Rees quotients of numerical semigroups

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Abstract. We introduce a class of finite semigroups obtained by considering Rees quotients of numerical semigroups. Several natural questions concerning this class, as well as particular subclasses obtained by considering some special ideals, are answered while others remain open. We exhibit nice presentations for these semigroups and prove that the Rees quotients by ideals of $\mathbb{N}$, the positive integers under addition, constitute a set of generators for the pseudovariety of commutative and nilpotent semigroups.

1. Introduction and motivation

A *numerical semigroup* is a co-finite subsemigroup of the non-negative integers, under addition. It is well known that a numerical semigroup has a (unique) minimal set of generators, which is finite. The smallest integer from which all the integers belong to a numerical semigroup is called the *conductor* of that semigroup. The (finite) set of elements of the semigroup not greater than the conductor, named *small elements*, also determines the semigroup. We just mentioned two (in general) different finite sets of integers that determine a given numerical semigroup but others could be considered. This motivates the following somehow vague question, which has been the starting point for the research presented in this paper:

Question 1.1. *Can a numerical semigroup be thought as a finite semigroup?*

We consider this question imprecise due to the usual fact that one does not distinguish between isomorphic semigroups. In the above motivation, concrete finite sets determining the numerical semigroups have been considered.

The following is a particular case of the approach we have taken. Given a numerical semigroup S we choose a cutting point $k \in S$ and a distinguished element ∞ not in S. Let $F = \{s \in S \mid s \leq k\}$ and take $Q = F \cup \{\infty\}$. In Q we define a (commutative and associative) operation as follows: $\infty \oplus q = q \oplus \infty = \infty$, for any $q \in Q$; for $a, b \in F$, $a \oplus b = a + b$ if $a + b \in F$ and $a \oplus b = \infty$ otherwise. The set F is called the *finite part* of the finite commutative semigroup Q. Notice that Q is a Rees quotient of S by an ideal (formed by the elements of S that are greater than k), which justifies the use of the terminology *Rees quotient semigroup*.

One could be tempted to ask whether taking as cutting point an integer greater than any of the minimal generators, or greater than the conductor, the finite quotient obtained by the above construction contains sufficient data to determine the numerical semigroup and somehow answering positively Question 1.1. This is far from being the case: we can have more than one numerical semigroup giving rise to the same quotient (up to isomorphism), as many of the forthcoming examples show.

This paper is written as follows: after this brief section mainly devoted to the motivation for the research presented, we give the main definitions and introduce the notation to be used. We proceed with a section containing examples and simple remarks that intend to give answers to several questions that can naturally be raised once one wants to consider the class of finite semigroups introduced in this paper: Rees quotients of numerical semigroups. Next we obtain a nice presentation for a Rees quotient numerical semigroup once it is known the defining pair (numerical semigroup, ideal). Observe that the study of presentations appears naturally in both the theories of numerical semigroups and of finite semigroups. In particular, classifying finite semigroups into pseudovarieties is one of the main subjects of study in finite semigroup theory. In Section 5 we determine the pseudovariety generated by the Rees quotients of numerical semigroups (in fact, the quotients of $\mathbb{N}$ suffice): it is the class of finite commutative nilpotent semigroups. In a final section we raise several questions for which we have not been able to get answers and left them as open problems, pointing out that some research work on this subject can be pursued.

2. Definitions and notation

Our reference for numerical semigroups is a book by Rosales and García-Sánchez [7]. One may use the GAP [9] package [3] for computations with numerical semigroups so as with (relative) ideals. Other relevant reference for this paper is [1]. It

contains everything we need on finite semigroups, namely the result on pseudo-varieties of commutative semigroups used to prove Theorem 5.1. For general background on semigroup theory we refer the reader to [6].

Except for numerical semigroups or their quotients, as usual, we always assume multiplicative notation.

