Convergence of a finite difference method for the KdV and modified KdV equations with L^2 data

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(Communicated by João Paulo Dias)

Abstract. We prove strong convergence of a semi-discrete finite difference method for the KdV and modified KdV equations. We extend existing results to non-smooth data (namely, in L^2), without size restrictions. Our approach uses a fourth order (in space) stabilization term and a special conservative discretization of the nonlinear term. Convergence follows from a smoothing effect and energy estimates. We illustrate our results with numerical experiments, including a numerical investigation of an open problem related to uniqueness posed by Y. Tsutsumi.

1. Introduction

This paper is concerned with the study of a numerical approximation of the equation

$$\partial_t u + \partial_x^3 u + \beta \partial_x u^{k+1} = 0, \quad \beta \neq 0, \, k = 1, 2, \tag{1.1}$$

with $u = u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$. When $k = 1$, the equation (1.1) is referred to as the Korteweg–de Vries (KdV) equation, and when $k = 2$ as the modified KdV (mKdV) equation.

As is well-known, the KdV equation describes the unidirectional propagation of small-but-finite amplitude waves in a nonlinear dispersive medium. It appears in several physical contexts, such as shallow water waves and ion-acoustic waves in a cold plasma. Also, the modified KdV equation has been used to describe acoustic waves and Alfén waves in plasmas without collisions. For a more complete description of the physical contexts concerning the Korteweg–de Vries equation and its generalizations, see [24] and the references therein.

A large amount of work on the KdV equation was initially directed toward the study of solitary waves, i.e., solutions of the form $u(x,t) = U(x - ct)$, especially the so-called soliton solutions, a class of solitary waves which preserve the form through nonlinear interaction (see [24], [18] for surveys on solitons). One of the most relevant results in soliton theory was the development of the inverse scattering method, initially applied to the KdV equation by Gardner et al. [7] and, in a general form, by Lax [15]. This technique was also used to obtain solutions of the KdV equation with low regularity [4], [5], [6].

Here, we concentrate on the numerical approximation of the solution of the Cauchy problem

$$\partial_t u + \partial_x^3 u + \beta \partial_x u^{k+1} = 0, \qquad \beta \neq 0, \, k = 1, 2, \tag{1.2a}$$

$$u(x, 0) = \varphi(x), \qquad \varphi \in L^2. \tag{1.2b}$$

The mathematical problem of well-posedness for (1.2a),(1.2b) has been extensively studied. We refer to the pioneering results in [1], [2], [22] and the improvements in [12], [13]. In these works, local well-posedness is proved in the Sobolev spaces H^s, $s > 3/2$, for generalized KdV (gKdV), in which the term $\partial_x u^{k+1}$ is replaced by $\partial_x V(u)$.

Existence and uniqueness was also obtained in [8], [9] with initial data in weighed L^2 and H^1 spaces. In our numerical approach, we follow the energy method used in those papers.

More recently, following the introduction by Bourgain [3] of certain Fourier spaces, the well-posedness result is strongly improved for data in negative Sobolev spaces (see the monograph [17] and the references therein), and uniqueness of solution in L^2 is proved in [26].

Regarding the numerical solution of the KdV equation, convergence results have been proven for a linearized equation [10] and for smooth solutions [20]. However (to our knowledge), the problem of proving rigorous convergence of numerical schemes without smoothness assumptions has only attracted attention in more recent years. Nixon [19] proves the convergence of approximate solutions for a discretized version of gKdV, but for small L^2 initial data, only. That work is the numerical counterpart of [14]. We refer also to the recent work by Pazoto et al. [21], dealing with the numerical treatment of the mKdV equation with critical exponent and a damping term, which shares some techniques with the present work. Finally, we refer to the recent work by Holden et al. [11], of which we learned as the present paper was already submitted for publication. There, the authors prove some results of a similar nature to ours by employing a fully discrete, implicit discretization.

Thus, to the best of our knowledge (apart from the mentioned results in [11]), the problem of rigorous convergence of numerical schemes for the KdV and

mKdV equations with general data in L^2 has remained unsolved. The purpose of this paper is to fill that gap.

Although the techniques we use to prove our convergence result are based on the ones in [9] (namely, the use of a fourth order stabilization term), their application to the numerical case is not trivial. Indeed, it is essential to use a special non-conservative discretization of the nonlinear term in (1.1). This idea dates back at least to [10], and is also used in [21]. Moreover, to obtain the necessary estimates for the numerical approximation, additional technical difficulties related to interpolators are encountered, with which we deal below.

An outline of the paper follows. After some notations and definitions in Section 2, we prove our main convergence result in Section 3. In Section 4 we present some numerical experiments to illustrate our convergence results and test the accuracy of our scheme.

Finally, in Section 5, we investigate numerically an open question posed by Y. Tsutsumi [25] relating to the uniqueness of solution to the Cauchy problem for the KdV equation with measure initial data. This is done by means of the Miura transformation (see [25]), which relates solutions of the KdV equation with measure initial data to solutions of the mKdV equation with L^2 initial data. As explained in more detail in Section 5, the numerical evidence we provide suggests that the Cauchy problem for the KdV equation with measure initial data is ill-posed. Note that, importantly, these numerical simulations involve discontinuous initial data in L^2 only, and, as such, are not covered by previous convergence results.

2. Notations and definitions

Let h denote our discretization parameter. We denote by $u_j^h = u_j^h(t)$ the (semi-discrete) difference approximation of $u(x_j, t)$, $x_j = jh$, $j \in \mathbb{Z}$. For $h > 0$, we define the Banach spaces

$$\ell_h^p(\mathbb{Z}) = \left\{ (z_j) : z_j \in \mathbb{R}, \|z\|_{p,h}^p \equiv \sum_{j \in \mathbb{Z}} h|z_j|^p < \infty \right\}.$$

For $p = 2$, we denote the usual scalar product by

$$(z, w)_h = \sum_{j \in \mathbb{Z}} h z_j w_j,$$

$z = (z_j)$, $w = (w_j)$. Let us also introduce the following standard notations for finite difference operators. For $u = (u_j)$,

$$D_+u_j = \frac{1}{h}(u_{j+1} - u_j), \qquad D_-u_j = \frac{1}{h}(u_j - u_{j-1}),$$

$$D_0u_j = \frac{1}{2h}(u_{j+1} - u_{j-1}) = \frac{1}{2}(D_+ + D_-)u_j,$$

$$\Delta_h u_j = D_+D_-u_j = D_-D_+u_j = \frac{1}{h^2}(u_{j+1} - 2u_j + u_{j-1}),$$

Also, denote the translation operators by

$$(u_+)_j = u_{j+1}, \qquad (u_-)_j = u_{j-1}.$$

We obtain the following formulas for the discrete differentiation of a product,

$$D_+(vu) = vD_+u + u_+D_+v \tag{2.1a}$$
$$D_-(vu) = vD_-u + u_-D_-v \tag{2.1b}$$

$$D_0(vu) = vD_0u + \frac{1}{2}(u_+D_+v + u_-D_-v) \tag{2.1c}$$

$$\Delta_h(vu) = v\Delta_h u + u\Delta_h v + D_+vD_+u + D_-vD_-u. \tag{2.1d}$$

Also, the difference operators verify in ℓ_h^2

$$(D_+u, v)_h = -(u, D_-v)_h, \qquad (D_0u, v)_h = -(u, D_0v)_h,$$

and so

$$(\Delta_h u, v)_h = (u, \Delta_h v)_h.$$

We will also need to denote for a sequence (u_j) and for a function w

$$\|u\|_{p,R,h}^p = \sum_{|jh| \le R} h|u_j|^p, \qquad R > 0,$$

$$\|w\|_{p,R} = \|w\|_{L^p(-R,R)}, \qquad 0 < R \le +\infty.$$

Finally, we introduce the continuous piecewise linear interpolator

$$P_1^h u(x) = u_j + (x - x_j)\frac{u_{j+1} - u_j}{x_{j+1} - x_j}, \qquad x \in (x_j, x_{j+1}), \tag{2.2}$$

and the piecewise constant interpolator $(P_0^h u)(x) = u_j$, $x \in (x_j, x_{j+1})$.

3. Convergence results

In this section, we prove our main result, Theorem 3.3, which establishes the convergence of a numerical approximation of problem (1.2).

