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Another approach on an elliptic equation of Kirchhoff type

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Abstract. This paper is concerned with the existence of solutions to the class of nonlocal boundary value problems of the type

$$-M\Big(\int_\Omega |\nabla u|^2\Big)\Delta u = f(x,u), \qquad \text{in } \Omega,\ u = 0,\ \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ and M is a positive continuous function. By assuming that $f(x,u)$ is a Carathéodory function which growths at most $|u|^{N/(N-2)}$, $N \geq 3$, and under a suitable growth condition on M, one proves an existence result by applying the Leray–Schauder fixed point theorem.

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1. Introduction

The purpose of this article is to investigate the existence of solutions to the class of nonlocal boundary value problems of the Kirchhoff type

$$\begin{aligned} -\Big[M\Big(\int_\Omega |\nabla u|^2\Big)\Big]\Delta u &= f(x,u) \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where, through this work, $\Omega \subset \mathbb{R}^N$, is a bounded smooth domain, $M : \mathbb{R} \to \mathbb{R}$ is a continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth that satisfy some conditions which will be stated later on.

Problem (1) is called nonlocal because of the presence of the term $M(\int_\Omega |\nabla u|^2)$ which implies that the equation (1), at each point, depends on the behavior of u on the whole Ω. This phenomenon provokes some mathematical difficulties which makes the study of such a class of problem particularly interesting. Besides of this, we have its physical motivation. Indeed, the operator $M(\int_\Omega |\nabla u|^2)\Delta u$ appears in the Kirchhoff equation (see, Lions [20], p. 307), which arises in nonlinear vibrations, namely

$$\begin{aligned} u_{tt} - \left[M\left(\int_\Omega |\nabla u|^2\right)\right]\Delta u &= f(x) \quad \text{in } \Omega \times (0, L) \\ u &= 0 \quad \text{on } \partial\Omega \times (0, L) \\ u(x, 0) &= u_0(x) \\ u_t(x, 0) &= u_1(x). \end{aligned} \tag{2}$$

Such a hyperbolic equation is a general version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2\tilde{L}} \int_0^{\tilde{L}} \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{3}$$

presented by Kirchhoff [18] (see also, Carrier [7]). This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the transverse vibrations, i.e., when it is supposed only vertical component for the tension on the string of length $\tilde{L}$. The parameters in equation (3) have the following meaning: $\tilde{L}$, a constant, is the total length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density, P_0 is the initial tension and $u(x, t)$ the vertical displacement of the point x of the string, at time t. It is worth mentioning that Eq. (2) received much attention after the work of Lions [20], where a functional analysis framework was proposed to the problem. To find solutions of (2), Lions [20] considers that $M \in C^1$ ($M(\lambda)$ and $\frac{d}{d\lambda}M(\lambda)$ are continuous on $\lambda \geq 0$, see, Lions [20], Remark 1.4 and Theorem 3.1). It is an interesting open problem in mathematics (see, Lions [20], Remark 3.7), what happens when M is only assumed

$$\lambda \to M(\lambda) \text{ is continuous on } \lambda \geq 0 \text{ and satisfies } M(\lambda) \geq m_0 > 0, \tag{4}$$

with m_0 a constant?.

Some other interesting results can be found, for example, in [6], [15] and [16].

We have to point out that nonlocal problems also appear in other fields as, for example, biological systems where u describes a process which depends on the average of itself (for example, population density). See, for example, [2] and its references.