2.1. Numerical Semigroups.

In this paper, for convenience, we shall use the terminology *numerical semigroup* for a co-finite subsemigroup of the non-negative integers, under addition. Traditionally, it is required that a numerical semigroup contains the 0, which works as an identity, and therefore numerical semigroups are monoids. The terminology "numerical monoid" also appears in the literature. This makes usually no difference in the theory development. Despite, the following notation is useful: given a numerical semigroup S, we denote by S^0 the monoid obtained from S by adjoining the integer 0 (which is the identity). In particular, denoting by $\mathbb{N}$ the semigroup of positive integers, $\mathbb{N}^0$ denotes the monoid of non-negative integers under addition.

Given a subsemigroup S of $\mathbb{N}$, let $d = \gcd(S)$. It is easy to see that $\left\{ \frac{s}{d} \mid s \in S \right\}$ is a co-finite subsemigroup of $\mathbb{N}$ (thus a numerical semigroup) that is isomorphic to S. Therefore, each isomorphism class of the set of subsemigroups of $\mathbb{N}$ contains a numerical semigroup. That contains exactly one (which implies that different numerical semigroups are non isomorphic) is well known and is a consequence of the following proposition which may be seen as a direct proof of this fact and is stated here for the sake of completeness.

Proposition 2.1. *Let $\varphi : S \to T$ be a surjective homomorphism from a numerical semigroup S to a numerical semigroup T. Then $S = T$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of generators of S. In particular, $\gcd(A) = 1$. Since φ is surjective, then $\varphi(A)$ is a set of generators of T and therefore, as T is a numerical semigroup, we obtain $\gcd(\varphi(A)) = 1$.

For each $i \in \{1, \ldots, n\}$, we have $a_1\varphi(a_i) = \varphi(a_1 a_i) = \varphi(a_i a_1) = a_i\varphi(a_1)$. It follows that $a_1 \mid a_i\varphi(a_1)$, for all $i \in \{1, \ldots, n\}$. Since $\gcd(a_1, a_2, \ldots, a_n) = 1$, we get that $a_1 \mid \varphi(a_1)$ and that $\frac{\varphi(a_1)}{a_1} \mid \varphi(a_i)$, for any $i \in \{1, \ldots, n\}$. This implies that $\frac{\varphi(a_1)}{a_1} \mid \gcd(\varphi(A))$ and so $a_1 = \varphi(a_1)$. Consequently $a_i = \varphi(a_i)$, for any $i \in \{1, \ldots, n\}$. Thus $S = T$, as required. $\square$

2.2. Notable elements.

Let S be a numerical semigroup. The greatest integer not belonging to S is called the *Frobenius number* of S and denoted $F(S)$. The successor of the Frobenius number is called the *conductor* of S. It is the least element of S such that all the integers greater than it belong to S and is denoted by $c(S)$. The least (positive) element of S, which is also the minimum of the unique minimal set of generators, is called the *multiplicity* of S and is denoted by

$m(S)$. The *embedding dimension* of S is the cardinality of the minimal generating set of S and is denoted by $e(S)$.

The positive integers that do not belong to S are called the *gaps* of S and the number of such elements is called the *genus* of S and denoted $g(S)$.

2.3. Ideals. When studying numerical semigroups it is common to consider relative ideals (see [2]). Given a numerical semigroup S, a relative ideal I_S of S has a (unique) finite minimal *ideal generating system*, say G, such that $I_S = G + S^0$. Since we are interested in forming quotients, we are only interested in those relative ideals that are contained in their ambient semigroups (and which are in fact the semigroup ideals). As in [7], these are called *ideals*. Notice that the ideals of a numerical semigroup S are precisely those relative ideals whose minimal ideal generating system is contained in S. Given a positive integer k, the set $I_k(S) = \{x \in S \mid x \geq k\}$ is an ideal of S; we say that it is an *ideal determined by the cutting point* k. When the ambient semigroup is understood, we write I (respectively I_k) for I_S (respectively $I_k(S)$).

