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When every principal ideal is flat

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Abstract. This paper deals with the well-known notion of PF-rings, that is, rings in which principal ideals are flat. We give a new characterization of PF-rings. Also we provide a necessary and sufficient condition for $R \bowtie I$ (resp. R/I when R is a Dedekind domain or I is a primary ideal) to be a PF-ring. The article includes a brief discussion of the scope and precision of our results.

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1. Introduction

Throughout this work, all rings are commutative with identity element, and all modules are unitary. We start by recalling some definitions.

A ring R is called a PF-ring if principal ideals of R are flat in [12]. Recall that R is a PF-ring if and only if R_Q is a domain for every prime (resp. maximal) ideal Q of R. For example, any domain, any ring R with $\operatorname{wgl.dim} R \leq 1$, and any semihereditary ring is a PF-ring (since a localization of a ring R with $\operatorname{wgl.dim} R \leq 1$ (resp. semihereditary) is locally a domain). Note that a PF-ring is reduced by [11], Theorem 4.2.2, p. 114. See for instance [11], [12], [15].

An R-module M is called P-flat if $x \in (0:s)M$ for any $(s,x) \in R \times M$ such that $sx = 0$, where $(0:s) = \operatorname{Ann}_R(s)$. If M is flat, then M is naturally P-flat. When R is a domain, M is P-flat if and only if it is torsion-free. When R is an arithmetical ring, then any P-flat module is flat (by [5], p. 236). Also, every P-flat cyclic module is flat (by [5], Proposition 1 (2)). See for instance [5], [11].

The amalgamated duplication of a ring R along an ideal I is a ring that is defined as the following subring with unit element $(1,1)$ of $R \times R$:

$$R \bowtie I = \{(r, r+i) \mid r \in R, i \in I\}.$$

This construction was studied in the general case and from the different point of view of pullbacks by D'Anna and Fontana [8]. Also, in [7], they considered the case of amalgamated duplication of a ring, along a multiplicative canonical ideal in the sense of [14], in a not necessarily Noetherian setting. In [6] D'Anna studied some properties of $R \bowtie I$ to construct reduced Gorenstein rings associated with Cohen–Macaulay rings, and applied this construction to curve singularities. On the other hand, Maimani and Yassemi in [18] studied the diameter and girth of the zero-divisor graph of the ring $R \bowtie I$. Some references are [7], [8], [9], [10], [18].

Let A and B be rings and let $\varphi : A \to B$ be a ring homomorphism making B an A-module. We say that A is a module retract of B if there exists a ring homomorphism $\psi : B \to A$ such that $\psi o \varphi = \mathrm{id}_A$. The homomorphism ψ is called retraction of φ. See for instance [11].

Our first main result in this paper is Theorem 2.1, which provides a new characterization of PF-rings. Also we provide a necessary and sufficient condition for $R \bowtie I$ (resp., R/I when R is a Dedekind domain or I is a primary ideal) to be a PF-ring. The results produce new and original examples of new families of PF-rings with zero-divisors.

2. Main results

Recall that an R-module M is called P-flat if $x \in (0:s)M$ for any $(s,x) \in R \times M$ with $sx = 0$. Now we give a new characterization for a class of PF-rings, which is the first main result of this paper.

Theorem 2.1. *Let R be a commutative ring. Then the following conditions are equivalent:*

(1) *Every ideal of R is P-flat.*

(2) *Every principal ideal of R is P-flat.*

(3) *R is a PF-ring, that is, every principal ideal of R is flat.*

(4) *For any elements $(s,x) \in R^2$ with $sx = 0$ there exists $\alpha \in (0:s)$ such that $x = \alpha x$.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) follows from [5], Proposition 1 (2).

(3) $\Rightarrow$ (4). Let (s,x) be an element of R^2 such that $sx = 0$. Our aim is to show that there exists $\beta \in (0:s)$ such that $x = \beta x$. The principal ideal generated by x is P-flat (since it is flat), so there exists $\alpha \in (0:s)$ and $r \in R$ such that $x = \alpha r x = \beta x$ with $\beta = \alpha r \in (0:s)$.