Now, a key work on nonlocal elliptic problems is the paper by Chipot and Rodrigues [10]. However, one of the first works on the Eq. (1), at least in a functional analysis setting, was given by Vasconcellos [24]. He considered the equation $N\big(x, a(x,u)\big)Lu = f$ in an unbounded domain of $\mathbb{R}^N$, by using a consequence of Browder fix point theorem, Galerkin approximations and weighted Sobolev Spaces. Motivated by this result, it was studied in Andrade and Ma [5] an operator equation of the type $M(\|A^s u\|_H^2)Au = Nu$, where $0 \leq s < 1$ and H is a Hilbert space. It corresponds to (1) when $A = -\Delta$, $s = \frac{1}{2}$ and $H = L^2(\Omega)$, and was also considered in Cousin *at all* [15] with $Nu = f$. More recently Chen, Kuo and Wu [8], investigated the multiplicity of solutions to a class of Dirichlet boundary value problems of the type (1), when $M(s) = as + b$ and $a, b > 0$ constants, by using Nehari manifold. It was studied in Martinez, Castelani, Silva and Shirabayashi [23], an equation of the type, $-M(\|u'\|_2^2)u'' = q(t)f(t,u,u')$, by using classical fixed point theorems, that is, they are compiling two theorems in order to establish strong conditions of existence for the equation described, namely Krasnoselskii's theorem and an alternative of Leray–Schauder's type. Other papers, [1], [2], [9], [11], [12], [13], [21], [22], studied nonlocal boundary value problems and unilateral problems with several applications.

As seen previously, the use of fixed point theorems, in nonlocal problems is not new, e.g., [24], [23], [13]. In Corrêa, Menezes and Ferreira [13] was proved a result on existence of positive solution for a nonlocal elliptic problem by using a result on Fixed Point Index Theory and the authors improved the results in [10], [9].

We will always work in the space $\mathbb{R}^N$, if $N \geq 3$, because in the another dimensions $N = 1$ and 2 everything follows by making standard modifications. Note that, in the case that we will consider we have $2^* = \frac{2N}{N-2}$, which is the well known critical Sobolev exponent.

The main tool in this work is the Leray–Schauder's fixed point theorem, this fixed point theorem is not usual in the literature to solve nonlocal elliptical type problems. In general, variational methods are more usual because, under appropiate assumptions on M and f, solutions of (1) can be obtained as critical points of the functional $\frac{1}{2}\tilde{M}(\|\nabla u\|_2^2) - \int_\Omega F(x,u)\,dx$, where $\tilde{M}(t) = \int_0^t M(s)\,ds$, $F(x,t) = \int_0^t f(x,s)\,ds$ (see [1] for more details).

2. The main result

The hypotheses on the functions M and f in (1) are the following.

(H1) $f(x,t)$ is a Carathéodory function such that

$$|f(x,t)| \leq b_\infty |t|^q + c_\infty.$$

where b_∞, c_∞ are positive constants, $1 < q \leq \frac{N}{N-2}$ if $N \geq 3$ and $1 < q < \infty$ if $N = 1, 2$.

(H2) $M : \mathbb{R} \to \mathbb{R}$ is a continuous monotonous nondecreasing function such that, there exists $a_0 > 0$ satisfying $M(t) \geq a_0$, for all $t \geq 0$ and

$$\lim_{t \to +\infty} \frac{M(t)}{t^{N/(N-2)}} = d,$$

where $d \in]0, +\infty]$.

Remark 2.1. We note that the hypotheses (H1)–(H2) does not include the case $M(t) = at + b$, with $a, b > 0$ constants. Really, if $N \geq 3$, we have that $\frac{N}{N-2} > 1$ and $\lim_{t \to +\infty} \frac{at+b}{t^{N/(N-2)}} = 0$.

Remark 2.2. When $N \geq 5$, the hypothesis to the limit in (H2) can be substituted by

$$\lim_{t \to +\infty} \frac{M(t)}{t} = d,$$

where $d \in]0, +\infty]$, and we can consider the most common of M's, which is behind the physical motivation of the problem: $M(s) = as + b$ with $a, b > 0$.

Recall that $u \in H_0^1(\Omega)$ is a weak solution of (1) if

$$M(\|u\|_{1,2}^2) \int_\Omega \nabla u \nabla v \, dx = \int_\Omega f(x, u) v \, dx, \qquad \forall v \in H_0^1(\Omega),$$

where $\| \cdot \|_{1,2}$ is the usual norm

$$\|u\|_{1,2}^2 = \int_\Omega |\nabla u|^2 \, dx$$

in the Sobolev space $H_0^1(\Omega)$.

