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Regularities and limit theorems of some additive functionals of symmetric stable process in some anisotropic Besov spaces

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Abstract. In this paper, we give some regularities and limit theorems of some additive functionals of symmetric stable process of index $1 < \alpha \leq 2$ in some anisotropic Besov spaces.

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1. Introduction

Relative compactness in the space of probability measures is a key tool in the study of weak convergence. A family $\mathscr{F}$ of probability measures on the general metric space S is said to be tight if for each positive ε, there is a compact set K such that $P(K) > 1 - \varepsilon$ for all P in $\mathscr{F}$. According to Prohorov's theorem, tightness is always a sufficient condition for relative compactness and is also necessary if S is separable and complete.

The space $\mathscr{C}[0,1]$ of continuous functions is a classical framework for many regularities and limit theorems in the theory of stochastic processes. The $\mathscr{C}[0,1]$-weak convergence of a sequence of stochastic processes X_n, gives useful results about the convergence in distribution of continuous functionals of the paths. In many situations the processes X_n are known to have almost surely paths with at least some Hölder regularity and the same happens for the limiting process X.

The recent developments in the theory of wavelets and their applications in probability and statistics show the interest in using more sophisticated function spaces like the Hölder space $\mathscr{C}^{\alpha}[0,1]$, $0 < \alpha < 1$, and Besov spaces.

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Our aim in this paper is to prove that some additive functionals of local times of symmetric stable processes of index $1 < \alpha \leq 2$ satisfy certain Hölder conditions in L_p-norms which are more precise than the classical Hölder conditions in the uniform norm. We generalize also some limit theorems obtained, in the space of continuous functions, by Rosen [15] for symmetric stable process of index $1 < \alpha \leq 2$ and Yor [19] for Brownian mtion, $\alpha = 2$. These will be done by recalling notions of anisotropic Besov spaces, we use a result of Kamont [13] who has proved the characterization of these spaces in terms of the coefficients of the expansion of a continuous function with respect to a basis which consists of tensor products of Schauder functions. For the one-parameter Besov spaces, Ciesielski et al. [12] have shown by using the techniques of constructive approximation of functions that Besov spaces are isomorphic to spaces of real sequences. These characterizations make the Besov topology easy to handle, and many applications have been given in stochastic calculus, such as the regularities of some additive functionals of local times of symmetric stable process of index $1 < \alpha \leq 2$ (see for example Ait Ouahra et al. [2] and [4]).

Most of the estimates in this paper contain unspecified positive finite constants. We use the same symbol C_p for these constants, even when they vary from one line to the next.

Throughout this paper, $X = \{X_t \mid t \geq 0\}$ denotes the real-valued symmetric stable process of index $1 < \alpha \leq 2$, which is known to have a jointly continuous local time $\{L(t,x) \mid t \geq 0, x \in \mathbb{R}\}$ (see Barlow [5] and Boylan [11]).

We have the well known regularity property of the local time and we refer to Marcus and Rosen [14] for a proof.

Lemma 1.1. *For any integer $p \geq 1$, there exists a constant $0 < C_p < \infty$ such that for all $0 \leq t, s \leq 1$ and all $x, y \in \mathbb{R}$,*

$$\|L(t,x) - L(s,x)\|_{2p} \leq C_p |t-s|^{(\alpha-1)/\alpha}, \tag{1}$$

$$\|L(t,x) - L(t,y)\|_{2p} \leq C_p t^{(\alpha-1)/2\alpha} |x-y|^{(\alpha-1)/2}, \tag{2}$$

where $\|\cdot\|_{2p} = (\mathbb{E}|\cdot|^{2p})^{1/2p}$.

The following lemma gives a regularity property of the local time as a random function of two variables, its proof can be found in Ait Ouahra and Eddahbi [3].

Lemma 1.2. *For any integer $p \geq 1$, there exists a constant $0 < C_p < \infty$ such that for all $0 \leq t, s \leq 1$ and all $x, y \in \mathbb{R}$,*

$$\|L(t,x) - L(s,x) - L(t,y) + L(s,y)\|_{2p} \leq C_p |t-s|^{(\alpha-1)/2\alpha} |x-y|^{(\alpha-1)/2}. \tag{3}$$

Remark 1.3. (1) Using (3) and the fact that a.s. $L(0,x)=0$, we get the spatial Hölder regularity of local time:

$$\|L(t,x)-L(t,y)\|_{2p}\leq C_p|x-y|^{(\alpha-1)/2}. \tag{4}$$

(2) Notice that for $\alpha=2$, X is a Brownian motion. Trotter [17] has proved the existence of an a.s. continuous version of the Brownian local time $l(t,x)$ as a random function of two variables. Moreover, by Boufoussi and Roynette [10], for each $t>0$ fixed, the random function $l(t,\cdot)$ satisfies the Hölder condition (4) with exponent $\frac{1}{2}$, and by Boufoussi [7], the function $(t,x)\to l(t,x)$ satisfies the mixed Hölder condition (3) with exponent $\frac{1}{4}$ in time and exponent $\frac{1}{2}$ in space.

In Sections 2 and 3, we study the Hölder properties of some additive functionals of the local time $L(t,x)$. We first recall the definition of slowly varying functions and some properties. For more details about slowly varying functions, we refer the reader to Bingham et al. [6].

Definition 1.4. We say that a measurable function $l:\mathbb{R}^+\to\mathbb{R}^+$ is *slowly varying at infinity* if for all t positive, we have

$$\lim_{x\to\infty}\frac{l(tx)}{l(x)}=1.$$

We are interested in the behavior of l at $+\infty$, then we can assume for example that l is bounded on each interval of the form $[0,a]$, where $a>0$.

The following theorem called Potter's Theorem has played a central role in the proof of our main results of regularities.

Theorem 1.5. (1) *If l is slowly varying function, then for any chosen constants $A>1$ and $\xi>0$, there exists $X=X(A,\xi)$ such that*

$$\frac{l(y)}{l(x)}\leq A\max\left\{\left(\frac{y}{x}\right)^{\xi},\left(\frac{y}{x}\right)^{-\xi}\right\}\qquad(x\geq X, y\geq X).$$

(2) *If moreover l is bounded away from 0 and ∞ on every compact subset of $[0,+\infty[$, then for every $\xi>0$, there exists $A'=A'(\xi)>1$ such that*

$$\frac{l(y)}{l(x)}\leq A'\max\left\{\left(\frac{y}{x}\right)^{\xi},\left(\frac{y}{x}\right)^{-\xi}\right\}\qquad(x>0, y>0).$$

We now introduce certain generalized fractional derivative transforms which play a central role in the sequel.