Let S be a numerical semigroup with multiplicity m and conductor c. Observe that if $k \geq c(S)$, then $I_k = \{x \in \mathbb{N} \mid x \geq k\}$. The minimal generating system of the numerical semigroup I_k is $\{k, k+1, \ldots, 2k-1\}$, while the minimal ideal generating system of $I_k(S)$ may be smaller. In fact, we have

$$I_k(S) = \{k, k+1, \ldots, k+m-1\} + S^0.$$

Moreover, when $k \geq c(S)$, we can easily conclude that $\{k, k+1, \ldots, k+m-1\}$ is the minimal ideal generating system of I_k.

2.4. Rees quotients. Given a numerical semigroup S and an ideal I_S we can form the Rees quotient

$$\texttt{rqns}(S, I_S) = S/I_S,$$

which is obtained from S by identifying all elements of I_S to a distinguished element.

Throughout this paper we will call *Rees quotient numerical semigroup* (`rqns`, for short) any semigroup that is isomorphic to a Rees quotient of a numerical semigroup by an ideal.

The distinguished element considered in the construction of a `rqns` is an absorbing element, usually named the *zero* in multiplicative language, and will be denoted by ∞. Despite we assume the additive notation for a `rqns`, we still designate its element ∞ by *zero*. The non-zero elements of a `rqns` are said to be *finite* and the set of finite elements is said to be the *finite part* of the `rqns`.

We denote by RQNS the class of all Rees quotient numerical semigroups.

2.5. Nilpotency. Let S be a finite semigroup. Let n be a positive integer. As usual, we write S^n for $\{s_1 \ldots s_n \,|\, s_1, \ldots, s_n \in S\}$. We say that S is *nilpotent* if $|S^n| = 1$, for some positive integer n. The least such n is called the *nilpotency class* of the nilpotent semigroup S. As usual when dealing with finite semigroups, x^ω denotes the idempotent that is a power of x, for any element x of the semigroup. The following fact is well known.

Fact 2.2. *Let S be a semigroup with zero. The following conditions are equivalent:*

  i) *S is nilpotent;*

 ii) *S satisfies an equation of the form $x_1 \ldots x_n = 0$, for some $n \in \mathbb{N}$;*

iii) *S satisfies an equation of the form $x^n = 0$, for some $n \in \mathbb{N}$;*

iv) *S satisfies the pseudoequation $x^\omega = 0$.*

Notice that, being z a (pseudo)word, the expression $z = 0$ is just an abbreviation for the (pseudo)equations $zy = yz = z$, with y a fixed variable not occurring in z.

The following is immediate.

Remark 2.3. A `rqns` is a nilpotent finite commutative semigroup. Furthermore, if S is a numerical semigroup with multiplicity m and conductor c, then the nilpotency class of $\texttt{rqns}(S, I_c)$ is the least positive integer k such that $km \geq c$.

2.6. Some classes of finite semigroups. A *pseudovariety* of semigroups is a class of finite semigroups closed under the formation of finite direct products, subsemigroups and homomorphic images.

Among the classes of finite semigroups considered in this paper are the classes N of nilpotent semigroups and Com of commutative semigroups. Remark 2.3 tells us that RQNS $\subseteq$ N $\cap$ Com.

Other classes also considered here are subclasses of RQNS, obtained by considering ideals determined by cutting points. Specifically, we consider

$$\text{CQNS} = \{S/I_k(S)\,|\,S \text{ is a numerical semigroup and } k \in \mathbb{N}\} \quad \text{and}$$
$$\text{C}\mathbb{N} = \{\mathbb{N}/I_k \,|\, k \in \mathbb{N}\}.$$

A semigroup in CQNS will be referred as a *quotient determined by the numerical semigroup S and the cutting point k*.