The main result of this paper is:

Theorem 2.3. *Assume that conditions* (H1)–(H2) *hold. Then, there exists a weak solution in* $H_0^1(\Omega) \cap W^{2,2^*/q}(\Omega)$ *for* (1).

In [21], Theorem 1, using the Galerkin's method, the author proved the existence of weak solution, in $H_0^1(\Omega)$, to problem (1) assuming that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x, s)| \leq C(1 + |s|^q) \qquad \forall x \in \Omega, \ \forall s \in \mathbb{R},$$

where $C > 0$, $1 < q < (N+2)/(N-2)$ if $N \geq 3$ and $1 < q < \infty$ if $N = 1, 2$ and exists $m_0 > 0$ such that $M(s) \geq m_0$ $\forall s \geq 0$. But an additional hypothesis f is considered, this is, there are constants $a, b > 0$ such that

$$f(x,s)s \leq a|s|^2 + b|s| \qquad \forall x \in \Omega,\ \forall s \in \mathbb{R} \tag{5}$$

with

$$a < m_0 \lambda_1,$$

where λ_1 is the principal eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

In [2], Theorem 3, using variational methods, the authors proved the existence of positive weak solution, in $H_0^1(\Omega)$, to problem (1) assuming that $f \in C(\overline{\Omega} \times \mathbb{R})$ is a locally Lipschitz function satisfying

$$|f(x,s)| \leq C(1 + |s|^q) \qquad \forall x \in \Omega,\ \forall s \in \mathbb{R},$$

where $C > 0$, $1 < q < (N+2)/(N-2)$ if $N \geq 3$ and $1 < q < \infty$ if $N = 1, 2$,

$$f(x,t) = o(t) \qquad (\text{as } t \to 0)$$

and, for some $\mu > 2$ and $R > 0$,

$$0 < \mu F(x,t) \leq f(x,t)t, \qquad \forall |t| > R,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$.

Example 2.4. Assuming $N = 3$ and functions $f(x,s) = 1 + |s|^3$ and $M(t) = a + bt^r$, with $a, b > 0$ constants, $t \geq 0$ and $r \geq 3$. We have that f and M satisfy the assumptions (H1)–(H2) because, $\lim_{t \to +\infty} \frac{M(t)}{t^3} = d$, where $d = b$ if $r = 3$ or $d = +\infty$ if $r > 3$.

Still, f does not satisfy (5) for $t > 0$ large enough. So, f does not satisfy the hypotheses on [21], Theorem 1. Also, f does not satisfy the assumptions of [2], Theorem 3 since, we don't have that $f(x,t) = o(t)$.

In this work using the Leray–Schauder's fixed point theorem and imposing some additional assumptions on M, we will prove an existence result for (1) and this result is different and complements early works.

3. Preliminary results

In order to make this presentation as self-contained as possible we introduce the result about the Leray–Schauder's fixed point. For more details see Friedman [17], pp. 189, Theorem 3.

Theorem 3.1 (Leray–Schauder). *Consider a mapping T goes from $X \times [a,b]$ to X, where $a, b \in \mathbb{R}$ and X is a Banach space. Assume that:*

(a) *For any fixed k, $T(x,k)$ is a compact transformation, i.e., it is continuous and maps bounded sets into relatively compacts sets.*

(b) *For x in bounded sets of X, $T(x,k)$ is uniformly continuous in k, i.e., for any bounded set $X_0 \subset X$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in X_0$, $|k_1 - k_2| < \delta$, $a \le k_1, k_2 \le b$, then $\|T(x,k_1) - T(x,k_2)\| < \varepsilon$.*

(c) *There exists a (finite) constant R such that every possible solution x of $x - T(x,k) = 0$ ($x \in X$, $k \in [a,b]$) satisfies: $\|x\| \le R$.*

(d) *The equation $x - T(x,a) = 0$ has a unique solution in X.*

Then there exists a solution of the equation $x - T(x,b) = 0$.