For any $\gamma \in]0, \beta[$ and $g \in \mathscr{C}^\beta \cap L^1(\mathbb{R})$, we define

$$K_{\pm}^{l,\gamma} g(x) := \frac{1}{\Gamma(-\gamma)} \int_0^{+\infty} l(y) \frac{g(x \pm y) - g(x)}{y^{1+\gamma}} dy.$$

Note that $l(x) = o(x^\beta)$ as $x \to +\infty$ for any $\beta > 0$ (see Bingham et al. [6], Prop. 1.3.6), so when $\gamma > 0$, $\int_1^{+\infty} \frac{l(y)}{y^{1+\gamma}} < +\infty$. Consequently, if $g \in \mathscr{C}^\beta \cap L^1(\mathbb{R})$ for some $\gamma \in]0, \beta[$, then $K_{\pm}^{l,\gamma} g(x)$ defined bounded continuous functions.

But for $\gamma = 0$, since $\frac{1}{y}$ is not integrable at $+\infty$, $K_{\pm}^{l,0}$ is defined by

$$K_{\pm}^{l,0} g(x) := -\int_0^{+\infty} l(y) \frac{g(x \pm y) - 1_{]0,1[}(y) g(x)}{y} dy,$$

for any $g \in \mathscr{C}^\beta \cap L^1(\mathbb{R})$ and $\beta > 0$.

We put

$$K^{l,\gamma} := K_{+}^{l,\gamma} - K_{-}^{l,\gamma},$$

for the symmetric generalized fractional derivatives.

Remark 1.6. If we take $l \equiv 1$, we recover the definitions of fractional derivative and Hilbert transform denoted by $D_{\pm}^{\gamma}$ and $D_{\pm}^{0}$ (see Yamada [18], Samko et al. [16] and the references therein).

Following the same arguments used in the proof of Lemma 1 in Ait Ouahra and Eddahbi [3] in the case of fractional derivatives of local time of symmetric stable process, we get the following time regularities.

Lemma 1.7. (1) *Let* $0 < \gamma < \frac{\alpha-1}{2}$ *and* $K \in \{K_{\pm}^{l,\gamma}, K^{l,\gamma}\}$. *Then, for any integer* $p \geq 1$, *there exists a constant* $0 < C_p < \infty$ *such that for all* $0 \leq t, s \leq 1$ *and all* $x, y \in \mathbb{R}$,

$$\|KL(t,\cdot)(x) - KL(s,\cdot)(x)\|_{2p} \leq C_p |t-s|^{(\alpha-1)/\alpha - \gamma/\alpha}.$$

(2) *In case* $\gamma = 0$ *and under the assumption* $\int_1^{+\infty} \frac{l(y)}{y} dy < \infty$, *we get*

$$\|KL(t,\cdot)(x) - KL(s,\cdot)(x)\|_{2p} \leq C_p |t-s|^{\xi},$$

where $0 < \xi < \frac{\alpha-1}{\alpha}$.

Proof. We treat only the case $K = K_{+}^{l,\gamma}$, the other cases are similar. Here we distinguish two cases.

(1) Case $\gamma > 0$. Let $b = |t-s|^{1/\alpha}$. By the definition of $K_+^{l,\gamma}$, we have

$$\begin{aligned}
&\|K_+^{l,\gamma}L(t,\cdot)(x) - K_+^{l,\gamma}L(s,\cdot)(x)\|_{2p} \\
&\quad \le \frac{1}{|\Gamma(-\gamma)|}\int_0^b l(u)\frac{\|L(t,x+u) - L(s,x+u) - L(t,x) + L(s,x)\|_{2p}}{u^{1+\gamma}}du \\
&\qquad + \frac{1}{|\Gamma(-\gamma)|}\int_b^{+\infty} l(u)\frac{\|L(t,x+u) - L(s,x+u) - L(t,x) + L(s,x)\|_{2p}}{u^{1+\gamma}}du \\
&\quad := I_1 + I_2.
\end{aligned}$$

We estimate I_1 and I_2 separately.

Estimate of I_1:

Since l is bounded on every compact subset of $[0,+\infty[$, it follows from (3) that

$$I_1 \le C_p|t-s|^{(\alpha-1)/2\alpha}b^{(\alpha-1)/2-\gamma} \le C_p|t-s|^{(\alpha-1)/\alpha-\gamma/\alpha}.$$

Now we return to estimate I_2:

Potter's Theorem with $0 < \xi < \gamma$ implies the existence of $A(\xi) > 1$ such that

$$l(u) \le A(\xi)l(b)\left(\frac{u}{b}\right)^{\xi}.$$

Combining this fact with (1), we obtain

$$I_2 \le C_p|t-s|^{(\alpha-1)/\alpha-\gamma/\alpha}.$$

The proof of this case is done.

(2) Case $\gamma = 0$. By the definition of $K_+^{l,0}$, we have

$$\|K_+^{l,0}L(t,\cdot)(x) - K_+^{l,0}L(s,\cdot)(x)\|_{2p} \le: J_1 + J_2,$$

where

$$J_1 = \int_0^1 l(y)\frac{\|L(t,x+y) - L(t,x) - L(s,x+y) + L(s,x)\|_{2p}}{y}dy,$$

and

$$J_2 = \int_1^{+\infty} l(y)\frac{\|L(t,x+y) - L(s,x+y)\|_{2p}}{y}dy.$$

Now let $b = |t-s|^{2\xi/(\alpha-1)}$. Using (1), (4) and the fact that l is bounded on every compact subset of $[0,+\infty[$, we get

$$\begin{aligned} J_1 &\le C_p \int_0^b \frac{\|L(t,x+y)-L(t,x)\|_{2p} + \|L(s,x+y)-L(s,x)\|_{2p}}{y}\,dy \\ &\quad + C_p \int_b^1 \frac{\|L(t,x+y)-L(s,x+y)\|_{2p} + \|L(t,x)-L(s,x)\|_{2p}}{y}\,dy \\ &\le C_p \Big[\int_0^b y^{(\alpha-1)/2-1}\,dy + |t-s|^{(\alpha-1)/\alpha} \log \frac{1}{|t-s|^{2\xi/(\alpha-1)}} \Big]. \end{aligned}$$

Then

$$\begin{aligned} J_1 &\le C_p |t-s|^{\xi} \left(1 + \frac{2\xi}{(\alpha-1)\left(\frac{\alpha-1}{\alpha}-\xi\right)} \big|\, |t-s|^{(\alpha-1)/\alpha-\xi} \log |t-s|^{(\alpha-1)/\alpha-\xi} \big| \right) \\ &\le C_p |t-s|^{\xi}, \end{aligned}$$

where we have used in the last estimation the elementary inequality: for any $x \in]0,1]$, we have $|x\log(x)| \le e^{-1}$.