Let us consider the semi-discrete finite difference scheme

$$\frac{d}{dt}u^h + D^3 u^h + \beta \frac{k+1}{k+2}[(u^h)^k D_0 u^h + D_0(u^h)^{k+1}] + h\Delta_h^2 u^h = 0, \quad (3.1a)$$

$$u^h(0) = \varphi^h, \quad (3.1b)$$

where $D^3 u_j = D_+ D_0 D_- u_j$ and $\Delta_h^2 = D_+ D_- D_+ D_-$ denotes the difference bilaplacian, and u^h denotes the unknown grid function $(u_j^h)_{j \in \mathbb{Z}}$, $u_j^h(t)$ being the approximation of the solution of (1.2) at the point (x_j, t).

The term $h\Delta_h^2 u^h$ is introduced in our scheme in order to obtain the uniform (in h) stability estimates necessary for the convergence proof. This term corresponds to the parabolic regularization used in [8] for the continuous problem. Also, the formally consistent discretization

$$\beta \partial_x u^{k+1} \sim \beta \frac{k+1}{k+2}[(u^h)^k D_0 u^h + D_0(u^h)^{k+1}] \quad (3.2)$$

is based on a corresponding one in [10] and is also essential in our proof. See also [21] for an application of the same idea in a different setting.

The following first result holds.

Proposition 3.1. *Let $h > 0$. Then, for each initial data $\varphi^h \in \ell_h^2(\mathbb{Z})$, there exists a unique global solution $u^h(t) \in C(\mathbb{R}; \ell_h^2(\mathbb{Z}))$ of (3.1).*

Proof. Let $S_h(t) = e^{-D^3 t}$ be the unitary group generated by the discrete operator $-D^3$ in the space $\ell_h^2(\mathbb{Z})$. The problem (3.1a),(3.1b) can be written in the usual semigroup framework, as an integral equation in $\ell_h^2(\mathbb{Z})$:

$$u^h(t) = S_h(t)\varphi^h + \int_0^t S_h(t-s)Ju^h(s)\,ds := \Phi(u^h) \quad (3.3)$$

where

$$J(u^h) = -\beta \frac{k+1}{k+2}[(u^h)^k D_0 u^h + D_0(u^h)^{k+1}] - h\Delta_h^2 u^h.$$

Now, for $R > \|\varphi^h\|_2$ and $T > 0$ we consider the space

$$B_R^T = \{w \in C([0, T]; \ell_h^2(\mathbb{Z})) : \|w\|_{L^\infty([0,T]; \ell_h^2(\mathbb{Z}))} \leq R\}.$$

By (3.3) and since $\ell_h^2(\mathbb{Z}) \subset \ell_h^\infty(\mathbb{Z})$ it is now a simple matter to prove that for $T > 0$ small enough, the map $u^h \in B_R^T \mapsto \Phi(u^h)$ is a strict contraction on the complete metric space B_R^T. Thus, by the Banach fixed-point theorem, we obtain a unique local solution u^h of the problem (3.1a),(3.1b) in the space $C([0, T]; \ell_h^2(\mathbb{Z}))$.

The global existence is an immediate consequence of the uniform bounds on the ℓ_h^2 norm, established below in Lemma 3.5. $\qquad\square$

Let P_1^h denote the continuous piecewise linear interpolator and let $\varphi \in L^2(\mathbb{R})$ be the initial data for the problem (1.2). Also, we denote by $C_w(I; X)$ the space of weakly continuous functions from the interval I to the Banach space X.

It is now convenient to introduce the notion of weak solution to the Cauchy problem (1.2a),(1.2b).

Definition 3.2. Let $\varphi \in L^2(\mathbb{R})$. A function $u(x, t)$, $x \in \mathbb{R}$, $t \geq 0$, is a solution to the Cauchy problem (1.2a),(1.2b) if

1. $u \in L_{\mathrm{loc}}^\infty((0, \infty); L^2(\mathbb{R}))$,

2. For each test function $\phi \in \mathscr{D}(\mathbb{R} \times (0, \infty))$ one has

$$\int_{\mathbb{R}^2} u(\partial_t \phi + \partial_x^3 \phi) + \beta u^{k+1} \partial_x \phi \, dx \, dt = 0, \qquad (3.4)$$

3. $u(t) \to \varphi$ in $L^2(\mathbb{R})$ as $t \to 0$ a.e.

We now state the main result of this paper.

Theorem 3.3 (Convergence of approximate solutions). *Let $\varphi^h \in \ell_h^2(\mathbb{Z})$ be the initial data for the discretized problem* (3.1), *such that $P_1^h \varphi^h \to \varphi$ in $L^2(\mathbb{R})$ when $h \to 0$. Then, for each $T > 0$, the sequence $P_1^h u^h$ satisfies*

$$P_1^h u^h \rightharpoonup u \quad \text{in } L^2([0, T]; H_{\mathrm{loc}}^1(\mathbb{R})) \quad weak,$$
$$P_1^h u^h \to u \quad \text{in } L^2([0, T]; L_{\mathrm{loc}}^2(\mathbb{R}))$$

with u a solution of (1.2). *Moreover, u satisfies*

$$u \in (L^\infty \cap C_w)([0, T]; L_{\mathrm{loc}}^2(\mathbb{R})), \qquad (3.5)$$
$$\|u(t)\|_2 \leq \|\varphi\|_2 \quad and \quad \|u(t) - \varphi\|_2 \to 0 \quad as \ h \to 0.$$

The proof will be postponed to the next section.

3.1. Main estimates. First, let us record some inequalities which will be of use throughout. Let $v = (v_j) \in \ell_h^q(\mathbb{Z})$, $q \in [1, \infty]$. From (2.2) we derive

$$\|P_1^h v - P_0^h v\|_{q,R} \le Ch\|D_0 v\|_{q,R,h}, \qquad 0 < R \le \infty,$$
$$\|P_1^h v - P_0^h v\|_{q,R} \le C\|v\|_{q,R,h}, \qquad 0 < R \le \infty, \ q \in [1, +\infty], \tag{3.6}$$

for some C independent of h. As a consequence, we obtain

$$\|P_1^h v\|_{q,R} \le C\|P_0^h v\|_{q,R} = C\|v\|_{q,R,h}, \qquad \forall v \in \ell_h^q(\mathbb{Z}), \ 0 < R \le +\infty. \tag{3.7}$$

We will need the following inequalities,

Lemma 3.4. *Let $\phi \in \ell_h^2(\mathbb{Z})$. Then,*

$$\|\phi\|_\infty \le C\|\phi\|_{2,h}^{1/2}\|D_\pm \phi\|_{2,h}^{1/2} \tag{3.8}$$

$$\|\phi\|_\infty \le \frac{1}{2}\|D_\pm \phi\|_{1,h} \tag{3.9}$$

Proof. The inequality (3.8) is a consequence of the Gagliardo-Niremberg inequality and $\partial_x P_1^h = P_0^h D_+$:

$$\|\phi\|_\infty = \|P_1^h \phi\|_\infty \le C\|P_1^h \phi\|_2^{1/2}\|\nabla P_1^h \phi\|_2^{1/2}$$
$$\le C\|\phi\|_{2,h}^{1/2}\|D_\pm \phi\|_{2,h}^{1/2},$$

while (3.9) is a consequence of the (continuous) inequality $\|\phi\|_\infty \le \frac{1}{2}\|\phi'\|_1$. $\qquad\square$

We are now ready to state our first stability estimate.

Lemma 3.5. *Let $u^h(t)$ be a solution of* (3.1). *Then, for all $t > 0$, it holds*

$$\|u^h(t)\|_{2,h} \le \|\varphi^h\|_{2,h}. \tag{3.10}$$

In particular, u^h is a global solution of (3.1).

Proof. In the next proofs, we will, for simplicity, omit h from the notation. Take the ℓ^2-scalar product of the equation (3.1a) with $u \equiv u^h$ to get

$$\left(\frac{d}{dt}u, u\right) + \beta\frac{k+1}{k+2}[(u^k D_0 u, u) + (D_0 u^{k+1}, u)] + h(\Delta_h^2 u, u) = 0.$$

Now,

$$(\Delta_h^2 u, u) = (\Delta_h u, \Delta_h u) = \|\Delta_h u\|_2^2,$$
$$(u^k D_0 u, u) = (u^{k+1}, D_0 u) = -(D_0 u^{k+1}, u),$$

and so

$$\frac{d}{dt}\|u(t)\|_2^2 = -h\|\Delta_h u\|_2^2 \le 0, \tag{3.11}$$

from which the conclusion follows. Notice how the discretization (3.2) leads to the non-increase of the ℓ^2 norm. $\qquad\square$

The next lemma is a fundamental identity which, as we will see in Proposition 3.7 below, implies a smoothing effect inherent to the equation: even though the initial data is only in $\ell_h^2(\mathbb{Z})$, the solution of (3.1) is actually in a more regular space (uniformly in h), namely, H_{loc}^1.

