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Outerplanarity without accumulation in the cylinder and the Möbius Band

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Abstract. In this paper we explicitly characterize the outer-embeddings without vertex accumulation points in the open cylinder and in the Möbius strip. In the first case, the list of forbidden minors consists of 11 graphs. In the second, we provide the list of 92 forbidden minors as well as the list of 182 forbidden subgraphs.

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1. Introduction

A number of research lines have been recently developed in the field of Topological Graph Theory. Some of them consider different properties over infinite graphs. In this sense, the characterization of graphs admitting embeddings with no vertex accumulation point in tubular surfaces of finite genus was given in [16], and graphs admitting embeddings in the plane without any vertex accumulation point and with all their vertices in a same face can be found in [3].

Here we face two characterizations of embeddings (in the open cylinder and the Möbius band), both of infinite, locally-finite graphs and on tubular surfaces, without accumulation and with all their vertices in one face.

Before that, we recall some necessary definitions and results. All graphs in this paper will be considered undirected and without loops or multiple edges. We will use the standard graph-theoretical terminology, as it is presented in [11], with the exception of *vertex* which will be employed instead of *point* and *edge* instead of *line*. When dealing with infinite graphs in this paper, we will use the terminology of [12], [15], [17], and we will refer to locally finite graphs with a countable vertex set, i.e., countable graphs such that the degree of any vertex is finite. The formal

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definition of an embedding for this kind of graphs in tubular surfaces can be found in [13], [14].

With regard to such tubular surfaces, we must consider that they are built from a compact surface S, of a finite genus, where n open discs are replaced by n open cylinders (i.e., a punctured compact surface, which is homeomorphic to the surface with n different points removed). So, $S(n)$ represents a non-compact surface of finite genus with n Freudenthal ends. For example, if S^2 is the sphere and P_2 is the projective plane, then $S^2(1)$ is homeomorphic to the plane, $S^2(2)$ to the open cylinder, and $P_2(1)$ to the Möbius band.

Going back to infinite graphs, it is worth mentioning that an infinite ray in a graph G is homeomorphic to the positive half-line $\mathbb{R}^+$ (see [9] to define a homeomorphism between an infinite graph and a metric space). An end of a graph defined by a ray is said to be *stable* (resp. *strongly stable*) if there exist a finite subgraph K such that every component of $G - K$ defined by the ray is a tree (resp. an infinite ray). Otherwise, the end is said to be *unstable*.

Later we will use a *decompactification* process, which consists of removing some points to make the graph non-compact (i.e., replacing an open disc from S by an open cylinder and replacing at least an edge from the graph by some infinite rays, as in Figure 1). If G is a countable graph with all its n ends strongly stable and addmiting an embedding without accumulation points in the tubular surface $S(n)$, then it is possible to obtain some graph G^* from which G is the decompactification of G^* by n points. In general, such a graph G^* is not unique, since it depends on the embedding chosen for G (moreover, it depends on the "remaining" vertices of degree two after contracting each end). Such a "compactification" is said to be the *main n-compactification* when the rays are replaced by n vertices and only one vertex of each ray remains; note that the main-n-compactification is unique for each embedding of G in $S(n)$, because all the graph ends going to the same end of the surface have to be replaced by only one vertex.

Graphs admitting an embedding in $S(n)$ with no vertex accumulation point are usually called VAP-free-$S(n)$. Clearly, if G is a minor of G' and G' is VAP-free-$S(n)$, then G is VAP-free-$S(n)$. In this way, the characterization of the

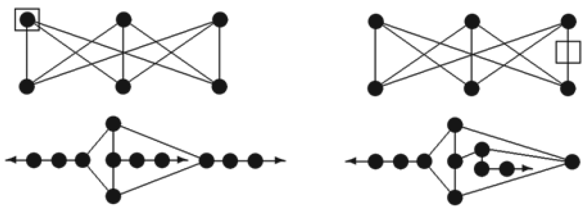

Figure 1. Decompactification of $K_{3,3}$ in a vertex (left) and in an inner point of an edge (right); note that the number of vertices represented in each ray is irrelevant.

VAP-free-$S(n)$ graphs can be given in terms of forbidden minors, and we denote by $\mathcal{K}_{\mathrm{VAP}}(S(n))$ the set of forbidden VAP-free-$S(n)$ minors. A graph G is in $\mathcal{K}_{\mathrm{VAP}}(S(n))$ if it is not VAP-free-$S(n)$ and it verifies that if H is a minor of G and G is not a minor of H then H is VAP-free-$S(n)$.