We now deal with J_2. Thanks to (1) and the assumption $\int_1^{+\infty} \frac{l(y)}{y}\,dy < \infty$, we obtain

$$J_2 \le C_p |t-s|^{(\alpha-1)/\alpha}.$$

which gives the desired estimate for $\gamma = 0$. □

In the same way we obtain the following space regularities.

Lemma 1.8. (1) *Let* $0 < \gamma < \frac{\alpha-1}{2}$ *and* $K \in \{K_{\pm}^{l,\gamma}, K^{l,\gamma}\}$. *Then for any integer* $p \ge 1$, *there exist a constant* $0 < C_p < \infty$ *such that for any* $0 \le t \le 1$, *all* $x, y \in [-M, M]$,

$$\|KL(t,\cdot)(x) - KL(t,\cdot)(y)\|_{2p} \le C_p t^{(\alpha-1)/2\alpha} |x-y|^{(\alpha-1)/2-\gamma}.$$

(2) *In the case* $\gamma = 0$ *and under the assumption* $\int_1^{+\infty} \frac{l(y)}{y}\,dy < \infty$, *we get*

$$\|KL(t,\cdot)(x) - KL(t,\cdot)(y)\|_{2p} \le C_p t^{(\alpha-1)/2\alpha} |x-y|^{(\alpha-1)/2}.$$

M is a finite positive constant.

Proof. We treat only the case $K = K^{l,\gamma}$, the other cases are similar. Here we distinguish two cases.

(1) Case $\gamma > 0$. Let $b = |x - y|$. By the definition of $K^{l,\gamma}$, we have

$$\begin{aligned}
&\|K^{l,\gamma}L(t,\cdot)(x) - K^{l,\gamma}L(t,\cdot)(y)\|_{2p} \\
&\quad \leq \frac{1}{|\Gamma(-\gamma)|}\int_0^b l(u) \\
&\qquad \times \frac{\|L(t,x+u) - L(t,x-u)\|_{2p} + \|L(t,y+u) - L(t,y-u)\|_{2p}}{u^{1+\gamma}}\,du \\
&\qquad + \frac{1}{|\Gamma(-\gamma)|}\int_b^{+\infty} l(u) \\
&\qquad \times \frac{\|L(t,x+u) - L(t,x-u)\|_{2p} + \|L(t,y+u) - L(t,y-u)\|_{2p}}{u^{1+\gamma}}\,du \\
&\quad := K_1 + K_2.
\end{aligned}$$

We estimate K_1 and K_2 separately.

Estimate of K_1:

Since l is bounded on every compact subset of $[0,+\infty[$, it follows from (2) that,

$$\begin{aligned}
K_1 &\leq C_p t^{(\alpha-1)/2\alpha}\int_0^b \frac{u^{(\alpha-1)/2}}{u^{1+\gamma}}\,du \\
&\leq C_p t^{(\alpha-1)/2\alpha}|x-y|^{(\alpha-1)/2-\gamma}.
\end{aligned}$$

Now we return to estimate K_2:

Potter's Theorem with $0 < \xi < \gamma$ implies the existence of $A(\xi) > 1$ such that

$$l(u) \leq A(\xi)l(b)\left(\frac{u}{b}\right)^{\xi}.$$

Combining this fact with (2), we obtain

$$K_2 \leq C_p t^{(\alpha-1)/2\alpha}|x-y|^{(\alpha-1)/2-\gamma}.$$

The proof of this case is done.

(2) Case $\gamma = 0$. Let us give the proof for $K_+^{l,0}$. The other case can be derived similarly and by linearity. By the definition of $K_+^{l,0}$, we have

$$\|K_+^{l,0}L(t,\cdot)(x) - K_+^{l,0}L(t,\cdot)(y)\|_{2p} \leq: L_1 + L_2,$$

where

$$L_1 = \int_0^1 l(u)\frac{\|L(t,x+u) - L(t,x) - L(t,y+u) + L(t,y)\|_{2p}}{u}\,du,$$

and

$$L_2 = \int_1^{+\infty} l(u) \frac{\|L(t, x+u) - L(t, y+u)\|_{2p}}{u} du.$$

Let us deal with L_1. We have

$$L_1 \le \int_0^b \frac{\|L(t, x+u) - L(t, x)\|_{2p} + \|L(t, y+u) - L(t, y)\|_{2p}}{u} du + \int_b^1 \frac{\|L(t, x+u) - L(t, x)\|_{2p} + \|L(t, y+u) - L(t, y)\|_{2p}}{u} du.$$

We consider the two cases $|x - y| > \frac{1}{e}$ and $|x - y| \le \frac{1}{e}$.

(a) Case $|x - y| > \frac{1}{e}$. Using (2) and choosing $\frac{1}{e} < b < |x - y|$, we have

$$L_1 \le C_p t^{(\alpha-1)/2\alpha} |x - y|^{(\alpha-1)/2}.$$

(b) Case $|x - y| \le \frac{1}{e}$. By choosing $0 < b < |x - y|$, (2) yields

$$L_1 \le C_p t^{(\alpha-1)/2\alpha} |x - y|^{(\alpha-1)/2}.$$

Therefore, we deduce that

$$L_1 \le C_p t^{(\alpha-1)/2\alpha} |x - y|^{(\alpha-1)/2}.$$

Now we deal with L_2. Thanks to (2) and the assumption $\int_1^{+\infty} \frac{l(y)}{y} dy < \infty$, we obtain

$$L_2 \le C_p t^{(\alpha-1)/2\alpha} |x - y|^{(\alpha-1)/2}.$$

which gives the desired estimate for $\gamma = 0$. □

As a consequence of Lemma 1.8 and the Markov property of symmetric stable processes, we get the following mixed regularities in time and space. (For more detail about proof, we refer to Ait Ouahra and Eddahbi [3], Theorem 1, for local time, and Ait Ouahra [1], p. 13, for fractional derivative of local time of symmetric stable process of index $1 < \alpha \le 2$.)