As follows from the definitions we have C$\mathbb{N}$ $\subseteq$ CQNS $\subseteq$ RQNS and therefore we have the following chain:

$$\text{C}\mathbb{N} \subseteq \text{CQNS} \subseteq \text{RQNS} \subseteq \text{N} \cap \text{Com}. \tag{1}$$

All the inclusions are strict, as we shall see in Section 3. In Section 5 we prove that the pseudovariety generated by the smallest of these classes, $C\mathbb{N}$, is $N \cap Com$, getting this way a nice set of generators for the pseudovariety $N \cap Com$. Before, in Section 4, we give presentations for semigroups of RQNS. Then, we devote Section 6 to open problems. Among these problems are possible definitions corresponding to the notable elements.

3. Examples and simple remarks

When we want to represent a set of integers that contains all the integers from, say, a_n on, we use the notation $\{a_1, \ldots, a_n^>\}$ instead of the more commonly used $\{a_1, \ldots, a_n, \rightarrow\}$ mainly because it is more compact and makes the table in Subsection 6.2 more readable. Singleton sets will be usually represented by the single element they contain. The exceptions occur when we want to stress out which are the elements in an equivalence class.

The following just intends to give a first example that helps gaining some intuition. We observe that the finite semigroup involved appears as quotient of two different numerical semigroups.

Example 3.1. Consider the following numerical semigroups, ideals and Rees quotients:

$$S = \langle 2, 5 \rangle, \quad I_S = \{6, 7\} + S = \{6^>\} \quad \text{and}$$
$$Q_1 = \texttt{rqns}(S, I_S) = \{\{2\}, \{4\}, \{5\}, \infty\};$$
$$T = \langle 3, 5 \rangle, \quad I_T = \{8, 9, 10\} + S = \{8^>\} \quad \text{and}$$
$$Q_2 = \texttt{rqns}(T, I_T) = \{\{3\}, \{5\}, \{6\}, \infty\}.$$

It is straightforward to observe that the function $\varphi : Q_1 \rightarrow Q_2$ defined by $\varphi(\{2\}) = \{3\}$, $\varphi(\{4\}) = \{6\}$, $\varphi(\{5\}) = \{5\}$ and $\varphi(\infty) = \infty$ is an isomorphism.

When a $\texttt{rqns}$ is defined by a numerical semigroup and a cutting point, there exists a numerical semigroup such that the cutting point is the conductor of the semigroup, as stated in the following remark.

Remark 3.2. Let S be a numerical semigroup and k a positive integer. There exists a numerical semigroup T such that $\texttt{rqns}(S, I_k) = \texttt{rqns}(T, I_{c(T)})$.

Proof. If $k \leq c(S)$, then take $T = S \cup \{n \in \mathbb{N} \mid n \geq k\}$. If $k > c(S)$, then take $T = 2S \cup \{n \in \mathbb{N} \mid n \geq 2k\}$. $\qquad\square$

Example 3.3. Let $S = \langle 3, 5 \rangle$ and take $k = 10$ as cutting point. We get $S/I_k = \{3, 5, 6, 8, 9, \infty\}$. One may take $T = \{6, 10, 12, 16, 18, 20^>\}$. As $c(T) = 20$, we get $T/I_{c(T)} = \{6, 10, 12, 16, 18, \infty\}$, which is isomorphic to S/I_k.

Notice that if a, b, c are finite elements of a `rqns` such that if $a + b = a + c$ is finite, then $b = c$. We state this property in the following proposition.

Proposition 3.4. *The cancellation law holds in the finite part of a* `rqns`.

This proposition is the main ingredient used in the examples of the following subsection which show that there exist quotients of numerical semigroups that are not Rees quotients by ideals, and that the direct product of Rees quotient numerical semigroups is not necessarily a `rqns`, respectively.

3.1. RQNS $\neq$ N $\cap$ Com. The following examples show that the class RQNS is not a pseudovariety. In fact, it fails to be closed under the formation of homomorphic images (Example 3.5) and under the formation of direct products (Example 3.6). The pictures show the structure in Green's $\mathscr{D}$-classes of the semigroups considered. The relevant elements are highlighted and the box containing the idempotent is marked by using a star.