(4) $\Rightarrow$ (1). Let I be an ideal of R and let $(s, x) \in R \times I$ such that $sx = 0$. Then there exists $\alpha \in (0 : s)$ such that $x = \alpha x$ and so $x \in (0 : s)I$. Therefore, I is P-flat, as desired. □

Corollary 2.2. *Let R be a ring. The following conditions are equivalent:*

(1) *Every ideal of R is P-flat.*

(2) *Every ideal of R_Q is P-flat for every prime ideal Q of R.*

(3) *Every ideal of R_m is P-flat for every maximal ideal m of R.*

(4) *R_Q is a domain for every prime ideal Q of R.*

(5) *R_m is a domain for every maximal ideal m of R.*

Proof. This is a consequence of Theorem 2.1, Lemma 3.1 [19], and [11], Theorem 4.2.2. □

Recall that a ring R is called an arithmetical ring if the lattice formed by its ideals is distributive and is said to have weak global dimension ≤ 1 ($\mathrm{wgl.dim}(R) \leq 1$) if every finitely generated ideal of R is flat. If $\mathrm{wgl.dim}(R) \leq 1$, then R is an arithmetical ring. See for instance [2], [3], [12]. In the domain context, all these forms coincide with the definition of a Prüfer domain.

Now we add a condition with arithmetical in order to have equivalence between arithmetical and $\mathrm{wgl.dim}(R) \leq 1$.

Proposition 2.3. *Let R be a ring. Then the following conditions are equivalent:*

(1) $\mathrm{wgl.dim}(R) \leq 1$.

(2) *R is arithmetical and a PF-ring.*

(3) *R is arithmetical and every principal ideal of R is flat.*

(4) *R is arithmetical and every principal ideal of R is P-flat.*

(5) *R is arithmetical and every ideal of R is P-flat.*

Proof. (1) $\Rightarrow$ (2). Assume that $\mathrm{wgl.dim}(R) \leq 1$. Then R is arithmetical by [13], Theorem 3.2.1. Let I be an ideal of R. As $\mathrm{wgl.dim}(R) \leq 1$, every finitely generated subideal of I is flat. Hence I is flat by [21], Proposition 3.48, and I is P-flat.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) by Theorem 2.1.

(5) $\Rightarrow$ (1). Assume that the ring R is arithmetical and every ideal of R is P-flat. Our aim is to show that $\mathrm{wgl.dim}(R) \leq 1$. Let I be a finitely generated ideal of R. Then I is P-flat, and so I is flat (since R is arithmetical by [5], p. 236), which completes the proof. □

Now we show that the localization of a PF-ring is always a PF-ring.

Proposition 2.4. *Let R be a PF-ring and let S be a multiplicative subset of R. Then, $S^{-1}(R)$ is a PF-ring.*

Proof. This is straightforward by [19], Lemma 3.1. □

Now we study the transfer of the PF-ring property to the direct product.

Proposition 2.5. *Let $(R_i)_{i\in\Lambda}$ be a family of commutative rings. Then $R = \prod_{i\in\Lambda} R_i$ is a PF-ring if and only if R_i is a PF-ring for all $i \in \Lambda$.*

Proof. Straightforward. □

Next we study the transfer of the PF-ring property to homomorphic images. First, the following example shows that the homomorphic images of a PF-ring is not always a PF-ring.

Example 2.6. Let A be a domain, X an indeterminate and let $R = A[X]$. Then

(1) R is a PF-ring since it is a domain,

(2) $R/(X^n)$ (for $n \geq 2$) is not a PF-ring since $\overline{X^n} = 0$ and $\overline{X} \neq 0$.

Recall that if R is a Dedekind domain and I is a nonzero ideal of R, then, by a celebrated Theorem by E. Noether, $I = P_1^{\alpha_1} \ldots P_n^{\alpha_n}$ for some distinct prime ideals $P_1, \ldots, P_n$ uniquely determined by I and some positive integers $\alpha_1, \ldots, \alpha_n$ uniquely determined by I (see also [16], Theorem 3.14).

Now, when R is a Dedekind domain or I is a primary ideal, we give a characterization of R and I such that R/I is a PF-ring.