Let $p : \mathbb{R} \to \mathbb{R}$ be a bounded, strictly increasing, smooth function, with all its derivatives bounded. Write $p_j = p(x_j)$, $j \in \mathbb{Z}$. For simplicity, we do not distinguish in our notation the continuous and the discrete p.

Lemma 3.6. *Let $u^h = u^h(t)$ be the solution of the discrete problem* (3.1). *Then, u^h satisfies the identity*

$$\frac{1}{2}\frac{d}{dt}\|p^{1/2}u^h\|_2^2 + (D_-u^h, D_+pD_-u^h) + \frac{1}{2}(D_+u^h, D_-pD_+u^h)$$
$$+ h(D_+D_-u^h, pD_+D_-u^h)$$
$$= -\frac{h}{2}(D_+D_-u^h, D_+pD_-u^h) + \frac{h}{2}(D_+u^hD_-p, D_+D_-u^h)$$
$$- (D_-u^h, u_-^hD_0D_-p) - h(D_+D_-u^h, D_-pD_-u^h)$$
$$- h(D_+D_-u^h, D_+pD_+u^h) - h(D_+D_-u^h, u^hD_+D_-p)$$
$$+ \frac{\beta}{2}\frac{k+1}{k+2}\left((u^h)^{k+1}, u_+^hD_+p + u_-^hD_-p\right). \tag{3.12}$$

Proof. We take (3.1a), multiply by $hp_ju_j^h$, and sum over $j \in \mathbb{Z}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|p^{1/2}u\|_2^2 + (D^3u, pu) + \beta\frac{k+1}{k+2}[(u^kD_0u, pu) + (D_0u^{k+1}, pu)]$$
$$+ h(\Delta_h^2u, pu) = 0. \tag{3.13}$$

We find from (2.1)

$$(D^3u, pu) = (D_+D_0D_-u, pu) = -\left(D_0D_-u, D_-(pu)\right)$$
$$= -(D_0D_-u, pD_-u) - (D_0D_-u, u_-D_-p) = A + B.$$

Since

$$A = (D_-u, pD_0D_-u) + \left(D_-u, \frac{1}{2}\left(D_+p(D_-u)_+ + D_-p(D_-u)_-\right)\right),$$

we obtain

$$2A = \left(D_-u, \frac{1}{2}D_+pD_+u\right) + \left(D_-u, \frac{1}{2}D_-p(D_-u)_-\right)$$

and so

$$\begin{aligned}
A &= \frac{1}{4}\left[(D_-u, D_+pD_+u) + (D_+u, D_+pD_-u)\right] \\
&= \frac{1}{2}(D_+u, D_+pD_-u) \\
&= \frac{1}{2}\left[(D_+u - D_-u, D_+pD_-u) + (D_-u, D_+pD_-u)\right] \\
&= \frac{h}{2}(D_+D_-u, D_+pD_-u) + \frac{1}{2}(D_-u, D_+pD_-u).
\end{aligned}$$

Similarly,

$$\begin{aligned}
B &= (D_-u, u_-D_0D_-p) + \left(D_-u, \frac{1}{2}[D_+u_-D_+p + D_-u_-(D_-p)_-]\right) \\
&= (D_-u, u_-D_0D_-p) + \frac{1}{2}(D_-u, D_+pD_-u) + \frac{1}{2}(D_+u, D_-uD_-p).
\end{aligned}$$

Since

$$(D_+u, D_-pD_-u) = (D_+u, D_-pD_+u) - h(D_+u, D_-pD_+D_-u),$$

we obtain

$$\begin{aligned}
B ={}& (D_-u, u_-D_0D_-p) + \frac{1}{2}(D_-u, D_+pD_-u) \\
&+ \frac{1}{2}(D_+u, D_-pD_+u) - \frac{h}{2}(D_+u, D_-pD_+D_-u).
\end{aligned}$$

For the term in (3.13) corresponding to the discrete bi-laplacian, we derive

$$\begin{aligned}
(\Delta_h^2 u, pu) &= -(D_-D_+D_-u, p_-D_-u + uD_-p) \\
&= -(D_-D_+D_-u, p_-D_-u) - (D_-D_+D_-u, uD_-p) \\
&= (D_+D_-u, pD_+D_-u) + (D_+D_-u, D_-pD_-u) \\
&\quad + (D_+D_-u, D_+pD_+u) + (D_+D_-u, uD_+D_-p).
\end{aligned}$$

As to the remaining term in (3.13), we find

$$(u^k D_0 u, pu) + (D_0 u^{k+1}, pu) = (D_0 u, pu^{k+1}) - (u^{k+1}, D_0(pu))$$

$$= -\left(u^{k+1}, \frac{1}{2}(u_+ D_+ p + u_- D_- p)\right).$$

All these results together give (3.12). This completes the proof of Lemma 3.6.

$\square$

As a consequence of the two preceding lemmas, we now prove the following result which states that, at the discrete level, u^h is in H^1_{loc}.

Proposition 3.7. *Let u^h be solution of the discretized problem* (3.1) *with initial data $\varphi = \varphi^h \in \ell^2_h(\mathbb{Z})$. Then, for each $T > 0$ and for each $R > 0$, there exists a constant $C = C(R, T, \|\varphi\|_{2,h})$ such that, for all $h > 0$,*

$$\int_0^T \sum_{|jh| \le R} h |D_\pm u_j^h|^2 \, dt \le C. \tag{3.14}$$

Proof. We apply Lemma 3.6 with a bounded, strictly increasing, smooth function p, with all its derivatives bounded, and such that, moreover, $p(x) \ge 1$ for all x, and $p'(x) = 1$ for $x \in [-R, R]$. Let us rewrite the identity (3.12), with obvious notation, as

$$\frac{1}{2}\frac{d}{dt}\|p^{1/2}u^h\|_2^2 + (D_- u^h, D_+ p D_- u^h) + \frac{1}{2}(D_+ u^h, D_- p D_+ u^h)$$

$$+ h(D_+ D_- u^h, p D_+ D_- u^h) = A_1 + \cdots + A_7. \tag{3.15}$$

Now observe that under our assumptions on p, the terms on the left-hand side (except the first) are non-negative, so we must bound the terms A_i.

The terms A_1 and A_2 are similar and yield

$$\frac{h}{2}|(D_+ D_- u^h, D_+ p D_- u^h)| \le \frac{h}{2}\|(D_+ D_- u^h)p^{1/2}\|_2 \|p^{-1/2}D_+ p D_- u^h\|_2$$

$$\le \eta \frac{h}{2}\|(D_+ D_- u^h)p^{1/2}\|_2^2 + \frac{Ch}{2\eta}\|(D_+ p)^{1/2}D_- u^h\|_2^2,$$

for all $\eta > 0$, where we have used the properties of p. Also, the terms A_4 and A_5 are similar and give

$$h(D_+D_-u^h, D_-pD_-u^h) = h(D_+D_-u^h, D_-pD_+u^h) - h^2(D_+D_-u^h, D_-pD_+D_-u^h)$$
$$= B_1 + B_2.$$

The term B_1 is similar to A_1, while

$$|B_2| \leq Ch^2(D_+D_-u^h, pD_+D_-u^h).$$

For the term A_3, we remark that

$$(D_-u^h, u_-^h D_0 D_- p) = (D_-u^h - D_-u_-^h, u_-^h D_0 D_- p) + (D_-u_-^h, u_-^h D_0 D_- p)$$
$$= h(D_-D_-u^h, u_-^h D_0 D_- p) - (u_-^h, D_-u^h D_0 D_- p)$$
$$- (u_-^h, u^h D_+ D_0 D_- p)$$

and so

$$|(D_-u^h, u_-^h D_0 D_- p)| \leq Ch\|p^{1/2}D_+D_-u^h\|_2\|u^h\|_2 + C\|u^h\|_2^2$$
$$\leq C_1 h^2 \|p^{1/2}D_+D_-u^h\|_2^2 + C_2\|u^h\|_2^2.$$