Example 3.5. Let $S = \mathtt{rqns}(\langle 4, 5\rangle, \{12^{>}\}) = \{4, 5, 8, 9, 10, \infty\}$ and let θ be the congruence of S generated by the pair $(9, 10)$. Then, it is easy to verify that $S/\theta = \{\{4\}, \{5\}, \{8\}, \{9, 10\}, \{12^{>}\}\}$ ($\{12^{>}\}$ is an absorbing element). We have that $\{4\} + \{5\} = \{9, 10\} = \{5\} + \{5\}$, whence S/θ does not satisfy the "cancellation law" (Proposition 3.4) and therefore is not the Rees quotient of a numerical semigroup.
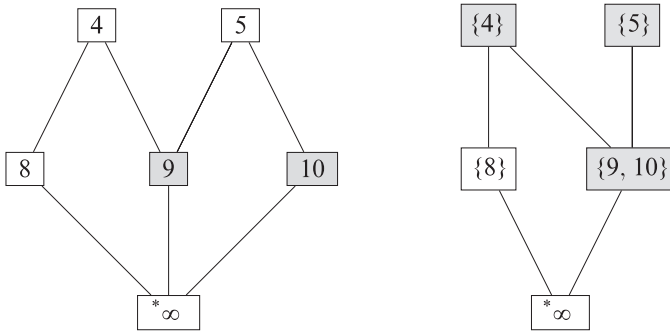
Example 3.6. Let $S = \mathtt{rqns}(\langle 2, 5\rangle, \{4^{>}\}) = \{2, \infty\}$ and $T = \mathtt{rqns}(\langle 2, 7\rangle, \{6^{>}\}) = \{2, 4, \infty\}$. Then, the direct product

$$S \times T = \{(2, 2), (\infty, 2), (2, 4), (\infty, 4), (2, \infty), (\infty, \infty) = \infty\}$$

is not isomorphic to a `rqns`. In fact, we have $(2, 2) + (2, 2) = (2, 2) + (\infty, 2) = (\infty, 2) + (\infty, 2) = (\infty, 4)$, which can not happen in a `rqns`, due (again) to the "cancellation law" (Proposition 3.4).

Corollary 3.7. *The class* RNQS *is not closed under neither the formation of homomorphic images nor the formation of direct products.*

3.2. CQNS ≠ RQNS. The following lemma, whose statement is inspired on Distler's work [4] on the classification of finite nilpotent semigroups, will allow us to prove that not every rqns is determined by a numerical semigroup and a cutting point. Observe that for an integer i, and a finite nilpotent semigroup S, the set S^i consists of the elements of S that can be written as the product of at least i minimal generators. Therefore, $S^i \backslash S^{i+1}$ consists of the elements that can be written as the product of exactly i minimal generators.

Lemma 3.8. *Let S be a commutative nilpotent semigroup minimally generated by $\{a, b\}$, such that $a^3 = b^3 = 0$, $|S^2 \backslash S^3| = 2$ and $|S^3 \backslash S^4| = 1$. Then S is not determined by a numerical semigroup and a cutting point.*

Prior to the proof we give an example of a semigroup fulfilling the conditions of the lemma.

Example 3.9. Let us consider the numerical semigroup $N = \langle 3, 5 \rangle = \{3, 5, 6, 8^>\}$ and its ideal $I_N = \{6\} + N^0 = \{6, 9, 11, 12, 14^>\}$. The quotient $S = N/I_N = \{3, 5, 8, 10, 13, \infty\}$ has the following multiplication table

$\oplus$	3	5	8	10	13	∞
3	∞	8	∞	13	∞	∞
5	8	10	13	∞	∞	∞
8	∞	13	∞	∞	∞	∞
10	13	∞	∞	∞	∞	∞
13	∞	∞	∞	∞	∞	∞
∞	∞	∞	∞	∞	∞	∞

and the following structure in Green's $\mathscr{D}$-classes
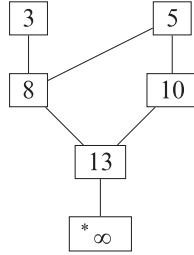
Observe that the constructed semigroup is a `rqns`, which proves the following Corollary:

Corollary 3.10. *A* `rqns` *is not necessarily determined by a numerical semigroup and a cutting point.*

Next we prove Lemma 3.8.