Lemma 1.9. (1) *Let* $0 < \gamma < \frac{\alpha-1}{2}$ *and* $K \in \{K_{\pm}^{l,\gamma}, K^{l,\gamma}\}$. *For any integer* $p \ge 1$, *there exists a constant* $0 < C_p < \infty$ *such that, for all* $0 \le t, s \le 1$ *and all* $x, y \in [-M, M]$,

$$\|KL(t,\cdot)(x) - KL(t,\cdot)(y) - KL(s,\cdot)(x) + KL(s,\cdot)(y)\|_{2p}$$
$$\leq C|t-s|^{(\alpha-1)/2\alpha}|x-y|^{(\alpha-1)/2-\gamma}.$$

(2) *In the case* $\gamma = 0$ *and under the assumption* $\int_1^{+\infty} \frac{l(y)}{y} dy < \infty$, *we get*

$$\|KL(t,\cdot)(x) - KL(t,\cdot)(y) - KL(s,\cdot)(x) + KL(s,\cdot)(y)\|_{2p}$$
$$\leq C|t-s|^{(\alpha-1)/2\alpha}|x-y|^{(\alpha-1)/\alpha}.$$

2. Besov spaces

We will firstly present a brief survey of Besov spaces. For more details, we refer the reader to Boufoussi [7] and Ciesielski et al. [12].

Let $I = [0,1]$. We denote by $L^p(I)$, $1 \leq p < +\infty$, the space of Lebesgue integrable real-valued functions defined on I with exponent p. The modulus of continuity of a Borel function $f : I \to \mathbb{R}$ in $L^p(I)$ norm is defined for all $h \in \mathbb{R}$ by

$$\omega_p(f,t) = \sup_{0 \leq h \leq t} \|\Delta_h f\|_p,$$

where

$$\Delta_h f(t) = \mathbf{1}_{[0,1-h]}(t)[f(t+h) - f(t)].$$

Definition 2.1. The Besov space denoted by $\mathbf{B}_{p,\infty}^{\omega_{\mu,\nu}}$, $1 \leq p < +\infty$, is a non-separable Banach space of real-valued continuous functions f on I, endowed with the norm

$$\|f\|_{p,\infty}^{\omega_{\mu,\nu}} = \|f\|_p + \sup_{0<t\leq 1} \frac{\omega_p(f,t)}{\omega_{\mu,\nu}(t)},$$

where

$$\omega_{\mu,\nu}(t) = t^{\mu}\left(1 + \log\left(\frac{1}{t}\right)\right)^{\nu},$$

for any $0 < \mu < 1$ and $\nu > 0$.

Ciesielski et al. [12] showed by using the techniques of constructive approximation of functions that Besov spaces are isomorphic to spaces of real sequences. These characterizations allows us to prove in the sequel some results of regularities of some additive functionals of local times of symmetric stable processes of index $1 < \alpha \leq 2$.

The Schauder basis on I is defined by

$$\begin{cases} \varphi_0(t) = \mathbf{1}_{[0,1]}(t), \qquad \varphi_1(t) = t\mathbf{1}_{[0,1]}(t), \\ n = 2^j + k, \qquad j \geq 0,\, k = 1, \dots, 2^j, \\ \varphi_{j,k}(t) = \varphi_n(t) = 2^{1-j/2}\Phi(2^j t - k), \end{cases}$$

where $\Phi(u) = u\mathbf{1}_{[0,1/2]}(u) + (1-u)\mathbf{1}_{]1/2,1]}(u)$.

In this basis, the decomposition and the coefficients of continuous functions f on I are respectively given by

$$f(t) = \sum_{n=0}^{\infty} C_n(f)\varphi_n(t).$$

and

$$\begin{cases} C_0(f) = f(0), \qquad C_1(f) = f(1) - f(0), \\ n = 2^j + k, \qquad j \geq 0,\, k = 1, \dots, 2^j, \\ C_n(f) = 2^{j/2}\big(2f\big(\frac{2k-1}{2^{j+1}}\big) - f\big(\frac{2k-2}{2^{j+1}}\big) - f\big(\frac{2k}{2^{j+1}}\big)\big). \end{cases}$$

We consider the separable Banach subspace of $\mathbf{B}_{p,\infty}^{\omega_{\mu,\nu}}$, $1 \leq p < +\infty$, defined by

$$\mathbf{B}_{p,\infty}^{\omega_{\mu,\nu},0} = \big\{f \in \mathbf{B}_{p,\infty}^{\omega_{\mu,\nu}} \,|\, \omega_p(f,t) = o\big(\omega_{\mu,\nu}(t)\big)\ (t \downarrow 0)\big\}.$$

The following characterization theorem is due to Ciesielski et al. [12], Theorem III.2.

Theorem 2.2. *The subspace* $\mathbf{B}_{p,\infty}^{\omega_{\mu,\nu},0}$, $1 \leq p < +\infty$, *corresponds to the sequences* $\big(C_n(f)\big)_n$ *such that*

$$\lim_{j\to+\infty} \frac{2^{-j(1/2-\mu+1/p)}}{(1+j)^{\nu}} \Big[\sum_{n=2^j+1}^{2^{j+1}} |C_n(f)|^p\Big]^{1/p} = 0.$$

For the proof of our results, we need the following tightness criterion in the subspace $\mathbf{B}_{p,\infty}^{\omega_{\mu,\nu},0}$, $2 \leq p < +\infty$ (see Ait Ouahra et al. [2], Lemma 4.3).

Theorem 2.3. *Let* $\{X_t^n \,|\, t \in [0,1]\}_{n\geq 1}$ *be a sequence of stochastic processes satisfying:*

(1) $X_0^n = 0$ *for all* $n \geq 1$.