The decompactification process provides a way to characterize VAP-free-$S(n)$ graphs. In 1966 Halin [10] already characterized VAP-free-$S^2(1)$ graphs in terms of forbidden subgraphs, and the list of forbidden minors for VAP-free-$P_2(1)$-embeddability was independently obtained by Revuelta [16] and Archdeacon, Bonnington, Debowsky, and Prestidge [1].

Moreover, a generalization of these characterizations can be found in [16]. For any compact surface S, the following three results hold. First, every graph in $\mathcal{K}_{\mathrm{VAP}}(S(1))$ is strongly stable. Second, if W is a non-empty set of vertices of G, then G admits an embedding in S such that all the vertices of W are in the same face if and only if G_W is VAP-free-$S(1)$, where G_W is the strongly stable graph built from G with one infinite ray starting from each vertex of W. Finally, $\mathcal{K}_{\mathrm{VAP}}(S(1))$ is composed by all the G_W such that G is a minimal element (in the minor ordering) regarding the W-S-embedddability (see [16] for more details).

The last results can be generalized in several ways for the general tubular surface $S(n)$; some of these generalizations are easy to prove and will be applied further ahead, when dealing with p-outer-$S(n)$ embeddings (i.e., of a locally finite graph which admits an embedding in $S(n)$ without vertex accumulation points and with all its vertices in only one face). Hence, every graph which does not admit a non-p-outer-$S(n)$ embedding is the decompactification of a non-outer-S graph by n points (see [5] if more details are needed). If $n = 1$, we are provided with a method to obtain lists of forbidden minors (and even lists of forbidden subgraphs). In fact, forbidden p-outer-$S(1)$ minors can be immediately generated when forbidden outer-S subgraphs are known. The difference with respect to the process of obtaining forbidden VAP-free graphs from the obstruction lists for finite graphs is that there may appear redundant graphs, which have to be removed from the final list (see [8] if more details are needed).

Let us call $\mathcal{K}_p(S(n))$ the set of forbidden p-outer-$S(n)$ minors. So, $\mathcal{K}_p(S^2(1))$ (forbidden p-outerplanar minors) was obtained (as well as the list of forbidden subgraphs) in [3], by applying the decompactification process to the classic list of forbidden outerplanar graphs given in [7]. This set is represented in Figure 2. Next we will deal with two particular cases: the open cylinder ($S = S^2$ and $n = 2$) and the Möbius band ($S = P_2$ and $n = 1$).

2. Outer-embeddings in the open cylinder

The next generalization of the Kuratowski graph planarity criterion is the main result of this paper. Theorem 2.6 states that a countable graph with vertices of

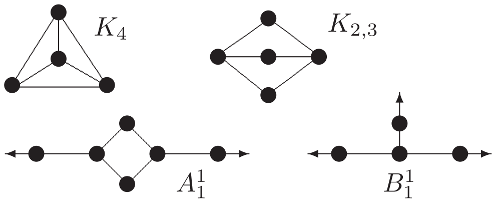

Figure 2. Forbidden p-outerplanar minors.

finite degrees can be embedded into the open annulus so that the set of vertices does not contain a convergent sequence and all the vertices are in one face if and only if the graph does not have any of 11 minors listed in its statement (Theorem 2.6 and Figures 2 and 4). In order to prove this characterization, some previous properties are required.

According to the previous comments, all ends in a forbidden p-outer-$S^2(2)$ minor are strongly stable. Moreover, each p-outer-$S^2(2)$ graph admits an $S^2(2)$-embedding without accumulation points of either vertices or edges and with all its vertices in the same face. However, the decompactification process presents some differences with respect to the case of only one tubular end. Hence, the following results are needed to characterize the p-outer-embeddings in $S^2(2)$ (i.e., p-outercylindrical), and they may have other topological applications.

Lemma 2.1. *If less than three rays are added to an outerplanar (finite) graph, then a p-outercylindrical graph is obtained.*

Proof. We start from an outerspherical embedding of an outerplanar graph G, and we choose two different points within the inner of a face where all the vertices lie. Thus, it is possible to add two rays, each of them with accumulation in exactly one of the two chosen points. The decompactification by these two points concludes the proof. □

The proofs for the following lemmas are similar to the aforementioned.

Lemma 2.2. *If less than five rays are added to a (finite) tree, then a p-outercylindrical graph is obtained.* □

Lemma 2.3. *If less than three rays are added to a same vertex in a p-outerplanar graph, then a p-outercylindrical graph is obtained.* □

Lemma 2.4. *If the graph G_1 of Figure 3 is added to a p-outerplanar graph G, by identifying a vertex of the cycle with a vertex of G, then a p-outercylindrical graph is obtained.* □

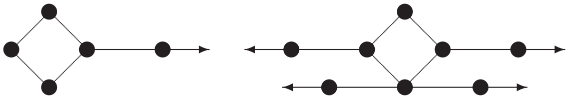

Figure 3. Two auxiliary graphs.