Proof. Suppose that there exist a numerical semigroup M, a positive integer k and an isomorphism $f : M/I_k \to S$. Let $p : M \to M/I_k$ be the canonical projection, i.e.

$$p(m) = \begin{cases} \{m\}, & m < k \\ \infty, & m \geq k, \end{cases}$$

for all $m \in M$, and take the surjective homomorphism $g = f \circ p : M \to S$. Clearly, being $\{n_1, n_2, \ldots, n_t\}$, with $n_1 < n_2 < \cdots < n_t$, the minimal generators of M, we have $t \geq 2$ and $\{g(n_1), g(n_2)\} = \{a, b\}$. Without loss of generality, we may admit that $g(n_1) = a$. Now, as S is commutative, we have $S^2 \backslash S^3 \subseteq \{a^2 = g(2n_1), ab = g(n_1 + n_2), b^2 = g(2n_2)\}$. Moreover, since $2n_1 < n_1 + n_2 < 2n_2$ and $|S^2 \backslash S^3| = 2$, we deduce that $S^2 \backslash S^3 = \{a^2, ab\}$. Thus, again by the commutativity of S, we obtain $S^3 \backslash S^4 \subseteq \{a^3 = g(3n_1), a^2 b = g(2n_1 + n_2), ab^2 = g(n_1 + 2n_1)\}$. Now, as $a^3 = 0$, we get $k \leq 3n_1 < 2n_1 + n_2 < n_1 + 2n_2$, whence $a^2 b = ab^2 = 0$ and so $S^3 \backslash S^4 = \emptyset$, a contradiction. $\square$

4. Presentations

We refer the reader to [8] for presentations of semigroups in general. Presentations of numerical semigroups may be found in our general reference [7].

Our aim is to find a nice presentation for a `rqns`. We use the presentation obtained to give a practical method to compute the automorphism group of a `rqns`.

4.1. Rees quotient semigroup presentations.

We start with some generalities. Let S be a semigroup without identity. We denote by S^I the semigroup S with an adjoined identity. An element $x \in S$ is called *indecomposable in S* if there are no elements $a, b \in S$ such that $x = ab$. Let I be a proper ideal of S. Then, clearly, an element $x \in S \backslash I$ is indecomposable in S if and only if $[x]_I = \{x\}$ is indecomposable in S/I. On the other hand, if X is a set of generators of S then $\{[x]_I = \{x\} \mid x \in X \backslash I\} \cup \{I\}$ is a set of generators of S/I. Notice that, if $S \backslash I$ is not a subsemigroup of S, then $\{[x]_I = \{x\} \mid x \in X \backslash I\}$ generates S/I (thus I does not need to be added as a generator).

Given a set X, we denote by X^+ the free semigroup on X.

Let $\langle X \mid R \rangle$ be a presentation of S. For each $y \in I$, denote by w_y a (fixed) word of X^+ representing the element y. Let Y be an *ideal generating system* of I, i.e. a non-empty subset Y of I such that $I = S^I Y S^I$ (notice that any set of generators of I is also an ideal generating system of I). Then, it is a routine matter to show:

Lemma 4.1. *The semigroup S/I is defined by the presentation with* zero *$\langle X \mid R, w_y = 0 \ (y \in Y) \rangle$.*

Recall that, in a presentation (of semigroups) $\langle A \mid \mathfrak{R} \rangle$ with zero, the *free semigroup with zero A_0^+* (i.e. the free semigroup A^+ with an absorbing element adjoined) plays the same role as A^+ in a usual presentation (of semigroups).