The term $A_6 = h(D_+D_-u^h, u^h D_+D_-p)$ is easily estimated using the Cauchy-Schwarz inequality. Finally, for the last term

$$A_7 = \frac{\beta}{2}\frac{k+1}{k+2}\left((u^h)^{k+1}, u_+^h D_+p + u_-^h D_-p\right),$$

let us consider only $k = 2$, since the case $k = 1$ is easier. Setting $q = (D_+p)^{1/2}$ and $u = u^h$, we obtain

$$|(u^3, u_+D_+p)| \leq C\|u\|_2^2\|qu^2\|_\infty$$

and from (3.9) it follows

$$\|qu^2\|_\infty \leq \|(D_-q)u_-^2 + q(uD_-u + u_-D_-u)\|_1 \leq C\|u\|_2^2 + \|qD_-u\|_2\|u\|_2.$$

Hence,

$$|(u^3, u_+D_+p)| \leq C\|u\|_2^4 + C\|u\|_2^3\|(D_+p)^{1/2}D_-u\|_2$$
$$\leq \varepsilon\|(D_+p)^{1/2}D_-u\|_2^2 + \frac{C}{\varepsilon}\|u\|_2^6 + C\|u\|_2^4,$$

for any $\varepsilon > 0$.

Choosing ε, η small enough, and for some small enough h_0 (which does not depend on R, T or φ), we find, after integrating (3.15) on $[0, T]$ and using the previous estimates,

$$\int_0^T \sum_{|jh| \le R} h|D_\pm u_j^h|^2\, dt \le C(R, T, \|\varphi\|_2).$$

This completes the proof of Proposition 3.7. □

3.2. Proof of Theorem 3.3. The proof of Theorem 3.3 relies on Aubin's compactness result, which we state here, in a simplified form, for the reader's convenience.

Lemma 3.8 ([16], p. 58). *Let $1 < p < \infty$, $T > 0$, and consider reflexive Banach spaces $B_0 \subset B \subset B_1$ such that B_0 is compactly embedded in B. Then, the space*

$$\left\{ v : v \in L^2(0, T; B_0), \frac{dv}{dt} \in L^p(0, T; B_1) \right\}$$

is compactly embedded in $L^2(0, T; B)$.

In order to apply Lemma 3.8, we will use the following estimates.

Lemma 3.9. *Let u^h be given by (3.1a),(3.1b) and let $T, R > 0$. Then, there exists $p > 1$ such that*

$$\int_0^T \|P_1^h u^h\|_{H^1(-R, R)}^2\, dt \le C, \tag{3.16}$$

$$\int_0^T \left\| \frac{d}{dt} P_1^h u^h \right\|_{H^{-5}(-R, R)}^p\, dt \le C, \tag{3.17}$$

uniformly in h, with $C = C(R, T, \|\varphi\|_{2,h})$.

Proof. First of all, note that since $\partial_x P_1^h = P_0^h D_+$, it follows from Lemma 3.5 and Proposition 3.7 that the estimate (3.16) holds for each $R > 0$.

Let us now prove the estimate (3.17). Let u^h be given by (3.1a),(3.1b). We apply the piecewise linear continuous interpolator P_1^h to the equation (3.1a) to obtain

$$\frac{d}{dt} P_1^h u^h + P_1^h D^3 u^h + \beta \frac{k+1}{k+2} \left(P_1^h[(u^h)^k D_0 u^h] + P_1^h D_0 (u^h)^{k+1} \right)$$
$$+ h P_1^h \Delta_h^2 u^h = 0, \tag{3.18}$$

with $u^h = u^h(t)$. We begin by estimating the term $P_1^h D^3 u^h$, for which it is convenient to consider the decomposition $P_1^h = (P_1^h - P_0^h) + P_0^h$ and analyze the two resulting terms. For each test function $\phi \in \mathscr{D}(-R, R)$ we have

$$(P_0^h D^3 u^h, \phi) = \sum_{|jh| \leq R} \int_{x_j}^{x_{j+1}} P_0^h D^3 u^h \phi \, dx$$

$$= \sum_{|jh| \leq R} D_+ D_0 D_- u_j \int_{x_j}^{x_{j+1}} \phi(x) \, dx$$

$$= \sum_{|jh| \leq R} D_- u_j \int_{x_j}^{x_{j+1}} \frac{1}{h^2} \left(\phi(x - 2h) - \phi(x - h) - \phi(x) + \phi(x + h) \right) dx$$

and so (by Taylor expansion of ϕ)

$$|(P_0^h D^3 u^h, \phi)| \leq C \sum_{|jh| \leq R} h |D_- u_j| \, \|\phi''\|_\infty$$

$$\leq C \left(\sum_{|jh| \leq R} h |D_- u_j|^2 \right)^{1/2} \|\phi\|_{H^3(-R, R)}.$$

Hence, by Proposition 3.7, we have

$$\int_0^T \|P_0^h D^3 u^h\|_{H^{-3}(-R, R)} \, dt \leq C. \qquad (3.19)$$

Next, if $x \in (x_j, x_{j+1})$ and $v_j = D^3 u_j$, we easily find

$$(P_1^h - P_0^h) v^h(x) = (x - x_j) \frac{v_{j+1} - v_j}{h},$$

and so, with obvious notation,

$$((P_1^h - P_0^h) D^3 u^h, \phi) = \sum_{|jh| \leq R} D_+ D^3 u_j \int_{x_j}^{x_{j+1}} (x - x_j) \phi(x) \, dt$$

$$= \frac{1}{h} (D_+ D^3 u_j, A_j)$$

$$= -\frac{1}{h} (D_- u_j, D_0 D_- D_- A_j).$$

A straightforward computation gives

$$D_0 D_- D_- A_j$$

$$= \frac{1}{2h^3} \int_{x_j}^{x_{j+1}} \big(\phi(x+h) - 2\phi(x) + 2\phi(x-2h) - \phi(x-3h)\big)(x-x_j)\, dx$$

from which we obtain by Taylor expansion of ϕ and Proposition 3.7

$$\int_0^T \big|\big((P_1^h - P_0^h)D^3 u^h, \phi\big)\big| \le C \int_0^T \sum_{|jh| \le R} h|D_- u_j|\, \|\phi'''\|_\infty \, dt$$

$$\le C\|\phi\|_{H^4(-R,R)} \int_0^T \Big(\sum_{|jh| \le R} h|D_- u_j|^2 \Big)^{1/2} dt$$

$$\le C\|\phi\|_{H^4(-R,R)}.$$

From this and (3.19) we obtain the estimate

$$\int_0^T \|P_1^h D^3 u^h\|_{H^{-4}(-R,R)}^2 \, dt \le C. \tag{3.20}$$

In an entirely similar way, we arrive at

$$\int_0^T \|P_1^h \Delta_h^2 u^h\|_{H^{-5}(-R,R)}^2 \, dt \le C. \tag{3.21}$$

It remains to estimate the nonlinear terms in (3.18). Choose a smooth function $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(x) = 1$ if $|x| \le R$ and $\theta(x) = 0$ if $|x| \ge R+1$. Using (3.6) and (3.8) we derive, for $R > 0$, $k = 1, 2$,

$$\|P_0^h[(u^h)^k D_0 u^h]\|_{3/2, R, h} \le \|D_0 u^h\|_{2, R+1, h} \|\theta u^h\|_{6k, h}^k$$

$$\le C\|D_0 u^h\|_{2, R+1, h} \|\theta u^h\|_\infty^{k-1/3}$$

$$\le \|D_0 u^h\|_{2, R+1, h}(C + C\|D_+ u^h\|_{2, K+1, h}^{k/2-1/6})$$

$$\le C + C\|D_+ u^h\|_{2, R+1, h}^{k/2+5/6},$$

with $C = C(\|\varphi\|_{2, h}, R)$. Choosing $p = 12/(3k+5) > 1$, we obtain from Proposition 3.7

$$\int_0^T \|P_0^h[(u^h)^k D_0 u^h]\|_{3/2}^p \, dt \le C + C \int_0^T \|D_+ u^h\|_{2, R+1, h}^2 \le C.$$

Since $\|(P_1^h - P_0^h)u^h\|_{3/2} \le C\|P_0^h u^h\|_{3/2}$ (cf. (3.6)), we conclude that

$$\int_0^T \|P_1^h[(u^h)^k D_0 u^h]\|_{3/2}^p \, dt \le C, \qquad k = 1, 2. \tag{3.22}$$