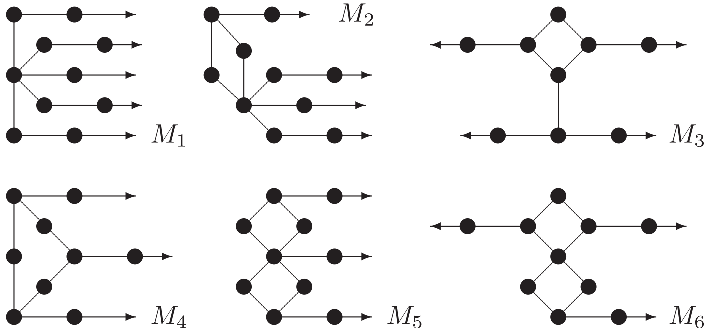

Figure 4. Some forbidden p-outer-cylindrical minors.

According to the previous definitions and results, it can be immediately inferred that a p-outer-$S(n)$ embedding implies an outerplanar embedding of its main n-compactification and conversely. So, the following result is verified:

Lemma 2.5. *A graph is p-outer-$S(n)$ if and only if its main n-compactification is outerplanar.* □

By taking into consideration these previous results as well as the graphs from Figures 2 and 4, we can prove the characterization theorem for p-outercylindrical graphs, whose formulation can be also found in [4]. Notice that $\sqcup$ stands for the disjoint union.

Theorem 2.6 ([8]). $\mathscr{K}_p(S^2(2)) = \{K_4, K_{2,3}, A_1^1 \sqcup A_1^1, A_1^1 \sqcup B_1^1, B_1^1 \sqcup B_1^1, M_i : i = 1, \dots, 6\}$.

Proof. Sufficiency. None of the eleven graphs above mentioned are p-outer-cylindrical, since their main 2-compactifications are not outerplanar. Besides, the set is minimal: none of the eleven graphs is a minor of any other.

Necessity. Let G be a non-p-outercylindrical graph. G has a non-p-outer-cylindrical minor having all its ends strongly stable. Thereby we can suppose that G has all its ends strongly stable. Moreover, if G has less than three ends, G

cannot be outerplanar not even when its rays were removed (according to Lemma 2.1). So, G has either K_4 or $K_{2,3}$ as a minor.

Let G be a non-p-outercyilindrical graph with at least three strongly stable ends. Then not all the connected components of G can be p-outerplanar. In fact, it can be supposed that G has not got any p-outerplanar connected component; therefore, if G is non-connected, then either it is not outerplanar (it has got a minor between K_4 or $K_{2,3}$) or it has got a minor among the disjoint union of two forbidden p-outerplanar minors ($A_1^1 \sqcup A_1^1$, $A_1^1 \sqcup B_1^1$, or $B_1^1 \sqcup B_1^1$). From now on, G is supposed to be connected.

If G has more than four ends, as G is connected, M_1 is a minor of G. Then, let G be a p-outerplanar graph which is not p-outercylindrical and has got three or four ends. It can be also supposed that any two cycles of G do not share more than one vertex: if they share one edge, this one can be contracted, obtaining a minor of G which can be named H; H is non-p-outercylindrical and has got less cycles sharing some elements (they can share no more elements because, in this case, it would exist a finite non-outerplanar subgraph of G).

From Lemma 2.2, there exists at least one cycle in G having between one and four cut points. We call these points *support vertices*: they are vertices of the cycle whose removing separates an infinite connected component from another in which the rest of vertices belonging to cycles remain. Here, cycles without support vertices do not affect the p-outer-$S^2(2)$-embeddability (i.e., a non-p-outer-$S^2(2)$ minor of G can be found with some support vertex in each cycle). Consequently, it can be supposed that every cycle of G has support vertices.

According to Lemma 2.3, if there are less support vertices than rays, then the resulting graph when removing two rays from the same support vertex cannot be p-outerplanar. So A_1^1 is a minor of this graph (B_1^1 is not because G would have five rays). Therefore, a minor of G is either M_2 (if the rays are added without other support vertices) or the graph G_2 from Figure 3. But G_2 is p-outercylindrical and it will be so although finite trees or cycles passing for some support vertices were added, and although some vertices were split, except in the case of the vertex of degree four, which produces the non-p-outercylindrical minor M_3. So, M_3 is a minor of G since there are less support vertices than rays and M_2 is not a minor of G.