Lemma 3.2. *Assume that condition* (H1) *hold. Then, if $\phi \in H_0^1(\Omega)$, then $\frac{f(x,\phi)}{M(\|\phi\|_{1,2}^2)} \in L^{2^*/q}(\Omega)$.*

Proof. Suppose $\phi \in H_0^1(\Omega)$, it follows from (H1) that

$$\int_\Omega \frac{|f(x,\phi)|^{2^*/q}}{[M(\|\phi\|_{1,2}^2)]^{2^*/q}} dx \le \frac{Cb_\infty^{2^*/q}}{a_0^{2^*/q}} \|\phi\|_{L^{2^*}(\Omega)}^{2^*} + \frac{Cc_\infty^{2^*/q}}{a_0^{2^*/q}} < \infty.$$

Therefore, $\frac{f(x,\phi)}{M(\|\phi\|_{1,2}^2)} \in L^{2^*/q}(\Omega)$.

For $\phi \in H_0^1(\Omega)$ and $0 \le \lambda \le 1$, we consider the following problem

$$\begin{aligned} -\left[M\left(\int_\Omega |\nabla\phi|^2\right)\right]\Delta u &= \lambda f(x,\phi) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6}$$

Note that finding a weak solution of the problem (6) in $H_0^1(\Omega)$ is equivalent to finding a weak solution of the following modified problem

$$\begin{aligned} -\Delta u &= \lambda \frac{f(x,\phi)}{M(\|\phi\|_{1,2}^2)} \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{7}$$

From Lemma 3.2, for each $\phi \in H_0^1(\Omega)$ and $\lambda \in [0,1]$, $\lambda \frac{f(x,\phi)}{M(\|\phi\|_{1,2}^2)} \in L^{2^*/q}(\Omega)$. As by (H1), $\frac{2^*}{q} \ge 2$, it follows from elliptic regularity that for every $\phi \in H_0^1(\Omega)$ and $\lambda \in [0,1]$, the problem (6) has a unique solution

$$u \in H_0^1(\Omega) \cap W^{2,2^*/q}(\Omega), \tag{8}$$

see, A. Ambrosetti and G. Prodi [4], Theorem 05.

The application $T : H_0^1(\Omega) \times [0,1] \to H_0^1(\Omega)$ defined by,

$$L(\phi, \lambda) = u \tag{9}$$

is well defined just because, for each ϕ and λ, u is in $H_0^1(\Omega)$ and unique.

Lemma 3.3. *The mapping $T : H_0^1(\Omega) \times [0,1] \to H_0^1(\Omega)$ has the following properties:*

(i) *$T(\cdot, \lambda) : H_0^1(\Omega) \to H_0^1(\Omega)$ is compact for every $\lambda \in [0,1]$, i.e., it is continuous and maps bounded sets into relatively compacts sets.*

(ii) *For every $\varepsilon > 0$ and every bounded set $A \subset H_0^1(\Omega)$ there exists $\delta > 0$ such that*

$$\|T(\phi, \lambda_1) - T(\phi, \lambda_2)\|_{H_0^1(\Omega)} < \varepsilon,$$

whenever $\phi \in A$ and $|\lambda_1 - \lambda_2| < \delta$.