(2) *For all* $2 \leq p < +\infty$, *there exists a constant* $0 < C_p < +\infty$ *such that*

$$\mathbb{E}|X_t^n - X_s^n|^p \leq C_p|t-s|^{p\mu} \quad \textit{for all } t, s \in [0,1],$$

where $0 < \mu < 1$. *Then the sequence* $\{X_t^n \mid t \in [0,1]\}$ *is tight in* $\mathbf{B}_{p,\infty}^{\omega_{\mu,\nu},0}$ *for all* $\nu > 0$ *and* $p > \max(\mu^{-1}, \nu^{-1})$.

Following the same arguments used in the proof of Lemma 9 in Ait Ouahra et al. [4], we have

Lemma 2.4. (1) *Let* $0 < \gamma < \frac{\alpha-1}{2}$ *and* $K \in \{K_{\pm}^{\gamma}, K^{\gamma}\}$. *The trajectory* $t \to KL(t,\cdot)(x)$ *belongs a.s. to* $\mathbf{B}_{p,\infty}^{\omega_{(\alpha-1)/\alpha-\gamma/\alpha,\nu},0}$, $1 \le p < +\infty$, *for any* $\nu > \frac{1}{p}$ *and all* $|x| \le M$.

(2) *In the case* $\gamma = 0$ *and under the assumption* $\int_1^{+\infty} \frac{l(y)}{y} dy < \infty$, *the trajectory* $t \to KL(t,\cdot)(x)$ *belongs a.s. to* $\mathbf{B}_{p,\infty}^{\omega_{\xi,\nu},0}$, $1 \le p < +\infty$, *for any* $\nu > \frac{1}{p}$ *and all* $|x| \le M$, *where* $0 < \xi < \frac{\alpha-1}{\alpha}$.

(3) *Let* $0 < \gamma < \frac{\alpha-1}{2}$, *and* $K \in \{K_{\pm}^{\gamma}, K^{\gamma}\}$. *The mapping* $x \to KL(t,\cdot)(x)$ *belongs a.s. to* $\mathbf{B}_{p,\infty}^{\omega_{(\alpha-1)/2-\gamma,\nu},0}$, $1 \le p < +\infty$, *for any* $\nu > \frac{1}{p}$ *and all* $t \in I$.

(4) *In the case* $\gamma = 0$ *and under the assumption* $\int_1^{+\infty} \frac{l(y)}{y} dy < \infty$, *the mapping* $x \to KL(t,\cdot)(x)$ *belongs a.s. to* $\mathbf{B}_{p,\infty}^{\omega_{(\alpha-1)/2,\nu},0}$, $1 \le p < +\infty$, *for any* $\nu > \frac{1}{p}$ *and all* $t \in I$, *where* M *is a positive finite constant.*

Proof. We are going to prove (1) since the other cases follow in the same manner. By Theorem 2.3, it suffices to show that a.s.

$$\lim_{j\to+\infty} \frac{2^{-j(1/2-((\alpha-1)/\alpha-\gamma/\alpha)+1/p)}}{(1+j)^{\nu}} \Big[\sum_{n=2^j+1}^{2^{j+1}} |C_n(KL(t,\cdot)(x)|^p \Big]^{1/p} = 0,$$

where

$$C_n(KL(t,\cdot)(x) = 2^{j/2}\left(2KL\left(\frac{2k-1}{2^{j+1}},\cdot\right)(x) - KL\left(\frac{2k-2}{2^{j+1}},\cdot\right)(x) - KL\left(\frac{2k}{2^{j+1}},\cdot\right)(x)\right).$$

For any $\lambda > 0$, we set

$$I = \mathbb{P}\left(\sup_{j\ge 0} \frac{2^{-j(1/2-((\alpha-1)/\alpha-\gamma/\alpha)+1/p)}}{(1+j)^{\nu}} \Big(\sum_{n=2^j+1}^{2^{j+1}} |C_n(KL(t,\cdot)(x)|^p \Big)^{1/p} > \lambda\right).$$

By Tchebychev's inequality, we have

$$I \le \frac{1}{\lambda^p} \sum_{j\ge 0} \frac{2^{-jp(1/2-((\alpha-1)/\alpha-\gamma/\alpha)+1/p)}}{(1+j)^{p\nu}} \sum_{n=2^j+1}^{2^{j+1}} \mathbb{E}|C_n(KL(t,\cdot)(x)|^p.$$

In view of the definition of $C_n(KL(t,\cdot)(x)$ and Lemma 1.7, we deduce that

$$I \le \frac{C}{\lambda^p} \sum_{j \ge 0} \frac{1}{(1+j)^{pv}} < \infty \qquad \text{for all } v > \frac{1}{p}.$$

The result is a simple application of the Borel–Cantelli lemma. □

In the following we generalize, in Besov spaces, the results obtained in the space of continuous functions by Rosen [15] in the case of symmetric stable processes of index $1 < \alpha \le 2$ and by Yor [19] in the case of Brownian motion. The fractional Brownian motion is obtained as a limit in law of linear local times of symmetric stable processes. To state this result, let $\{B_t^H(x) \mid t \ge 0, x \in \mathbb{R}\}$ denote a fractional Brownian sheet with index $H \in]0,1[$. It the continuous centered Gaussian process with covariance function

$$\mathbb{E}\big(B_t^H(x), B_s^H(y)\big) = (s \wedge t)\frac{1}{2}(|x|^H + |y|^H - |x-y|^H).$$

Let $p_t(x,y)$ be the transition probability density for the symmetric stable processes and write $p_t(0, x-y) = p_t(x-y) = p_t(|x-y|)$. The α-potential density is defined by

$$u^\alpha(x) = \int_0^{+\infty} e^{-\alpha t} p_t(x)\, dt.$$

Theorem 2.5. *Let ζ an independent exponential random variable of mean* 1. *Then as $\varepsilon \to 0$ the sequence of processes*

$$\left\{ \frac{1}{\varepsilon^{(\alpha-1)/2}} \big(L(\zeta, \varepsilon x) - L(\zeta, 0)\big) \mid x \in \mathbb{R} \right\},$$

converges in law to the process

$$\{2\sqrt{c_\alpha u^1(0)} B_\zeta^{\alpha-1}(x) \mid x \in \mathbb{R}\}$$

in the Besov space $\mathbf{B}_{p,\infty}^{\omega_{(\alpha-1)/2,v},0}$, $2 \le p < +\infty$, for all $v > \frac{1}{p}$. $B^{\alpha-1}$ is independent of ζ, where

$$c_\alpha = \int_0^{+\infty} \left(p_1(0) - p_1\left(\frac{1}{s^{1/\alpha}}\right) \right) \frac{ds}{s^{1/\alpha}}$$

and

$$u^1(0) = \frac{1}{2\pi}\int \frac{dp}{1+|p|^\alpha}$$

is the 1*-potential at* 0*.*

Proof. By Theorem 1.3 in Rosen [15], we have the convergence of the finite-dimensional distributions. It remains to show tightness.