Theorem 2.7. *Let R be a ring and let I be an ideal of R. Then:*

(1) *Assume that R is a Dedekind domain and $I = P_1^{\alpha_1} \ldots P_n^{\alpha_n}$ is a non-zero ideal of R, where $P_1, \ldots, P_n$ are the prime ideals defined by I. Then R/I is a PF-ring if and only if $\alpha_i = 1$ for all $i \in \{1, \ldots, n\}$.*

(2) *I is a primary ideal of R and R/I is a PF-ring if and only if I is a prime ideal of R.*

Proof. (1) Let R be a Dedekind domain and let $I = P_1^{\alpha_1} \ldots P_n^{\alpha_n}$ for $P_1, \ldots, P_n$ be a nonzero prime ideals of R. Then $R/I = \prod_{i=1}^n (R/P_i^{\alpha_i})$.

Assume that $\alpha_i = 1$ for all $1 \leq i \leq n$. Hence R/P_i is a PF-ring since R/P_i is an integral domain, and so $R/I = \prod_{i=1}^n (R/P_i^{\alpha_i})$ is a PF-ring by Proposition 2.5.

Conversely, assume that $R/I = \prod_{i=1}^n (R/P_i^{\alpha_i})$ is a PF-ring. Let $i \in \{1, \ldots, n\}$. Then $R/P_i^{\alpha_i}$ is a PF-ring by Proposition 2.5. Hence $R/P_i^{\alpha_i}$ is reduced, and so the intersection of all prime ideals Q of $R/P_i^{\alpha_i}$ is zero (i.e., $\bigcap_{Q\in \mathrm{Spec}(R/P_i^{\alpha_i})} Q = \{0\}$)

by [1], Proposition 1.8. On the other hand, for every prime ideal Q of $R/P_i^{\alpha_i}$ there exists a prime ideal Q' of R such that $P_i^{\alpha_i} \subset Q'$ and $Q = Q'/P_i^{\alpha_i}$. Thus $P_i/P_i^{\alpha_i} \subset Q$. It follows that $\{0\} = \bigcap_{Q \in \mathrm{Spec}(R/P_i^{\alpha_i})} Q = P_i/P_i^{\alpha_i}$ and so $P_i = P_i^{\alpha_i}$. Since R is Dedekind domain, $\alpha_i = 1$.

(2) It is obvious that if I is a prime ideal, then R/I is a PF-ring and I is a primary ideal.

Conversely, assume that I is a primary ideal and R/I is a PF-ring. Our aim is to show that I is a prime ideal of R. Let $(x, y) \in R^2$ such that $xy \in I$. We claim that $x \in I$ or $y \in I$. Without loss of generality, we may assume that $x \notin I$. Since $xy \in I$, there exists an integer $n > 0$ such that $y^n \in I$ (since I is a primary ideal). Thus $\bar{y}^n = 0$ and so $\bar{y} = 0$ since R/I is a PF-ring. Then $y \in I$. Therefore, $x \in I$ or $y \in I$, and so I is a prime ideal of R, as desired. □

As a consequence of Theorem 2.7 (1) we are able to give examples of PF-rings and non-PF-rings.

Example 2.8. (1) $\mathbb{Z}/4\mathbb{Z}$ is not a PF-ring.

(2) $\mathbb{Z}/30\mathbb{Z}$ is a PF-ring.

Now we study the transfer of a PF-property to an amalgamated duplication of a ring R along an ideal I.

Let R be a ring. An ideal I of R is called a pure submodule of R if for every R module M the sequence $0 \to I \otimes_R M \to R \otimes_R M \to R/I \otimes_R M \to 0$ is exact; equivalently, $I_m = 0$ or R_m for any maximal ideal m of R.

Theorem 2.9. *Let R be a ring, and let I be an ideal of R. Then $R \bowtie I$ is a PF-ring if and only if R is a PF and I is pure.*

We need the following lemma before proving this theorem.

Lemma 2.10. *Let R and S be a rings and let $\varphi : R \to S$ be a ring homomorphism making R a module retract of S. If S is a PF-ring, then so is R.*

Proof. Let $\varphi : R \to S$ be a ring homomorphism and let $\psi : S \to R$ be a ring homomorphism such that $\psi o \varphi = \mathrm{id}_R$. Let $(x, y) \in R^2$ such that $xy = 0$. Then $\varphi(x)\varphi(y) = \varphi(xy) = 0$. Hence there exists an element $\alpha \in S$ such that $\alpha\varphi(x) = 0$ and $\varphi(y) = \alpha\varphi(y)$ (since S is a PF-ring) and so $y = \psi(\varphi(y)) = \psi(\alpha\varphi(y)) = \psi(\alpha)y$ and $\psi(\alpha)x = \psi(\alpha\varphi(x)) = \psi(0) = 0$, as desired. □

Proof of Theorem 2.9. Assume that $R \bowtie I$ is a PF-ring. We must to show that R is a PF-ring and I is a pure ideal of R. We can easily show that R is a module retract of $R \bowtie I$, where the retraction map φ is defined by $\varphi(r, r + i) = r$, and so R is a PF-ring by Lemma 2.10.