Proof of (i). Take $(\phi_n)_{n\geq 1} \in H_0^1(\Omega)$, such that $\phi_n \to \phi$ in $H_0^1(\Omega)$ and consider $T(\phi_n, \lambda) = u_n$ and $T(\phi, \lambda) = u$, then, $u_n - u$ satisfies

$$\begin{aligned} -\Delta(u_n - u) &= \lambda\left(\frac{f(x,\phi_n)}{M(\|\phi_n\|_{1,2}^2)} - \frac{f(x,\phi)}{M(\|\phi\|_{1,2}^2)}\right) \quad \text{in } \Omega \\ (u_n - u) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{10}$$

By Lemma 3.2, elliptic regularity and continuity of immersion $W_0^{1,2^*/q}(\Omega) \cap W^{2,2^*/q}(\Omega) \hookrightarrow H_0^1(\Omega)$, we conclude that

$$\begin{aligned} \|T(\phi_n, \lambda) - T(\phi, \lambda)\|_{1,2} = \|u_n - u\|_{1,2} &\leq C\left\|\frac{f(x,\phi_n)}{M(\|\phi_n\|_{1,2}^2)} - \frac{f(x,\phi)}{M(\|\phi\|_{1,2}^2)}\right\|_{2^*/q} \\ &\leq C\frac{1}{M(\|\phi_n\|_{1,2}^2)}\|f(x,\phi_n) - f(x,\phi)\|_{2^*/q} \\ &\quad + C\left\|\frac{f(x,\phi)\big(M(\|\phi_n\|_{1,2}^2) - M(\|\phi\|_{1,2}^2)\big)}{M(\|\phi_n\|_{1,2}^2)M(\|\phi\|_{1,2}^2)}\right\|_{2^*/q} \\ &\leq C\frac{1}{a_0}\|f(x,\phi_n) - f(x,\phi)\|_{2^*/q} \\ &\quad + \frac{C}{a_0^2}|M(\|\phi_n\|_{1,2}^2) - M(\|\phi\|_{1,2}^2)|\,\|f(x,\phi)\|_{2^*/q}. \end{aligned}$$

It follows from (H2) (the continuity of M) that $|M(\|\phi_n\|_{1,2}^2) - M(\|\phi\|_{1,2}^2)| \to 0$. As $|f(x,s)| \leq b_\infty |s|^q + c_\infty$, it follows from (H1) and Ambrosetti and Prodi [4], Theorem 2.2 that the Nemytskii operator f is a continuous map from $L^{2^*}(\Omega)$ to $L^{2^*/q}(\Omega)$ and we have that $\|f(x,\phi_n) - f(x,\phi)\|_{2^*/q} \to 0$ consequently. By the last inequality we conclude that $T(\cdot,\lambda)$ is continuous, for each $\lambda \in [0,1]$.

To prove the compactness of $T(\cdot,\lambda)$, we note that $T(\cdot,\lambda)$ can be written as the composition of operator solution $H_0^1(\Omega) \to W_0^{1,2^*/q}(\Omega) \cap W^{2,2^*/q}(\Omega)$ with the inclusion operator $W_0^{1,2^*/q}(\Omega) \cap W^{2,2^*/q}(\Omega) \hookrightarrow H_0^1(\Omega)$, that is compact, indeed, by A. Ambrosetti and G. Prodi [4], Theorem 0.4, $W^{2,2^*/q}(\Omega) \hookrightarrow H^1(\Omega)$ is compact and by continuity of the trace operator $H^1(\Omega) \mapsto H^{1/2}(\partial\Omega)$ it follows the assertion. This proves the item (i).

Proof of (ii). We note that for each $\phi \in A$, there exists $R > 0$ such that $\|\phi\|_{L^{2^*}(\Omega)} \le R$ (from [4], Theorem 0.4 the immersion $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous). We consider $\phi \in A$, $T(\phi,\lambda_1) = u_1$, $T(\phi,\lambda_2) = u_2$, then, $u = u_1 - u_2$ satisfies

$$\begin{aligned} -\Delta u &= (\lambda_1 - \lambda_2)\frac{f(x,\phi)}{M(\|\phi\|_{1,2}^2)} \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{11}$$