By virtue of (4), for any $1 \le p < +\infty$, we obtain

$$\begin{aligned}\mathbb{E}[L(\zeta, \varepsilon x) - L(\zeta, \varepsilon y)]^{2p} &= \int_0^{+\infty} e^{-s}\mathbb{E}[L(s, \varepsilon x) - L(s, \varepsilon y)]^{2p}\, ds \\ &\le C_p(\varepsilon|x-y|)^{2p((\alpha-1)/2)}.\end{aligned}$$

This together with Theorem 2.3 completes the proof of Theorem 2.5. □

3. Anisotropic Besov spaces

Now we denote by $L^p(I^2)$ the space of Lebesgue integrable functions with exponent p $(1 \le p < \infty)$. For any function $f : I^2 \to \mathbb{R}$, any $h \in \mathbb{R}$, the progressive difference in direction x_1 (resp. x_2), is defined by

$$\begin{aligned}\Delta_{h,1} f(x_1, x_2) &= f(x_1 + h, x_2) - f(x_1, x_2), \\ \Delta_{h,2} f(x_1, x_2) &= f(x_1, x_2 + h) - f(x_1, x_2).\end{aligned}$$

For any $(h_1, h_2) \in \mathbb{R}^2$, we set

$$\begin{aligned}\Delta_{h_1,h_2} f &= \Delta_{h_1,1} \circ \Delta_{h_2,2} f, \\ \Delta^2_{h,i} f &= \Delta_{h,i} \circ \Delta_{h,i} f, \qquad i = 1, 2.\end{aligned}$$

For any Borel function $f : I^2 \to \mathbb{R}$ such that $f \in L^p(I^2)$, one can measure its smoothness by its modulus of continuity computed in $L^p(I^2)$ norm.

To this end let us define, for any $t \in I$ and $(t_1, t_2) \in I^2$,

$$\begin{aligned}\omega_{(1,0)\cdot p}(f, t_1) &= \sup_{|h_1| \le t_1} \|\Delta_{h_1,1} f\|_p, \\ \omega_{(0,1)\cdot p}(f, t_2) &= \sup_{|h_2| \le t_2} \|\Delta_{h_2,2} f\|_p, \\ \omega_{(1,1)\cdot p}(f, t_1, t_2) &= \sup_{|h_1| \le t_1, |h_2| \le t_2} \|\Delta_{h_1,h_2} f\|_p.\end{aligned}$$

Definition 3.1. Let $0 < \alpha_1, \alpha_2 < 1$ and $\nu \in \mathbb{R}$. The anisotropic Besov space, denoted by $\mathrm{Lip}_p(\alpha_1, \alpha_2, \nu)$, $1 \leq p < +\infty$, is a non-separable Banach space of real-valued continuous functions f on I^2, endowed with the norm

$$\|f\|_{\beta}^{\omega_\nu^{\alpha_1,\alpha_2}} := \|f\|_p + \sup_{0<t_1\leq 1} \frac{\omega_{(1,0).p}(f, t_1)}{\omega_\nu^{\alpha_1,\alpha_2}(t_1, 1)} + \sup_{0<t_2\leq 1} \frac{\omega_{(1,0).p}(f, t_2)}{\omega_\nu^{\alpha_1,\alpha_2}(1, t_2)} + \sup_{0<t_1,t_2\leq 1} \frac{\omega_{(1,1).p}(f, t_1, t_2)}{\omega_\nu^{\alpha_1,\alpha_2}(t_1, t_2)},$$

where

$$\omega_\nu^{\alpha_1,\alpha_2}(t_1, t_2) = t_1^{\alpha_1} t_2^{\alpha_2}\left(1 + \log\left(\frac{1}{t_1 t_2}\right)\right)^\nu.$$

We consider the separable Banach subspace of $\mathrm{Lip}_p(\alpha_1, \alpha_2, \nu)$, $1 \leq p < +\infty$, defined by

$$\begin{aligned}\mathrm{Lip}_p^*(\alpha_1, \alpha_2, \nu) := \big\{f \in \mathrm{Lip}_p(\alpha_1, \alpha_2, \beta) \,|\, &\omega_{(1,0)\cdot p}(f, t_1) = o\big(\omega_\nu^{\alpha_1,\alpha_2}(t_1, 1)\big) \text{ as } t_1 \to 0,\\ &\omega_{(0,1).p}(f, t_2) = o\big(\omega_\nu^{\alpha_1,\alpha_2}(1, t_2)\big) \text{ as } t_2 \to 0,\\ &\omega_{(1,1).p}(f, t_1, t_2) = o\big(\omega_\nu^{\alpha_1,\alpha_2}(t_1, t_2)\big) \text{ as } t_1 \wedge t_2 \to 0\big\},\end{aligned}$$

where $t_1 \wedge t_2 := \min(t_1, t_2)$.

Now, for any continuous functions f on I^2, we have the decomposition

$$f(t_1, t_2) = \sum_{m=0}^{\infty} \sum_{\max(n,n')=m} C_{n,n'}(f)\varphi_n \otimes \varphi_{n'}(t_1, t_2),$$

where $C_{n,n'}(f) = C_n^1 \circ C_n^2(f)$ with

$$\begin{cases} C_n^1(f)(t) = C_n\big(f(.,t)\big),\\ C_n^2(f)(t) = C_n\big(f(t,\cdot)\big).\end{cases}$$

In order to state our main results, we need the following characterization theorem. (See Kamont [13], Theorem A.2, who described anisotropic Besov spaces in terms of the coefficients of the expansion of a continuous function with respect to a basis which consists of tensor products of Schauder functions.)