We claim that $I_m \in \{0, R_m\}$ for every maximal ideal m of R. Let m be an arbitrary maximal ideal of R. Then $I \subseteq m$ or $I \nsubseteq m$. If $I \nsubseteq m$, then $I_m = R_m$. If $I \subseteq m$, assume by contradiction that $I_m \notin \{0, R_m\}$ and so $(R \bowtie I)_M = R_m \bowtie I_m$, where M a maximal ideal of $R \bowtie I$ such that $M \cap R = m$. Since R_m is a domain, $R_m \bowtie I_m$ is reduced and $O_1 (= \{0\} \times I_m)$ and $O_2 (= I_m \times \{0\})$ are the only minimal prime ideals of $(R \bowtie I)_M$ by [8], Proposition 2.1. Hence it is not a PF-ring by [11], Theorem 4.2.2 (since $(R \bowtie I)_M$ is local), the desired contradiction. Therefore, $I_m \in \{0, R_m\}$ for every maximal ideal m of R.

Conversely, assume that R is a PF-ring and I is a pure ideal of R, i.e., $I_m \in \{0, R_m\}$ for every maximal ideal m of R. Our aim is to prove that $R \bowtie I$ is a PF-ring. Using Corollary 2.2, we need to prove that $(R \bowtie I)_M$ is a PF-ring whenever M is a maximal ideal of $R \bowtie I$. Let M be an arbitrary maximal ideal of $R \bowtie I$ and set $m = M \cap R$. Then $M \in \{M_1, M_2\}$, where $M_1 = \{(r, r+i) \mid r \in m, i \in I\}$ and $M_2 = \{(r+i, r) \mid r \in m, r \in I\}$, by [7], Theorem 3.5. On the other hand, $I_m \in \{0, R_m\}$. Then, testing all cases of [6], Proposition 7, we have two cases:

(a) $(R \bowtie I)_M \cong R_m$ if $I_m = 0$ or $I \nsubseteq m$.

(b) $(R \bowtie I)_M \cong R_m \times R_m$ if $I_m = R_m$ and $I \subseteq m$.

Since R_m is a PF-ring (by Corollary 2.2), so is $R_m \times R_m$ by Proposition 2.5 and hence $(R \bowtie I)_M$ is a PF-ring. □

Corollary 2.11. *Let R be a domain and let I be a proper ideal of R ($I \neq R$ and $I \neq 0$). Then $R \bowtie I$ is never a PF-ring.*

Corollary 2.12. *Let (R, m) be a local ring and let I be a proper ideal of R ($I \neq R$ and $I \neq (0)$). Then $R \bowtie I$ is never a PF-ring.*

Now we are able to construct a class of PF-rings.

Example 2.13. Let R be a PF-ring and let $I = Re$, where e is an idempotent element of R. Then $R \bowtie I$ is a PF-ring by Theorem 2.9.

The following example shows that a subring of PF-ring is not always a PF-ring. For any ring R, we denote by $T(R)$ the total ring of quotients of R.

Example 2.14. Let R be an integral domain, I a proper ideal of R and let $S = R \bowtie I$. Then:

(1) $S (= R \bowtie I)$ is not a PF-ring by Corollary 2.11.

(2) $R \bowtie I \subseteq R \times R$ and $R \times R$ is a PF-ring by Proposition 2.5 (since R is a PF-ring).

(3) $T(S) = T(R \times R) = K \times K$, where $K = T(R)$.

We end this paper by showing that the transfer of the PF-ring property to pullback is not always a PF-ring.

Example 2.15. Let R be a domain and I a proper ideal of R. Then:

(1) The ring $R \bowtie I$ can be obtained as a pullback of R and $R \times R$ over $R \times (R/I)$.

(2) The ring $R \bowtie I$ is not a PF-ring by Corollary 2.11.

(3) The rings R and $R \times R$ are PF-rings.

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