Next, let $\langle X \mid R \rangle$ be a presentation of S and let Y be any subset of I (not necessarily an ideal generating system) such that

$$\langle X \mid R, w_y = 0 \ (y \in Y) \rangle$$

is a presentation (with *zero*) of S/I. We define from R the following sets of relations on $(X \backslash I)_0^+$:

$$R_1 = \{(u, v) \mid (u, v) \in R \text{ and } u, v \in (X \backslash I)^+\},$$
$$R_2 = \{(u, 0) \mid (u, v) \in R \text{ or } (v, u) \in R, \text{ with } u \in (X \backslash I)^+ \text{ and } v \in X^+ \backslash (X \backslash I)^+\}$$

and

$$R' = R_1 \cup R_2. \tag{2}$$

Let $Y' = \{y \in Y \mid w_y \in (X \backslash I)^+\}$ (in addition, we may assume that, for each element $y \in I \cap X$, we have taken w_y as the word y and so, in this case, $Y' \cap X = \emptyset$). Under these conditions, we have:

Lemma 4.2. *The semigroup S/I is defined by the presentation with* zero *$\langle X \backslash I \mid R', w_y = 0 \ (y \in Y') \rangle$.*

Proof. First, observe that, clearly, all the relations of $\langle X \backslash I \mid R', w_y = 0 \ (y \in Y') \rangle$ are satisfied by S/I. Therefore, it remains to prove that all the equalities, between words of $(X \backslash I)^+$, satisfied by S/I are consequences of $R' \cup \{w_y = 0 \mid y \in Y'\}$.

Let $w, w' \in (X \backslash I)^+$ be such that the equality $w = w'$ is satisfied by S/I. Then w and w' both represent elements of I or $w = w'$ is satisfied by S (and, in this case, we may suppose that w and w' represent an element of $S \backslash I$).

If this last case occurs, since S is presented by $\langle X \mid R \rangle$, then there exists a finite sequence of elementary R-transitions $w \to w_1 \to \cdots \to w_{n-1} \to w'$ over X^+. As each word of this sequence represents the same element of $S \backslash I$, then all the letters involved must belong to $X \backslash I$. Hence $w \to w_1 \to \cdots \to w_{n-1} \to w'$ is also a finite sequence of elementary R_1-transitions over $(X \backslash I)^+$ and so, in particular, $w = w'$ is a consequence of $R' \cup \{w_y = 0 \mid y \in Y'\}$.

Next, suppose that w represents an element of I. Since $\langle X \mid R, w_y = 0 \ (y \in Y) \rangle$ is a presentation of S/I and the equality $w = 0$ is satisfied by S/I, then there exist a finite sequence of elementary R-transitions $w \to w_1 \to \cdots \to w_{n-1}$ over X^+ and an elementary $\{w_y = 0 \mid y \in Y\}$-transition $w_{n-1} \to 0$ over X^+. Now, we consider two possibilities. First, if $w_1, \ldots, w_{n-1} \in (X \backslash I)^+$, then $w \to w_1 \to \cdots \to w_{n-1}$ is also a finite sequence of elementary R_1-transitions over $(X \backslash I)^+$ (notice that, we have taken $w \in (X \backslash I)^+$) and $w_{n-1} \to 0$ is an elementary $\{w_y = 0 \mid y \in Y'\}$-transition over $(X \backslash I)^+$, whence $w = 0$ is a consequence of $R' \cup \{w_y = 0 \mid y \in Y'\}$. Secondly, we suppose that there exists an index $i \in \{1, \ldots, n-1\}$ such that $w_i \in X^+ \backslash (X \backslash I)^+$ and take the smallest of such indexes. Thus $w \to w_1 \to \cdots \to w_{i-1}$ (where w_{i-1} denotes the word w, for $i = 1$) is a finite sequence of elementary R_1-transitions over $(X \backslash I)^+$ and $w_{i-1} \to w_i$ is an elementary $\{(u, v)\}$-transition, for some $u \in (X \backslash I)^+$ and $v \in X^+ \backslash (X \backslash I)^+$ such that $(u, v) \in R$ or $(v, u) \in R$. Hence $w \to w_1 \to \cdots \to w_{i-1} \to 0$ is a finite sequence of elementary $R_1 \cup R_2$-transitions over $(X \backslash I)^+$ and so again we conclude that $w = 0$ is a consequence of $R' \cup \{w_y = 0 \mid y \in Y'\}$.