Theorem 3.2. *The subspace* $\mathrm{Lip}_p^*(\alpha_1, \alpha_2, v)$, $1 \le p < +\infty$, *corresponds to the sequences* $\big(C_{n,n'}(f)\big)$ *such that*

$$\lim_{j\to+\infty} \frac{2^{-j(1/2-\alpha_1+1/p)}}{(1+j)^v} \Big[\sum_{n=2^j+1}^{2^{j+1}} |C_{n,l'}(f)|^p\Big]^{1/p} = 0, \qquad l' = 0, 1,$$

$$\lim_{j\to+\infty} \frac{2^{-j(1/2-\alpha_2+1/p)}}{(1+j)^v} \Big[\sum_{n=2^j+1}^{2^{j+1}} |C_{l,n}(f)|^p\Big]^{1/p} = 0, \qquad l = 0, 1,$$

$$\lim_{j,j'\to+\infty} \frac{2^{-j(1/2-\alpha_1+1/p)}2^{-j'(1/2-\alpha_2+1/p)}}{(1+j+j')^v} \Big[\sum_{n=2^j+1}^{2^{j+1}} \sum_{n'=2^{j'}+1}^{2^{j'+1}} |C_{n,n'}(f)|^p\Big]^{1/p} = 0.$$

The first result of this section is the following.

Theorem 3.3. (1) *The trajectory* $(t, x) \to L(t, x)$ *belongs a.s. to anisotropic Besov space* $\mathrm{Lip}_p^*\big(\frac{\alpha-1}{2\alpha}, \frac{\alpha-1}{2}, v\big)$, $1 \le p < +\infty$, *for any* $v > \frac{1}{p}$.

(2) *Let* $0 < \gamma < \frac{\alpha-1}{2}$, *and* $K \in \{K_{\pm}^{\gamma}, K^{\gamma}\}$. *The trajectory* $(t, x) \to KL(t, \cdot)(x)$ *belongs a.s. to* $\mathrm{Lip}_p^*\big(\frac{\alpha-1}{2\alpha}, \frac{\alpha-1}{2} - \gamma, v\big)$, $1 \le p < +\infty$, *for any* $v > \frac{1}{p}$.

(3) *In the case* $\gamma = 0$ *and under the assumption* $\int_1^{+\infty} \frac{l(y)}{y} dy < \infty$, *the trajectory* $(t, x) \to KL(t, \cdot)(x)$ *belongs a.s. to* $\mathrm{Lip}_p^*\big(\frac{\alpha-1}{2\alpha}, \frac{\alpha-1}{2}, v\big)$, $1 \le p < +\infty$, *for any* $v > \frac{1}{p}$.

Proof. We are going to prove (1) since the other cases follow in the same manner. Notice that a.s., for all $x \in \mathbb{R}$, $L(0, x) = 0$, thus $C_{0,n}(L) = 0$. Therefore by Theorem 3.2, it suffices to show that

$$\lim_{j\to+\infty} \frac{2^{-j(1/2-(\alpha-1)/2+1/p)}}{(1+j)^{\beta}} \Big[\sum_{n=2^j+1}^{2^{j+1}} |C_{1,n}(L)|^p\Big]^{1/p} = 0,$$

$$\lim_{j\to+\infty} \frac{2^{-j(1/2-(\alpha-1)/2\alpha+1/p)}}{(1+j)^v} \Big[\sum_{n=2^j+1}^{2^{j+1}} |C_{n,0}(L)|^p\Big]^{1/p} = 0,$$

$$\lim_{j\to+\infty} \frac{2^{-j(1/2-(\alpha-1)/2\alpha+1/p)}}{(1+j)^v} \Big[\sum_{n=2^j+1}^{2^{j+1}} |C_{n,1}(L)|^p\Big]^{1/p} = 0,$$

$$\lim_{j,j'\to+\infty} \frac{2^{-j(1/2-(\alpha-1)/2\alpha+1/p)}2^{-j'(1/2-(\alpha-1)/2+1/p)}}{(1+j+j')^v} \Big[\sum_{n=2^j+1}^{2^{j+1}} \sum_{n'=2^{j'}+1}^{2^{j'+1}} |C_{n,n'}(L)|^p\Big]^{1/p} = 0.$$

where

$$C_{1,n}(L) = 2^{j/2}\left[2L\left(1,\frac{2k'-1}{2^{j'}}\right) - L\left(1,\frac{2k'-2}{2^{j'}}\right) - L\left(1,\frac{2k'}{2^{j'}}\right)\right],$$

$$C_{n,0}(L) = 2^{j/2}\left[2L\left(\frac{2k-1}{2^{j}},0\right) - L\left(\frac{2k-2}{2^{j}},0\right) - L\left(\frac{2k}{2^{j}},0\right)\right],$$

$$C_{n,1}(L) = 2^{j/2}\left[2L\left(\frac{2k-1}{2^{j}},1\right) - L\left(\frac{2k-2}{2^{j}},1\right) - L\left(\frac{2k}{2^{j}},1\right)\right].$$

The first three inequalities follow immediately by the same arguments used in proof of Lemma 2.4. We will now prove the last inequality. We write

$$\begin{aligned}
&2^{-((j+j')/2)}C_{n,n'}(L)\\
&\quad= \Delta^2_{1/2^{j+1},1}\circ\Delta^2_{1/2^{j+1},2}L\left(\frac{2k-2}{2^{j+1}},\frac{2k'-2}{2^{j'+1}}\right)\\
&\quad= 4L\left(\frac{2k-1}{2^{j+1}},\frac{2k'-1}{2^{j'+1}}\right) - 2L\left(\frac{2k-1}{2^{j+1}},\frac{2k'-2}{2^{j'+1}}\right) - 2L\left(\frac{2k}{2^{j+1}},\frac{2k'-1}{2^{j'+1}}\right)\\
&\qquad - 2L\left(\frac{2k-1}{2^{j+1}},\frac{2k'}{2^{j'+1}}\right) - 2L\left(\frac{2k-2}{2^{j+1}},\frac{2k'-1}{2^{j'+1}}\right) + L\left(\frac{2k}{2^{j+1}},\frac{2k'-2}{2^{j'+1}}\right)\\
&\qquad + L\left(\frac{2k}{2^{j+1}},\frac{2k'}{2^{j'+1}}\right) + L\left(\frac{2k-2}{2^{j+1}},\frac{2k'}{2^{j'+1}}\right) + f\left(\frac{2k-2}{2^{j+1}},\frac{2k'-2}{2^{j'+1}}\right)\\
&\quad:= 2E_{n,n'}(L) + 2F_{n,n'}(L) + G_{n,n'}(L),
\end{aligned}$$