For the remaining nonlinear term $P_1^h D_0(u^h)^{k+1}$, we split it as above into $(P_1^h - P_0^h) + P_0^h$. First, note that

$$\|P_0^h(u^h)^{k+1}\|_{2,R} \le C\|\theta u^h\|_{2k+2}^{k+1} \le C\|\theta u^h\|_\infty^k \le C + C\|D_+ u^h\|_{2,R+1,h}^{k/2}, \tag{3.23}$$

and, by (3.7), the same estimate is obtained for $\|P_1^h(u^h)^{k+1}\|_{2,R}$. Next, since $P_0^h D_0 = \partial_x P_1^h$, we obtain for $k = 1, 2$

$$\int_0^T \|P_0^h D_0(u^h)^{k+1}\|_{H^{-1}(-R,R)}^{4/k} \, dt = \int_0^T \|\partial_x P_1^h(u^h)^{k+1}\|_{H^{-1}(-R,R)}^{4/k} \, dt$$

$$\le \int_0^T \|P_1^h(u^h)^{k+1}\|_{2,R}^{4/k} \, dt$$

$$\le C + C \int_0^T \|D_+ u^h\|_{2,R+1,h}^2 \, dt \le C.$$

We now need to estimate $(P_1^h - P_0^h)D_0(u^h)^{k+1}$. With computations similar to the ones after (3.19), we find

$$\left|\left((P_0^h - P_1^h)D_0(u^h)^{k+1}, \phi\right)\right| \le C \sum_{|jh| \le R} h|u_j|^{k+1}\|\phi''\|_\infty$$

$$\le C\|P_0^h(u^h)^{k+1}\|_{2,R}\|\phi\|_{H^3(-R,R)},$$

and by (3.23),

$$\int_0^T \|(P_0^h - P_1^h)D_0(u^h)^{k+1}\|_{H^{-3}(-R,R)}^{4/k} \, dt \le C + C \int_0^T \|D_+ u^h\|_{2,R+1,h}^2 \, dt \le C.$$

Thus we conclude that

$$\int_0^T \|P_1^h D_0(u^h)^{k+1}\|_{H^{-3}(-R,R)}^{4/k} \, dt \le C, \qquad k = 1, 2. \tag{3.24}$$

The desired estimate (3.17), with $p = 12/(3k + 5) > 1$, now follows from the estimates (3.18), (3.20), (3.21), (3.22), and (3.24). This completes the proof of Lemma 3.9. □

Proof of Theorem 3.3. In view of the estimates in Lemma 3.9, we apply Lemma 3.8 with $p = 12/(3k + 5)$, $B_0 = H^1(-R, R)$, $B = L^q(-R, R)$, $q \in (1, \infty]$, and $B_1 = H^{-5}(-R, R)$ (note that $B_0 \subset B$ with compact embedding). We conclude that, up to a subsequence, $P_1^h u^h$ converges weakly in $L^2(0, T; H^1(-R, R))$ and strongly in $L^2(0, T; L^q(-R, R))$. Using a diagonal argument, we obtain for a further subsequence

$$
\begin{aligned}
P_1^h u^h \rightharpoonup u \quad &\text{in } L^2([0, T]; H^1(-R, R)) \quad \text{weak}, \\
P_1^h u^h \to u \quad &\text{in } L^2([0, T]; L^q(-R, R)), \quad R > 0, q \in (1, \infty],
\end{aligned}
\tag{3.25}
$$

for some $u \in L^\infty(0, T; L^2(\mathbb{R})) \cap L^2(0, T; H^1_{\text{loc}}(\mathbb{R}))$, as $h \to 0$. Also, from (3.6) and Proposition 3.7 we can conclude that

$$
P_0^h u^h \to u \quad \text{in } L^2(0, T; L^q(-R, R)), \quad 1 < q \le 2. \tag{3.26}
$$

Now we must prove that u is a weak solution of the problem (1.2a),(1.2b), in the sense of Definition 3.2. Let us apply the piecewise constant interpolator P_0^h to the discrete equation (3.1a):

$$
\frac{d}{dt} P_0^h u^h + P_0^h D^3 u^h + \beta \frac{k+1}{k+2} \left(P_0^h [(u^h)^k D_0 u^h] + P_0^h D_0 (u^h)^{k+1} \right)
$$
$$
+ h P_0^h \Delta_h^2 u^h = 0. \tag{3.27}
$$

First, consider the linear terms. We take a test function $\phi \in \mathscr{D}(\mathbb{R} \times (0, \infty))$ and compute in the sense of distributions

$$
\begin{aligned}
(P_0^h D^3 u^h, \phi) &= \int_0^\infty \sum_{j \in \mathbb{Z}} D^3 u_j \int_{x_j}^{x_{j+1}} \phi(x, t) \, dx \, dt \\
&= -\int_0^\infty \sum_{j \in \mathbb{Z}} u_j \int_{x_j}^{x_{j+1}} \left(\partial_x^3 \phi + \mathcal{O}(h) \right) dx \, dt \\
&= -(P_0^h u^h, \partial_x^3 \phi) + \mathcal{O}(h) \to -(u, \partial_x^3 \phi) = (\partial_x^3 u, \phi)
\end{aligned}
$$

as $h \to 0$. The term $h P_0^h \Delta_h^2 u^h$ is treated similarly and tends to zero as $h \to 0$ in the sense of distributions.

Now consider the nonlinear terms. Note that $P_0^h D_0 = \partial_x P_1^h$ and write $P_0^h = (P_0^h - P_1^h) + P_1^h$. Using (3.6) and (3.8) we find

$$
\|(P_1^h - P_0^h)(u^h)^{k+1}\|_{1, R, h} \le Ch \|D_+(u^h)^{k+1}\|_{1, R, h} \le Ch(1 + \|D_+ u_j\|_{2, R+1, h}^{3/2})
$$

and so

$$
(P_0^h - P_1^h)(u^h)^{k+1} \to 0 \tag{3.28}
$$

in $L^{4/3}(0, T; L^1_{\text{loc}}(\mathbb{R}))$ as $h \to 0$. Since P^h_0 commutes with the nonlinearity, it follows from (3.26) that

$$P^h_0(u^h)^{k+1} \to u^{k+1} \quad \text{in } L^1(0, T; L^1_{\text{loc}}(\mathbb{R})). \tag{3.29}$$

Hence, we deduce from (3.28),(3.29) that

$$P^h_0 D_0(u^h)^{k+1} = \partial_x P^h_1(u^h)^{k+1} \to \partial_x u^{k+1}$$

in the sense of distributions. For the other nonlinear term, we note that $P^h_0[(u^h)^k D_0 u^h] = P^h_0(u^h)^k P^h_0 D_0 u^h$. We have

$$P^h_0(u^h)^k \to u^k \quad \text{in } L^2(0, T; L^2_{\text{loc}}(\mathbb{R}))$$

and, from (3.25),

$$P^h_0 D_0 u^h = \partial_x P^h_1 u^h \rightharpoonup \partial_x u \quad \text{in } L^2(0, T; L^2_{\text{loc}}(\mathbb{R})).$$

Therefore,

$$P^h_0[(u^h)^k D_0 u^h] \to u^k \partial_x u \quad \text{in } L^1(0, T; L^1_{\text{loc}}(\mathbb{R})).$$

Multiplying (3.27) by a test function in $\mathscr{D}(\mathbb{R} \times (0, \infty))$, the above convergences allow us to conclude that u verifies the property (3.4) of Definition 3.2.

It remains to prove the weak $L^2(\mathbb{R})$-valued continuity property, (3.5), and that $u(t) \to \varphi$ in $L^2(\mathbb{R})$ as $t \to 0$ a.e. To prove the weak continuity property, we remark that, for $\phi \in \mathscr{D}(\mathbb{R})$, $t \in [0, T]$,

$$\left(P^h_1 u^h(t + \rho) - P^h_1 u^h(t), \phi\right) = \int_t^{t+\rho} \left(\frac{du^h}{dt}, \phi\right) ds$$

and, from (3.17), we get

$$\left|\left(P^h_1 u^h(t + \rho) - P^h_1 u^h(t), \phi\right)\right| \leq C\rho$$

with $C = C(T, \phi)$, and so the family $t \mapsto P^h_1 u^h(t)$ is uniformly bounded in L^2 (see (3.7)) and weakly equicontinuous. Therefore, the Arzelà–Ascoli Theorem implies that $u \in C_w([0, \infty); L^2(\mathbb{R}))$.

