lemma 10]. Thus $T_2 = 0$. Similarly, S^* is class $wA(s,t)$ and $S_1^* = S^*|_{\ker(X)^\perp}$ is normal, therefore $\ker(X)^\perp$ reduces S^* by Lemma 4.4 of [29]. Hence, $S_2 = 0$. Since the pair (T_1, S_1) satisfies the Fuglede–Putnam theorem, $X \in \delta_{T_1^*, S_1^*}$. Consequently, $X \in \ker(\delta_{T^*,S^*})$. □

4. Nonnormal derivation

Recall that each unitarily invariant norm $\|.\|_I$ is defined on a natural subclass $J_{\|.\|_I}$ of $\mathscr{L}(\mathscr{H})$ called the norm ideal associated with the norm $\|.\|_I$ and satisfies the invariance property $\|UTV\|_I = \|T\|$ for all $J_{\|.\|_I}$ and for all unitary operators $U, V \in \mathscr{L}(\mathscr{H})$.

Let $T \in \mathscr{L}(\mathscr{H})$ be compact, and let $s_1(T) \geq s_2(T) \geq \cdots \geq 0$ denote the singular values of T, i.e., the eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p-class C_p if

$$\|T\|_p = \Big(\sum_{j=1}^{\infty}\big(s_j(T)\big)^p\Big)^{1/p} = \big(\mathrm{tr}|T|^p\big)^{1/p} < \infty, \qquad 1 \leq p < \infty,$$

where $\mathrm{tr}(\,\cdot\,)$ denote the trace functional. Hence $C_1(\mathscr{H})$ is the trace class, $C_2(\mathscr{H})$ is the Hilbert-Schmidt class, and C_∞ is the class of compact operator with $\|T\|_\infty = s_1(T)$ denoting the usual norm.

Theorem 4.1. *Let $T, S, X \in \mathscr{L}(\mathscr{H})$ such that the pairs (T, S) satisfies the Fuglede–Putnam property, that is, $X \in \ker(\delta_{T^*,S^*})$ whenever $X \in \ker(\delta_{T,S})$. If $R \in \mathscr{L}(\mathscr{H})$ such that $\delta_{T,S}(R) + X \in J_{\|.\|_I}$, then $X \in J_{\|.\|_I}$ and*

$$\|\delta_{T,S}(R) + X\|_I \geq \|X\|_I.$$

To prove this theorem, we need the following lemma from [26].

Lemma 4.2. *Let $N, M, X \in \mathscr{L}(\mathscr{H})$ such that N and M are normal and $X \in \ker(\delta_{N,M})$. If $R \in \mathscr{L}(\mathscr{H})$ such that $\delta_{N.M}(R) + X \in J_{\|.\|_I}$, then $X \in J_{\|.\|_I}$ and*

$$\|\delta_{N,M}(R) + X\|_I \geq \|X\|_I.$$

Proof of Theorem 4.1. Since the pairs (T, S) satisfies the Fuglede–Putnam property, it follows from Lemma 2.5 that $\overline{\mathrm{Re}(X)}$ reduces T, $\ker(X)^\perp$ reduces S, and $T|_{\overline{\mathrm{Re}(X)}}$ and $S|_{\ker(X)^\perp}$ are unitarily equivalent normal operators. Then with respect to the orthogonal decomposition $\mathscr{H} = \overline{\mathrm{Re}(X)} \oplus \overline{\mathrm{Re}(X)}^{\perp}$ and $\mathscr{H} = \ker(X)^{\perp} \oplus \ker(X)$, T and S can be respectively represented as

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}.$$

Now assume that the operators $X, R : \ker(X)^{\perp} \oplus \ker(X) \to \overline{\mathrm{Re}(X)} \oplus \overline{\mathrm{Re}(X)}^{\perp}$ have the matrix representations

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}.$$

Then T_1 and S_1 are normal, and $T_1X_1 = X_1S_1$. Applying Lemma 4.2 to the operators T_1, S_1, X_1, and R_1 we see that $X_1 \in J_{\|.\|_I}$. Hence $X \in J_{\|.\|_I}$ and

$$\begin{aligned} \|\delta_{T,S}(R) + X\|_I &= \left\| \begin{pmatrix} \delta_{T_1,S_1}(R_1) + X_1 & * \\ * & * \end{pmatrix} \right\|_I \\ &\geq \|\delta_{T_1,S_1}(R_1) + X_1\|_I \geq \|X_1\|_I = \|X\|_I, \end{aligned}$$

so the proof of the theorem is achieved. □

Definition 4.3. Given subspaces $\mathscr{M}$ and $\mathscr{N}$ of a Banach space $\mathscr{V}$ with norm $\|.\|$. $\mathscr{M}$ is said to be orthogonal to $\mathscr{N}$ if $\|m + n\| \geq \|n\|$ for all $m \in \mathscr{M}$ and $n \in \mathscr{N}$.