where

$$\begin{aligned}
E_{n,n'}(L) &= L\left(\frac{2k-1}{2^{j+1}},\frac{2k'-1}{2^{j'+1}}\right) - L\left(\frac{2k-1}{2^{j+1}},\frac{2k'-2}{2^{j'+1}}\right)\\
&\quad - L\left(\frac{2k}{2^{j+1}},\frac{2k'-1}{2^{j'+1}}\right) + L\left(\frac{2k}{2^{j+1}},\frac{2k'-2}{2^{j'+1}}\right),\\
F_{n,n'}(L) &= L\left(\frac{2k-1}{2^{j+1}},\frac{2k'-1}{2^{j'+1}}\right) - L\left(\frac{2k-1}{2^{j+1}},\frac{2k'}{2^{j'+1}}\right)\\
&\quad - L\left(\frac{2k-2}{2^{j+1}},\frac{2k'-1}{2^{j'+1}}\right) + L\left(\frac{2k-2}{2^{j+1}},\frac{2k'}{2^{j'+1}}\right),\\
G_{n,n'}(L) &= L\left(\frac{2k}{2^{j+1}},\frac{2k'}{2^{j'+1}}\right) - L\left(\frac{2k}{2^{j+1}},\frac{2k'-2}{2^{j'+1}}\right)\\
&\quad - L\left(\frac{2k-2}{2^{j+1}},\frac{2k'}{2^{j'+1}}\right) + L\left(\frac{2k-2}{2^{j+1}},\frac{2k'-2}{2^{j'+1}}\right).
\end{aligned}$$

It suffices then to show that

$$\lim_{j,j'\to+\infty}\frac{2^{-j(-((\alpha-1)/2\alpha)+1/p)}2^{-j'(-((\alpha-1)/2)+1/p)}}{(1+j+j')^{v}}\Big[\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}}|E_{n,n'}(L)|^p\Big]^{1/p}=0,$$

$$\lim_{j,j'\to+\infty}\frac{2^{-j(-((\alpha-1)/2\alpha)+1/p)}2^{-j'(-((\alpha-1)/2)+1/p)}}{(1+j+j')^{v}}\Big[\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}}|F_{n,n'}(L)|^p\Big]^{1/p}=0,$$

$$\lim_{j,j'\to+\infty}\frac{2^{-j(-((\alpha-1)/2\alpha)+1/p)}2^{-j'(-((\alpha-1)/2)+1/p)}}{(1+j+j')^{v}}\Big[\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}}|G_{n,n'}(L)|^p\Big]^{1/p}=0.$$

Let us for example give the proof of the first equality.

For any $\lambda > 0$, we set

$$I=\mathbb{P}\bigg(\sup_{j\geq 0}\sup_{j'\geq 0}\frac{2^{-j(-((\alpha-1)/2\alpha)+1/p)}2^{-j'(-((\alpha-1)/2)+1/p)}}{(1+j+j')^{v}}\times\Big[\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}}|E_{n,n'}(L)|^p\Big]^{1/p}>\lambda\bigg).$$

By Tchebychev's inequality, we have

$$I\leq\frac{1}{\lambda^p}\sum_{j\geq 0}\sum_{j'\geq 0}\frac{2^{-j(-((\alpha-1)/2\alpha)+1/p)}2^{-j'(-((\alpha-1)/2)+1/p)}}{(1+j+j')^{v}}\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}}\mathbb{E}|E_{n,n'}(L)|^p.$$

In view of the definition of $E_{n,n'}(L)$ and (3), we deduce that

$$I\leq\frac{C}{\lambda^p}\sum_{j\geq 0}\sum_{j'\geq 0}\frac{1}{(1+j+j')^{pv}}<\infty\qquad\text{for all } v>\frac{1}{p}.$$

The result is a simple application of the Borel–Cantelli lemma. This completes the proof of Theorem 3.3. □

For the proof of the next theorem, we need the following tightness criterion in the subspace $\mathrm{Lip}_p^*(\alpha_1,\alpha_2,v)$, $2\leq p<+\infty$ (see Boufoussi and Lakhel [9], Lemma 2.5).

Theorem 3.4. *Let* $\{X^n_{s,t}\,|\,(s,t)\in[0,1]^2\}_{n\geq 1}$ *be a sequence of random fields satisfying:*

(1) $X^n_{\cdot,0}=X^n_{0,\cdot}=x$ *for some* $\in\mathbb{R}$.

(2) *For all* $2\leq p<+\infty$, *there exists a constant* $0<C_p<+\infty$ *such that*

$$\mathbb{E}|X^n_{s,t}-X^n_{s',t}-X^n_{s,t'}+X^n_{s',t'}|^p\leq C_p|s-s'|^{\alpha_1 p}|t-t'|^{\alpha_2 p}\qquad\textit{for all } t,s\in[0,1],$$

where $0 < \alpha_1, \alpha_2 < 1$. *Then the sequence* $\{X^n\}_{n\geq 1}$ *is tight in* $\mathrm{Lip}_p^*(\alpha_1, \alpha_2, \nu)$ *for all* $\nu > \frac{2}{p}$.

Now we are ready to state and prove our second result.

Theorem 3.5. *The sequence of processes*

$$\left\{\frac{1}{\varepsilon^{(\alpha-1)/2}}\big(L(t,\varepsilon x) - L(t,0)\big) \mid (t,x) \in [0,1]^2\right\}$$

converges in law as $\varepsilon \to 0$ *to the process*

$$\{2\sqrt{c_\alpha} B^{\alpha-1}_{L(t,0)}(x) \mid (t,x) \in [0,1]^2\}$$

in $\mathrm{Lip}_p^*\big(\frac{\alpha-1}{2\alpha}, \frac{\alpha-1}{2}, \nu\big)$, $2 \leq p < +\infty$, *for all* $\nu > \frac{2}{p}$.

Proof. By Theorem 1.2 of Rosen [15], we have the convergence of the finite-dimensional distributions. The tightness follows from (1) and Theorem 3.4. □

Remark 3.6. In Section 3, the local time $L(t,x)$ is analyzed in both its variables through the anisotropic Besov Space, in contrast to Section 2 where one is interested in the regularity in t.

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