Finally, since $\|u(t)\|_2 \leq \|\varphi\|_2$ a.e. in t, we have

$$\|u(t) - \varphi\|_2^2 \leq \|u(t)\|_2^2 - 2(u(t), \varphi) + \|\varphi\|_2^2$$
$$\leq 2\|\varphi\|_2^2 - 2(u(t), \varphi) \to 0$$

for almost every $t \to 0^+$. This completes the proof of Theorem 3.3. □

4. Numerical experiments

4.1. A fully discrete, fully implicit scheme. In this section, we present some numerical experiments to test the accuracy of our scheme and to illustrate our results. In order to fully discretize the semi-discrete equations (3.1a), we use a fully implicit Euler scheme, as follows. Given a time step τ, a space step h, and $J \in \mathbb{N}$, solve for each $n = 1, 2, \ldots$ the system of $J - 1$ equations

$$
\frac{u_j^{n+1} - u_j^n}{\tau} + D^3 u_j^{n+1} + \frac{k+1}{k+2} [(u_j^{n+1})^k D_0 u_j^{n+1} + D_0 (u_j^{n+1})^{k+1}]
$$
$$
+ \eta h \Delta_h^2 u_j^{n+1} = 0, \qquad j = 1, \ldots J - 1, \tag{4.1}
$$

with boundary conditions

$$
u_0^{n+1} = u_J^{n+1} = 0,
$$

where $u_j^n \simeq u(a + jh, n\tau)$ for some $a \in \mathbb{R}$. The computational domain is thus the interval $[a, a + Jh]$. We have set $\beta = 1$ and introduced a new viscosity parameter $\eta > 0$ allowing us to explicitly control the amount of viscosity in the scheme.

As is standard in the numerical simulation of dispersive equations, we consider a sufficiently large spatial domain and initial data exponentially small outside some bounded region, ensuring that spurious wave reflection at the boundary of the domain remains negligible.

Written in full, the scheme (4.1) reads

$$
\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{2h^3} (u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1})
$$
$$
+ \frac{k+1}{2h(k+2)} \left((u_j^{n+1})^k (u_{j+1}^{n+1} - u_{j-1}^{n+1}) + (u_{j+1}^{n+1})^{k+1} - (u_{j-1}^{n+1})^{k+1}\right)
$$
$$
+ \frac{\eta}{2h^3} (u_{j+2}^{n+1} - 4u_{j+1}^{n+1} + 6u_j^{n+1} - 4u_{j-1}^{n+1} + u_{j-2}^{n+1}) = 0,
$$
$$
u_0^{n+1} = u_J^{n+1} = 0.
$$

Due to the nonlinear terms, it is necessary to perform a Newton iteration at each time step, which we carry out with a tolerance of 10^{-6} in the simulations below. To solve the pentadiagonal linear system at each iteration of Newton's method, we employ a standard LU decomposition method. Notice that the Jacobian matrix needed for Newton's method is easily computed explicitly from the method.

4.2. Comparison with exact solutions. The first step is to test our scheme with the known soliton solutions of the KdV equation (1.1) [17], p. 140.

These read

$$u(x,t) = \left[\frac{1}{2}(k+2)c^2 \operatorname{sech}^2\left(\frac{k}{2}c(x-d-c^2 t)\right)\right]^{1/k} \tag{4.2}$$

for arbitrary $c > 0$, $d \in \mathbb{R}$, and consist of traveling waves with speed c^2. We observe in passing that these exact solutions actually solve the equation (1.1) and not the slightly different version in [17], p. 139.

In Figures 1 and 2, we present the relative $L^2([0,T]; L^2(\mathbb{R}))$ error between the exact solution (4.2) with $c = 1$, $d = 20$, and the computed solution, for $T = 10$, computed on the domain $x \in (10, 50)$, as a function of the spatial step h, respectively, for $k = 1$ and $k = 2$. Note that for the values of h presented here, the number of spatial points varies between 5000 and 45000. As can be readily seen from the plot, we observe a convergence of order one, that is, proportional to h. In Figure 3 we present the result of varying τ for a fixed value of $h = 0.004$, corresponding to 10000 spatial points, and we observe the same linear convergence rate.

In Figure 4 we show the evolution of the ℓ_h^2 norm of the discrete solution, computed with 10000 spatial points on the interval $[0, 50]$ (thus $h = 0.005$) up to $T = 5$ with $\tau = 0.0001$. The viscosity parameter was chosen as $\eta = 0.001$. This quantity should not increase, according to (3.10). Some decrease is observed, due to the dissipation in the scheme. Of course ideally there should be no decrease of the discrete L^2 norm of the approximate solution, since the exact solution does not dissipate. However, we stress again that the results presented here serve to illus-
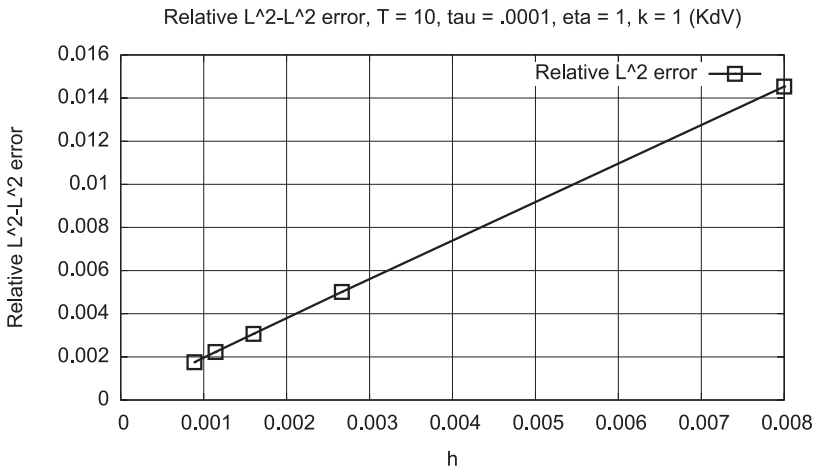

Figure 1. Relative $L^2([0,T]; L^2(\mathbb{R}))$ error as a function of the spatial step h, $k = 1$ (KdV equation).

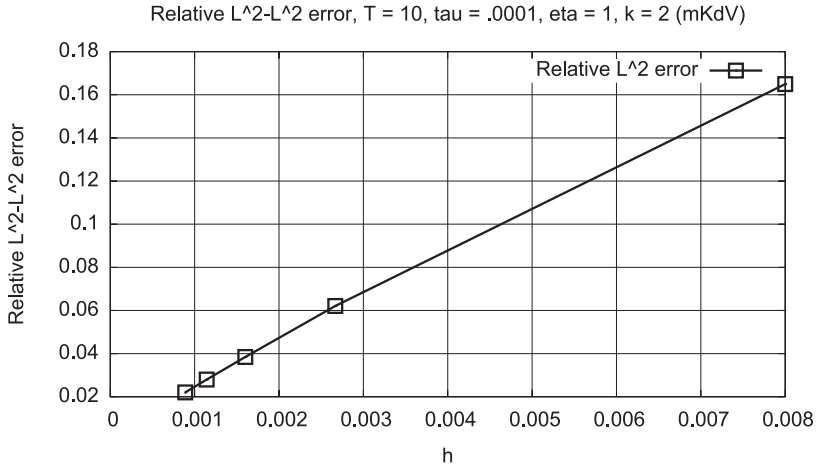Relative L^2-L^2 error, T = 10, tau = .0001, eta = 1, k = 2 (mKdV)

Figure 2. Relative $L^2([0, T]; L^2(\mathbb{R}))$ error as a function of the spatial step, $k = 2$ (mKdV equation).
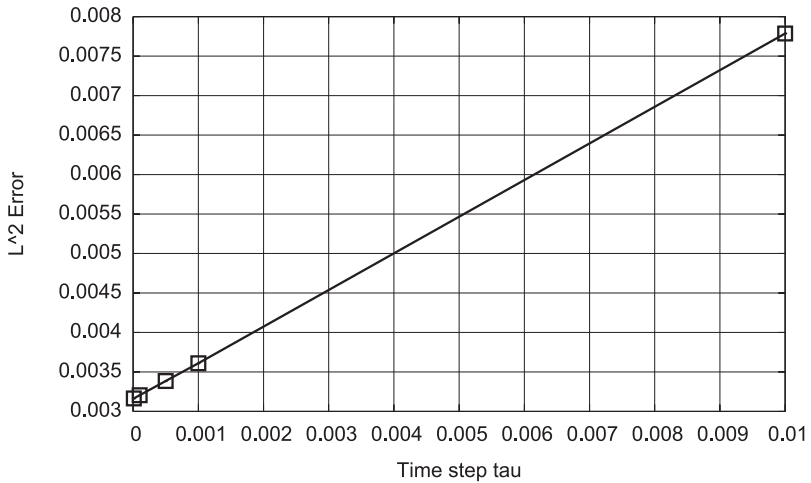

Figure 3. Relative $L^2([0, T]; L^2(\mathbb{R}))$ error as a function of the time step, $k = 1$ (KdV equation).

trate our rigorous convergence results, and we do not claim that the proposed scheme is particularly efficient. Still, observe that for $T = 5$ the decrease in the ℓ_h^2 norm is very small.