J. H. Anderson and C. Foias [4] proved that if T and S are normal, R is an operator such that $TR = RS$, then

$$\|\delta_{T,S}(R) + X\| \geq \|X\|$$

Where $\|.\|$ is the usual operator norm. Hence the range of $\delta_{T,S}$ is orthogonal to the null space of $\delta_{T,S}$. The orthogonality here is understood to be in the sense of Definition 4.3.

Theorem 4.4. *Let $T, S \in \mathscr{L}(\mathscr{H})$ and (T, S) satisfy the Fuglede–Putnam property. Then the range of $\delta_{T,S}$ is orthogonal to the kernel of $\delta_{T,S}$, that is, $\|\delta_{T,S}(R) + X\| \geq \|X\|$ for all $R \in \mathscr{L}(\mathscr{H})$ and $X \in \ker(\delta_{T,S})$.*

Proof. Since the pairs (T, S) satisfies the Fuglede–Putnam property, it follows from Lemma 2.5 that $\overline{\mathrm{Re}(X)}$ reduces T, $\ker(X)^{\perp}$ reduces S, and $T|_{\overline{\mathrm{Re}(X)}}$ and $S|_{\ker(X)^{\perp}}$ are unitarily equivalent normal operators. Then with respect to the orthogonal decomposition $\mathscr{H} = \overline{\mathrm{Re}(X)} \oplus \overline{\mathrm{Re}(X)}^{\perp}$ and $\mathscr{H} = \ker(X) \oplus \ker(X)^{\perp}$, T and S can be respectively represented as

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}.$$

Now assume that the operators $X, R : \ker(X)^{\perp} \oplus \ker(X) \to \overline{\mathrm{Re}(X)} \oplus \overline{\mathrm{Re}(X)}^{\perp}$ have the matrix representations

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}.$$

Then T_1 and S_1 are normal, and $T_1X_1 = X_1S_1$. Hence

$$\delta_{T,S}(R) + X = \begin{pmatrix} \delta_{T_1,S_1}(R_1) + X_1 & * \\ * & * \end{pmatrix}.$$

Since $X_1 \in \ker(\delta_{T_1,S_1})$ and T_1, S_1 are normal, it follows by [4] that

$$\begin{aligned} \|\delta_{T,S}(R) + X\| &= \left\| \begin{pmatrix} \delta_{T_1,S_1}(R_1) + X_1 & * \\ * & * \end{pmatrix} \right\| \\ &\geq \|\delta_{T_1,S_1}(R_1) + X_1\| \geq \|X_1\| = \|X\|. \end{aligned}$$

That is, the range of $\delta_{T,S}$ is orthogonal to the kernel of $\delta_{T,S}$. □

For each pairs of operators A and B in $\mathscr{L}(\mathscr{H})$, an operator τ in $\mathscr{L}\big(C_2(\mathscr{H})\big)$ is defined by

$$\tau X = AXB.$$

Evidently $\|\tau\| \leq \|A\|\,\|B\|$. And the adjoint of τ is given by the formula $\tau^* X = A^* X B^*$. In particular, if A and B are both positive, then τ is positive and $\tau^{1/2}X = A^{1/2}XB^{1/2}$, as one sees from the calculation

$$\begin{aligned} \langle \tau X, X \rangle &= \mathrm{tr}(AXBX^*) = \mathrm{tr}(A^{1/2}XBX^*A^{1/2}) \\ &= \mathrm{tr}\big((A^{1/2}XB^{1/2})(A^{1/2}XB^{1/2})^*\big) \geq 0. \end{aligned}$$

Since $|\tau|^2 X = |A|^2 X |B^*|^2$ and $|\tau^*|^2 X = |A^*|^2 X |B|^2$, we have

$$|\tau|^{1/2^n} = |A|^{1/2^n} X |B^*|^{1/2^n}$$

and

$$|\tau^*|^{1/2^n} = |A^*|^{1/2^n} X |B|^{1/2^n}$$

for each integer $n \geq 1$.

Now we need the following lemma.