By Lemma 3.2, elliptic regularity and continuity of immersion $W_0^{1,2^*/q}(\Omega) \cap W^{2,2^*/q}(\Omega) \hookrightarrow H_0^1(\Omega)$, we get

$$\begin{aligned} \|T(\phi,\lambda_1) - T(\phi,\lambda_2)\|_{1,2} = \|u_1 - u_2\|_{1,2} &\le C_1|\lambda_1 - \lambda_2| \left\| \frac{f(x,\phi)}{M(\|\phi\|_{1,2}^2)} \right\|_{L^{2^*/q}(\Omega)} \\ &\le C_1|\lambda_1 - \lambda_2| \left(C_2 \frac{b_\infty^{2^*/q}}{a_0^{2^*/q}} \|\phi\|_{L^{2^*}(\Omega)}^{2^*} + \frac{C_2 c_\infty^{2^*/q}}{a_0^{2^*/q}} \right)^{q/2^*} \\ &\le C_3|\lambda_1 - \lambda_2| \frac{(b_\infty^{2^*/q} R^{2^*} + c_\infty^{2^*/q})^{q/2^*}}{a_0} \\ &\le C_3|\lambda_1 - \lambda_2| \frac{(b_\infty R^q + c_\infty)}{a_0}. \end{aligned}$$

Choosing $\delta = \varepsilon\left[C_3 \frac{(b_\infty R^q + c_\infty)}{a_0}\right]^{-1}$, we have that $|\lambda_1 - \lambda_2| < \delta$ implies

$$\|T(\phi,\lambda_1) - T(\phi,\lambda_2)\|_{H_0^1(\Omega)} < \varepsilon,$$

for each $\phi \in A$.

Lemma 3.4. *Suppose that assumptions* (H1)–(H2) *are satisfied. Then, there exists a number* $\rho > 0$*, such that any fixed point* $u \in H_0^1(\Omega)$ *of* $T(\cdot,\lambda)$ *for any* $\lambda \in [0,1]$, *i.e., $T(u,\lambda) = u$ for some $\lambda \in [0,1]$, satisfies*

$$\|u\|_{1,2} < \rho. \tag{12}$$

Proof. Testing the equation

$$\begin{aligned} -\left[M\left(\int_\Omega |\nabla u|^2\right)\right]\Delta u = \lambda f(x,u) \quad &\text{in } \Omega \\ u = 0 \quad &\text{on } \partial\Omega \end{aligned} \tag{13}$$

by u and using integration by parts, Young's inequality and (H1), we conclude that

$$\begin{aligned} M(\|u\|_{1,2}^2)\int_\Omega |\nabla u|^2\,dx &= \lambda\int_\Omega f(x,u)u\,dx \\ &\le \frac{q}{2^*}\int_\Omega (b_\infty|u|^q + c_\infty)^{2^*/q}\,dx + \frac{2^*-q}{2^*}\int_\Omega |u|^{2^*/(2^*-q)}\,dx \\ &\le C_1(b_\infty^{2^*/q}, q, 2^*)\int_\Omega |u|^{2^*}\,dx + C_2(c_\infty^{2^*/q}, |\Omega|, q, 2^*) \end{aligned}$$

If $\|u\|_{L^{2^*}(\Omega)} \le 1$, we have that

$$\int_\Omega |\nabla u|^2\,dx \le \frac{C_1(b_\infty^{2^*/q}, q, 2^*)}{a_0} + \frac{C_2(c_\infty^{2^*/q}, |\Omega|, q, 2^*)}{a_0}.$$

Suppose that $\|u\|_{L^{2^*}(\Omega)} > 1$. As the immersion $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous, in other words, there is a constant $C_0 > 0$ such that $\|u\|_{L^{2^*}(\Omega)}^2 \le C_0\|u\|_{1,2}^2$. Therefore, it follows from (H2) that

$$M(C_0^{-1}\|u\|_{L^{2^*}(\Omega)}^2)\int_\Omega |\nabla u|^2\,dx \le C_1(b_\infty^{2^*/q}, q, 2^*)\int_\Omega |u|^{2^*}\,dx + C_2(c_\infty^{2^*/q}, |\Omega|, q, 2^*).$$