Finally, let G be a connected and outerplanar non-p-outercylindrical graph with three or four rays corresponding to different support vertices. Indeed, it can be supposed that there are not finite rays in G and that each ray is incident with a support vertex. Note that two support vertices of a same cycle are not adjacent: if so, when the edge between them was contracted, the resulting minor would remain non-p-outercylindrical (due to the fact that the number of faces in which all vertices can be represented does not decrease), although it would have less support vertices than rays, having a minor between M_2 or M_3. According to this, if there

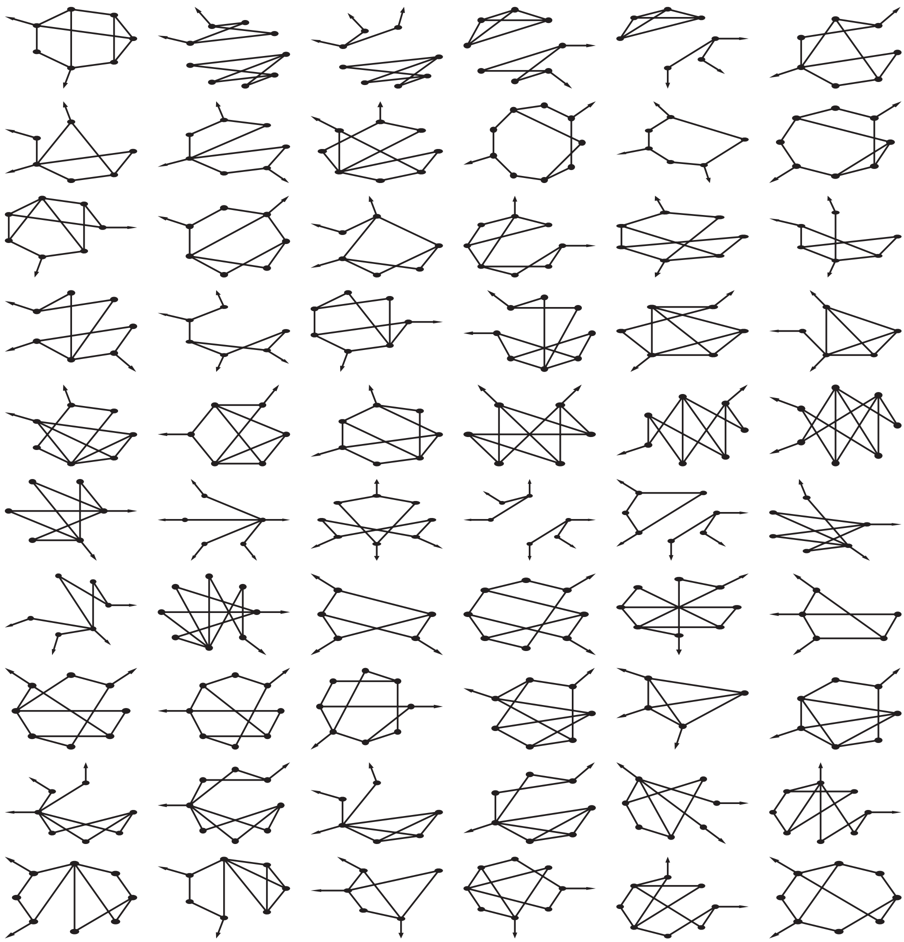

Figure 5. Infinite forbidden p-outer-$P_2(1)$ minors.

are three support vertices in a same cycle (as in the case when G has a unique cycle), M_4 is a minor of G.

If there are not three support vertices in a same cycle, then there will exist two cycles and two support vertices in these cycles such that each of these support vertices only belongs to one of the two cycles and each of these cycles only shares a vertex with other cycles. If there are two support vertices in each of both cycles, then M_5 (if there are only three support vertices) or M_3 is a minor of G. In the other case, there is a cycle which only shares a vertex with other cycles having a unique support vertex. It could be that the support vertex is not adjacent with the shared vertex (if they are adjacent, the edge between them could be removed