Lemma 4.5. *Let $s > 0$ and $t > 0$. Let T and S be operators in $\mathscr{L}(\mathscr{H})$. If T and S^* are class $wA(s,t)$ operators. Then the operator $\tau : C_2(\mathscr{H}) \to C_2(\mathscr{H})$ defined by $\tau X = TXS$ is class $wA(s,t)$ operator.*

Proof. For $X \in \mathscr{L}(\mathscr{H})$, we have

$$\begin{aligned}
&\big((|\tau^*|^t|\tau|^{2s}|\tau^*|^t)^{t/(t+s)} - |\tau^*|^{2t}\big)X \\
&\quad = \big((|T^*|^t|T|^{2s}|T|^t)^{t/(t+s)} - |T^*|^{2t}\big)X(|S|^t|S^*|^{2s}|S|^t)^{t/(t+s)} \\
&\qquad + |T^*|^{2t}X\big((|S|^t|S^*|^{2s}|S|^t)^{t/(t+s)} - |S|^{2t}\big),
\end{aligned}$$

and

$$\begin{aligned}
&\big(|\tau|^{2s} - (|\tau|^s|\tau^*|^{2t}|\tau|^s)^{s/(t+s)}\big)X \\
&\quad = \big(|T|^{2s} - (|T|^s|T^*|^{2t}|T|^s)^{s/(t+s)}\big)X|S^*|^{2s} \\
&\qquad + (|T|^s|T^*|^{2t}|T|^s)^{s/(t+s)}X\big(|S^*|^{2s} - (|S^*|^s|S|^{2t}|S^*|^s)^{s/(t+s)}\big).
\end{aligned}$$

Since T and S^* are class $wA(s,t)$, we have

$$\big((|\tau^*|^t|\tau|^{2s}|\tau^*|^t)^{t/(t+s)} - |\tau^*|^{2t}\big) \geq 0$$

and

$$\big(|\tau|^{2s} - (|\tau|^s|\tau^*|^{2t}|\tau|^s)^{s/(t+s)}\big) \geq 0.$$

Therefore, $\tau X = AXB$ is class $wA(s,t)$ operator. □

Theorem 4.6. *Let $0 < s, t \leq 1$. Let T be class $wA(s,t)$ operator and S^* be an invertible class $wA(s,t)$ operator. If $X \in \ker(\delta_{T,S})$ for $X \in C_2(\mathscr{H})$, then $X \in \ker(\delta_{T^*,S^*})$.*

Proof. Let τ be defined on $C_2(\mathscr{H})$ by $\tau X = TXS^{-1}$. Since S^* is an invertible class $wA(s,t)$ operator, then it follows from [16] that S^* is also a class $wA(s,t)$ operator for each $s > 0$ and t [37]. Since T is class $wA(s,t)$ operator and $(S^{-1})^* = (S^*)^{-1}$ is class $wA(s,t)$ operator, we have that τ is class $wA(s,t)$ operator on $C_2(\mathscr{H})$ by Lemma 4.5. Moreover, we have $\tau X = TXS^{-1} = X$ because of $X \in \ker(\delta_{T,S})$. Hence X is an eigenvector of τ. By [37], we have $\tau^* X = T^*X(S^{-1})^* = X$, that is, $X \in \ker(\delta_{T^*,S^*})$. So the proof is achieved. □

Theorem 4.7. *Let $T, S \in \mathscr{L}(\mathscr{H})$. Then*

$$\|\delta_{T,S}(R) + S\|_2^2 = \|\delta_{T,S}(R)\|_2^2 + \|X\|_2^2 \tag{5}$$

and

$$\|\delta_{T^*,S^*}(R) + S\|_2^2 = \|\delta_{T^*,S^*}(R)\|_2^2 + \|X\|_2^2 \tag{6}$$

for all $R \in C_2(\mathscr{H})$ *and* $X \in \ker(\delta_{T,S}) \cap C_2(\mathscr{H})$ *if and only if the pair* (T,S) *satisfies Fuglede–Putnam property.*

Proof. Since the Hilbert-Schmidt class $C_2(\mathscr{H})$ is a Hilbert space under the inner product $\langle Y, Z\rangle = \mathrm{tr}(Z^*Y) = \mathrm{tr}(YZ^*)$. Then we have

$$\begin{aligned}\|\delta_{T,S}(R) + X\|_2^2 &= \|\delta_{T,S}(R)\|_2^2 + \|X\|_2^2 + 2\,\mathrm{Re}\langle\delta_{T,S}(R), X\rangle \\ &= \|\delta_{T,S}(R)\|_2^2 + \|X\|_2^2 + 2\,\mathrm{Re}\langle R, \delta_{T^*,S^*}(X)\rangle\end{aligned}$$

and

$$\|\delta_{T^*,S^*}(R) + X\|_2^2 = \|\delta_{T^*,S^*}(R)\|_2^2 + \|X\|_2^2 + 2\,\mathrm{Re}\langle R, \delta_{T,S}(X)\rangle$$

Hence, Equations (5) and (6) hold if and only if the pair (T,S) satisfies Fuglede–Putnam property. □

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