One advantage of the present method is that it allows direct control of the amount of dissipation my means of the parameter η in (4.1). We first note that
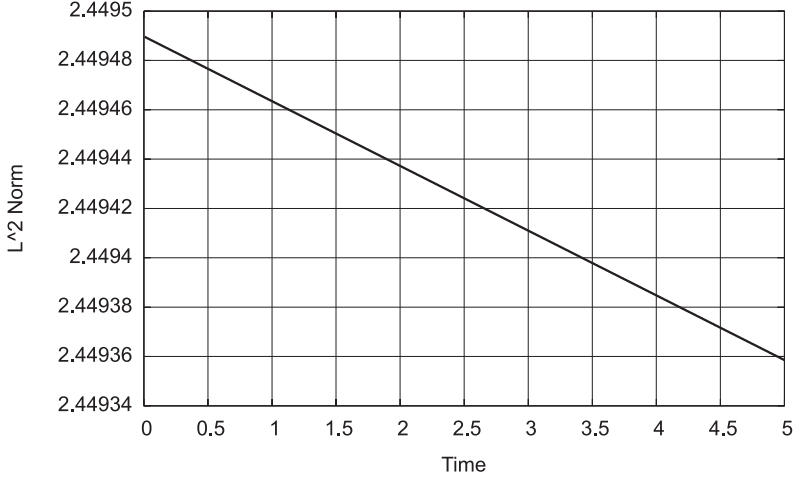
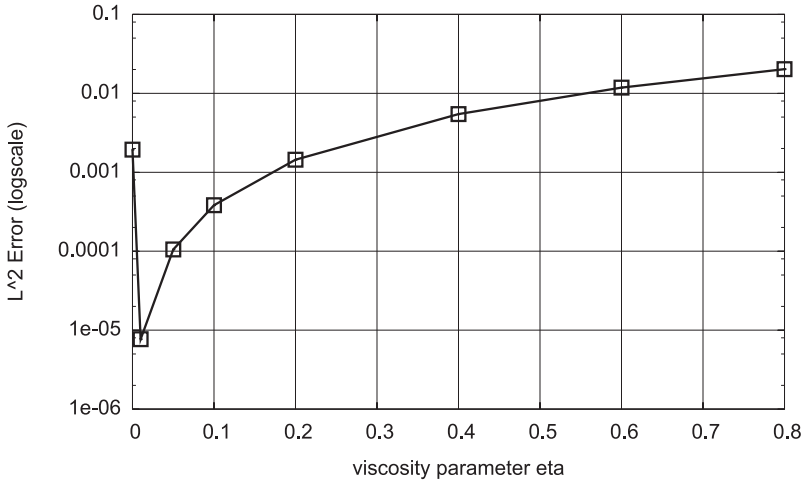Figure 4. ℓ_h^2 norm of the approximate solution.

Figure 5. Relative L^2 error at $T = 10$ as a function of the viscosity parameter η (KdV equation).

our convergence results remain valid for any $\eta > 0$ (but not for $\eta = 0$). As would be expected, reducing the value of η provides a sharper, less dissipative approximation. This is confirmed by our simulations, and in Figure 5 we present the L^2 error at $T = 10$ for various values of η. Interestingly, setting $\eta = 0$ sometimes provides a very good approximation, but not always, which is perhaps a consequence of the instability of the scheme without dissipation.

It would be of interest, but falls outside the scope of the present paper, to investigate the behavior of the present scheme if in (4.1) the factor h multiplying the double laplacian is replaced by some power h^α. One can then endeavor to find the optimal power α, as was done by Sepúlveda [24] for a related problem.

5. On an open question of Y. Tsutsumi

In [25], the Cauchy problem for the KdV equation (1.1) with measure initial data is considered. In that work, the author addresses the open question of uniqueness of solution to the Cauchy problem for the KdV equation with measure initial data in the following way.

It is well known that a solution of the mKdV equation with L^2 initial data may be transformed, by the Miura transform $u \mapsto \mathcal{M}(u) = \partial_x u + u^2$, into a solution of the KdV equation with a measure as initial data. Now, the family of functions

$$u_c^0(x) = \begin{cases} \dfrac{c+1}{(c+1)x + c}, & x > 0, \\[2mm] \dfrac{1}{x+c}, & x < 0, \end{cases} \tag{5.1}$$

with $c \leq -1$, all verify $\mathcal{M}(u_c^0) = \delta(0)$, where δ denotes the Dirac delta. Therefore, if $u_c(x,t)$ is the solution of the mKdV equation with initial data $u_c^0(x)$, the question arises whether the Miura transform maps each of these different solutions to the same solution of the KdV equation with $\delta(0)$ as initial data, or if, on the contrary, $\mathcal{M}(u_c(x,t))$ varies with c, which would establish non-uniqueness. If the latter case is observed numerically, it would support the conjecture that the Cauchy problem for the KdV equation with measure initial data *does not* enjoy the uniqueness property.

We have investigated this question numerically, and found that our numerical experiments support this lack of uniqueness conjecture. Thus, we have considered the mKdV equation with initial data given by u_c^0 (see (5.1)) and Dirichlet boundary conditions for various values of $c \leq -1$, computed the solution $u_c(x,t)$ up to some time $T > 0$, applied the Miura transform $\mathcal{M}(u_c(x,t))$, and finally compared the solutions obtained.

We have observed a clear dependence of $\mathcal{M}(u_c(x,t))$ as $c \leq -1$ varies, see Figure 6. This provides strong numerical evidence in support of a non-uniqueness property for the KdV equation with measure initial data and also a non-trivial test of the robustness of our numerical method: recall that the initial data (5.1) are discontinuous functions in L^2 only.

Miura transform of solution, c=-1, c=-5, c=-20, c=-500, 30000 points, dt = 0.001, h = 1/30, T = 5 (zoom)
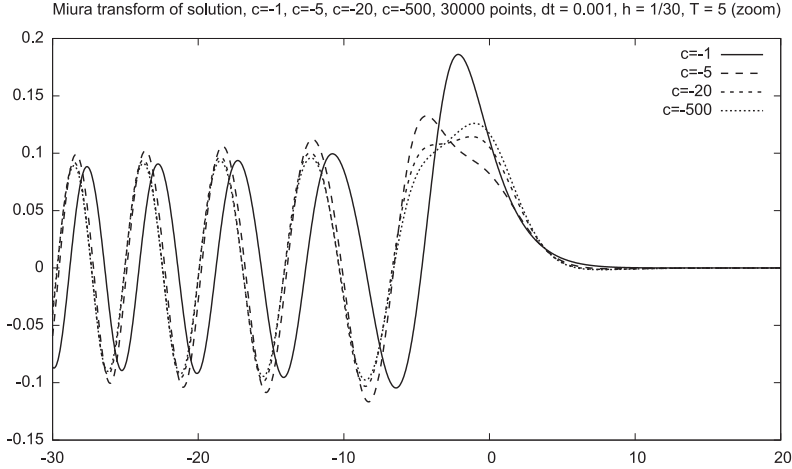

Figure 6. Miura transform of solution for various values of c. $T = 5$, 30000 spatial points, $\tau = 0.001$.

Note that the simulations in Figure 6 were computed with an accuracy of 30000 spatial points. In this case, the boundary conditions are a delicate problem, since the initial data decays only as x^{-1}. For this reason, we chose the large computational domain $[-500, 500]$, giving a value of $h = 1/30$. We have also performed the computations with a coarser grid of 5000 points, and have found that the variation in the results does not affect the overall qualitative aspect of the solution. In other words, the lack of uniqueness conjecture is supported by our numerical experiments.