Again by (H2) we conclude that

$$\begin{aligned} \int_\Omega |\nabla u|^2\,dx &\le C_1(b_\infty^{2^*/q}, q, 2^*)\frac{[\|u\|_{L^{2^*}(\Omega)}^2]^{N/(N-2)}}{M(C_0^{-1}\|u\|_{L^{2^*}(\Omega)}^2)} + \frac{C_2(c_\infty^{2^*/q}, |\Omega|, q, 2^*)}{a_0} \\ &= \frac{C_1(b_\infty^{2^*/q}, q, 2^*)}{C_0^{-N/(N-2)}}\frac{1}{\frac{M(C_0^{-1}\|u\|_{L^{2^*}(\Omega)}^2)}{[C_0^{-1}\|u\|_{L^{2^*}(\Omega)}^2]^{N/(N-2)}}} + \frac{C_2(c_\infty^{2^*/q}, |\Omega|, q, 2^*)}{a_0} \le C_4 < \infty. \end{aligned}$$

Choosing, $\rho > \left(\max\left\{\frac{C_1(b_\infty^{2^*/q}, q, 2^*)}{a_0} + \frac{C_2(c_\infty^{2^*/q}, |\Omega|, q, 2^*)}{a_0}, C_4\right\}\right)^{1/2}$ follows the lemma.

Remark 3.5. When $N \geq 5$, as $0 < \frac{N}{N-2} - 1 = \frac{2}{N-2} \leq \frac{2}{3} < 1$. It follows from Remark 2.2 and rewriting the last inequality we obtain that

$$\begin{aligned}\int_\Omega |\nabla u|^2\, dx &\leq \frac{C_5}{C_0^{-1}} \frac{1}{\frac{M(C_0^{-1}\|u\|^2_{L^{2^*}(\Omega)})}{C_0^{-1}\|u\|^2_{L^{2^*}(\Omega)}}} [\|u\|^2_{L^{2^*}(\Omega)}]^{2/(N-2)} + C_6 \\ &\leq \frac{C_5}{C_0^{-1}} \frac{1}{\frac{M(C_0^{-1}\|u\|^2_{L^{2^*}(\Omega)})}{C_0^{-1}\|u\|^2_{L^{2^*}(\Omega)}}} \left(C_0 \int_\Omega |\nabla u|^2\, dx\right)^{2/(N-2)} + C_6 \\ &\leq C_7 \left(\int_\Omega |\nabla u|^2\, dx\right)^{2/(N-2)} + C_6.\end{aligned}$$

Thus, there is a constant C_8, independent of u, such that

$$\int_\Omega |\nabla u|^2\, dx \leq C_8.$$

Choosing, $\rho > \left(\max\left\{\frac{C_1(b_\infty^{2^*/q}, q, 2^*)}{a_0} + \frac{C_2(c_\infty^{2^*/q}, |\Omega|, q, 2^*)}{a_0}, C_8\right\}\right)^{1/2}$ it follows again the conclusion of Lemma 3.4.

4. Proof of Theorem 2.3

From Lemma 3.4, we know the existence of a number $\rho > 0$ which satisfies the property stated in (12) and we have that the equation, $u - T(u, 0) = 0$, admits a unique solution, i.e., the unique solution of

$$\begin{aligned} -\left[M\left(\int_\Omega |\nabla u|^2\right)\right]\Delta u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

is $u = 0$. Indeed, it is enough to multiply the equation by u and integrate by parts to get (as $M \geq a_0 > 0$) $\int_\Omega |\nabla u|^2\, dx = 0$, and $u = 0$ subsequently.

It follows from Leray–Schauder's fixed point theorem, Theorem 3.1 with $X = H_0^1(\Omega)$ and $[a, b] = [0, 1]$, that the problem

$$u - T(u, 1) = 0,$$

has a solution u.

It follows from (9) that u is a solution of the problem (1), and by (8) follows the theorem.

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