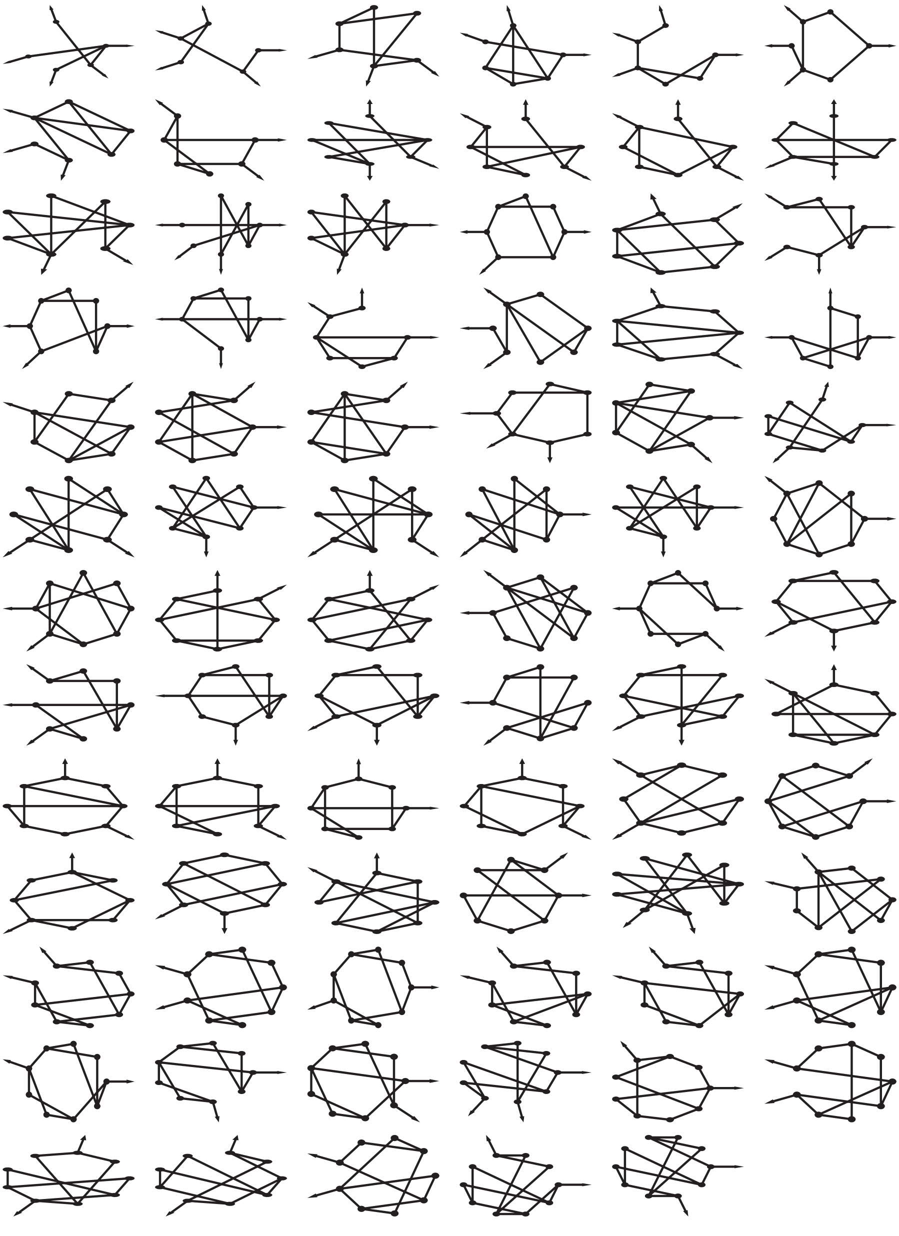

Figure 6. Remaining forbidden p-outer-$P_2(1)$ subgraphs.

and the number of cycles would be reduced, but the graph does not become p-outercylindrical), therefore (according to Lemma 2.4) a cycle (except for a vertex) and a ray can be removed in such a way that the resulting graph is not p-outerplanar. So, a minor of G is the union of A_1^1 or B_1^1 with G_1 (from Figure 3) by a vertex, and a minor of G is M_2, M_5, or M_6.

As all possible non-p-outercylindrical graphs have already been considered, the proof concludes. □

3. Outer-embeddings in the Möbius band

As there is only one end in $P_2(1)$, this case is easier than the previous one. Although the starting lists are lengthier, the characterization of outer-projective planar subgraphs (which can be found in [2], [6], [16]) provides us with the opportunity of applying a straightforward procedure. After decompactificating the forbidden outer-projective planar subgraphs (from [2]) and deleting redundant minors, the 92 forbidden p-outer-$P_2(1)$ minors are obtained: the list $\mathscr{K}_p\big(P_2(1)\big)$ is composed of the forbidden outer-projective minors plus the graphs from Figure 5. Similarly to the process followed in [6], the forbidden p-outer-$P_2(1)$ subgraphs are obtained from those 92 minors; this list of forbidden subgraphs is composed of 182 forbidden p-outer-$P_2(1)$ graphs (the forbidden outer-P_2 subgraphs and all the graphs from Figures 5 and 6).

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