We have verified as well that the result does not depend on the viscosity parameter η appearing in (4.1). The simulations presented take $\eta = 0.001$, but considering larger values of η (up to $\eta = 0.1$) gives virtually indistinguishable results.

In fact, it is easy to check that the more general family

$$u_{c,\varepsilon}^0(x) = \begin{cases} \dfrac{c + \varepsilon}{(c + \varepsilon)x + c}, & x > 0, \\ \dfrac{1}{x + c}, & x < 0 \end{cases} \tag{5.2}$$

verifies $\mathcal{M}(u_{c,\varepsilon}^0) = N(c,\varepsilon)\delta_0$, with $N = (c + \varepsilon - 1)/c$. The same remarks about uniqueness apply, and so as a last test we have carried out simulations with $(c, \varepsilon) = (-1/4, 1/4)$ and $(c, \varepsilon) = (-1, -2)$ (for which $N(c, \varepsilon) = 4$), performing the same comparison of the Miura transform of the computed solutions.

For these simulations we have taken a fine grid of 50000 spatial points, which corresponds to $h = 0.02$. The viscosity parameter is $\eta = 0.001$. In Figure 7 we plot the Miura transform of the solution for two different values of (c, ε), with $T = 10$, and 10000 and 50000 spatial points for each value of (c, ε).

Again, some variation with h is observed for the same values of (c, ε), which is natural since the scheme includes dissipation. But the main thing to note are the appearance of two distinct solutions, one for each set of values of the pair (c, ε), clearly apparent in Figure 7. The same distinction between the two solutions is also apparent for intermediate values of the number of grid points, whose solutions are seen to lie smoothly between the ones presented here.

Moreover, in an effort to show that the boundary effects are negligible, we performed very precise simulations reported in Figures 8 and 9 with an accuracy of 150000 spatial points on the interval $[-1000, 1000]$, and $\tau = 0.0001$. As far as we can tell, the boundary effects remain negligible. Further confirmation is given in Figure 10. Here, we compare the approximation in Figure 8 with an approximation of the same problem on the smaller interval $[-500, 500]$ and with only 10000 spatial points (thus, with zero boundary conditions at different points). As can be seen form the figure, the two solutions are clearly similar, suggesting that boundary effects contribute little to the solution.

We can therefore conclude that our numerical experiments strongly indicate lack of uniqueness for the Cauchy problem for the KdV equation with a measure initial data.
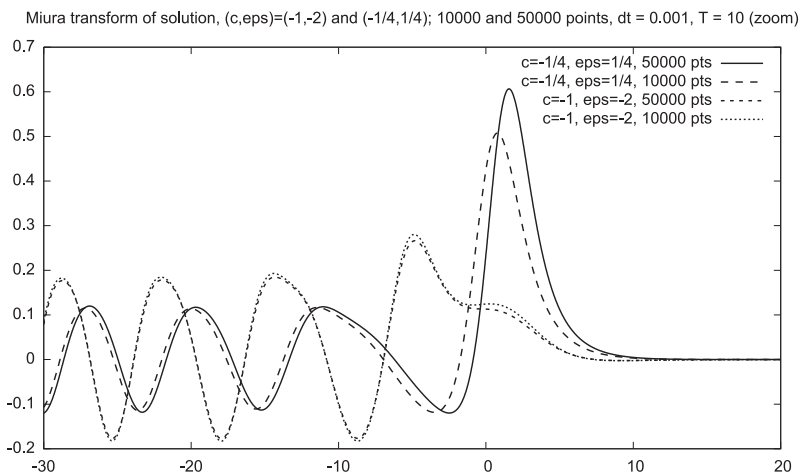

Figure 7. Miura transform of solution for various values of c, ε. $T = 10$, 10000 and 50000 spatial points. $\tau = 0.001$.
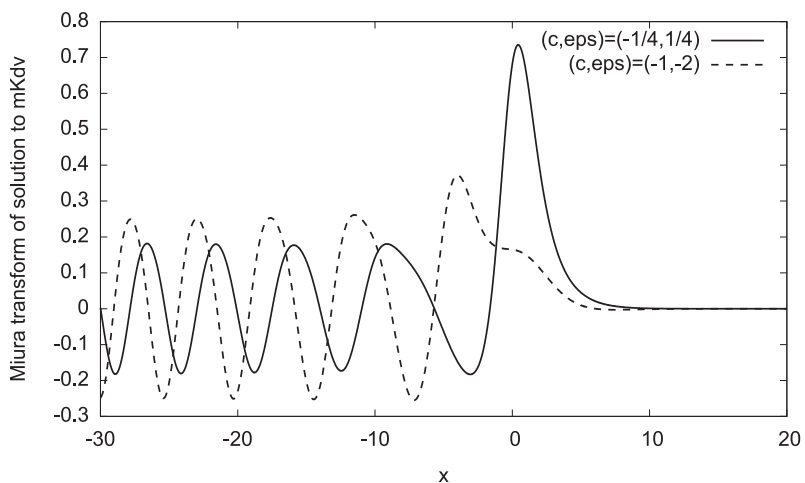
Figure 8. Miura transform of solution for various values of c, ε, $x \in [-1000, 1000]$ (zoom). $T = 5$, 150,000 spatial points. $\tau = 0.0001$.
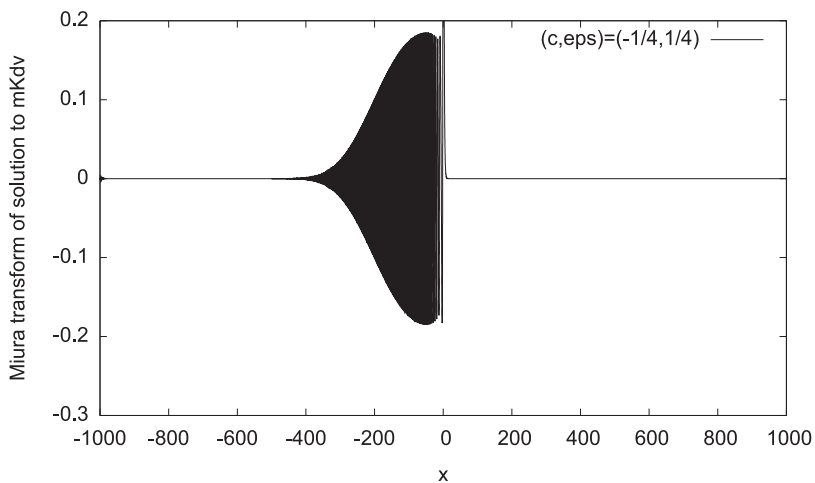

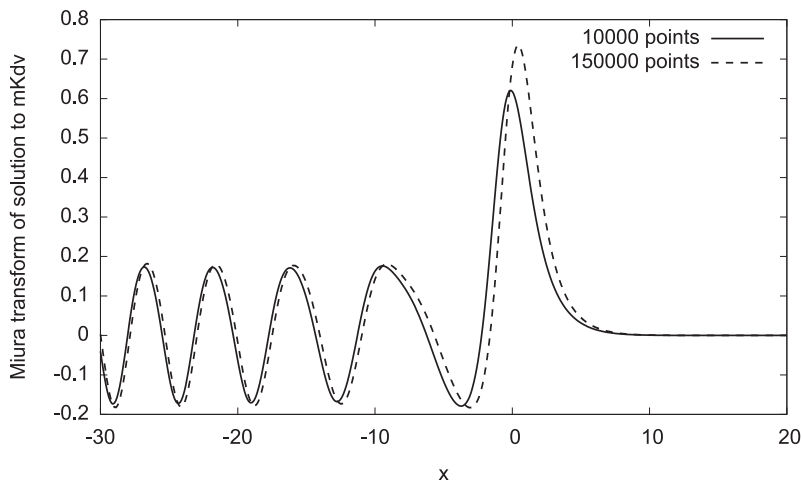Figure 9. Miura transform of solution (global view). $T = 5$, 150,000 spatial points. $\tau = 0.0001$.

Figure 10. Miura transform of solution, comparison between 150,000 points on $[-1000, 1000]$ ($h = 0.0133$) and 10000 points on $[-500, 500]$ ($h = 0.1$), T $= 5$, $\tau = 0.0001$.

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