Therefore, if w and w' both represent elements of I, from the last paragraph we deduce that $w = w'$ is a consequence of $R' \cup \{w_y = 0 \mid y \in Y'\}$, as required. □

4.2. Presentations of `rqns`'s.

Now, let S be a numerical semigroup and let I be a proper ideal of S. Take an ideal generating system G of I (whence $I = G + S^0$) and a presentation $\langle X \mid R \rangle$ of S. Then, as a particular case of Lemma 4.1, we have:

Corollary 4.3. *The* `rqns` *S/I is defined by the presentation with* zero *$\langle X \mid R, w_g = 0 \ (g \in G) \rangle$.*

Example 4.4. Let $m = m(S)$ and let $k \geq c(S)$ be an integer. Recall that the minimal ideal generating system of the ideal I_k of S is $\{k, k+1, \ldots, k+m-1\}$. Thus

$$\langle X \mid R, w_k = w_{k+1} = \cdots = w_{k+m-1} = 0 \rangle$$

is a presentation of S/I_k.

Next, we suppose that X is the minimal generating system of S. Then (the set of classes of the elements of) $X \backslash I$ is a set of generators of S/I (since $S \backslash I$ is never a subsemigroup of S). Moreover, as $X \backslash I \subseteq S/I$ is also a set of indecomposable elements of S/I, then $X \backslash I$ is a minimum set (for set inclusion) of generators of $S \backslash I$. We call to $X \backslash I$ the *minimal generating system* of the `rqns` S/I.

In particular, if the ideal I does not contain any element of the minimal generating system X of S, then X is also the minimal generating system of S/I. In this case, the presentation of S/I given by Corollary 4.3 (considering there X as being the minimal generating system of S) is already over its minimal generating system.

In what follows, we aim to determine a presentation over the minimal generating system of any `rqns`.

Let G be the minimal ideal generating system of I. Recall that, for each $g \in G$, we denote by w_g a fixed word of X^+ representing the element g. Then, we have:

Lemma 4.5. *For any $g \in G$, $w_g \in (X \backslash I)^+$ if and only if $g \in G \backslash X$.*

Proof. If $g \in X$ then g is indecomposable, whence w_g should be the word g (despite, in any case, we could have chosen w_g equal g) and so $w_g \notin (X \backslash I)^+$.

Conversely, suppose that $w_g \notin (X \backslash I)^+$. Then $g = x_1 + \cdots + x_n$, with $x_1, \ldots, x_n \in X$ $(n \geq 1)$ and $x_i \in I$, for some $1 \leq i \leq n$. Then, as x_i is indecomposable and $x_i \in G + S^0$, we must have $x_i \in G$ (since $x_i = x_i + 0$ is the unique decomposition permitted). Now, if $g \neq x_i$, then $I = G + S^0 \subseteq (G \backslash \{g\}) + S^0 \subseteq I$ (observe that, given $x \in S^0$, we have $g + x = x_i + (x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n + x) \in (G \backslash \{g\}) + S^0$), which contradicts the minimality of G. Thus $g = x_i \in X$ and so the lemma is proved. $\qquad \square$

Let R' be the set of relations over $(X \backslash I)_0^+$ obtained from R as defined in general in (2). Thus, combining Lemma 4.2 with Lemma 4.5, we immediately have:

Theorem 4.6. *The `rqns` S/I is defined by the presentation with zero $\langle X \backslash I \mid R', w_g = 0 \ (g \in G \backslash X) \rangle$ over its minimal generating system.*

4.3. Isomorphisms of `rqns`'s. In what follows, all Rees quotient numerical semigroups considered are constructed from proper ideals.

Let S_1 and S_2 be two Rees quotient numerical semigroups with minimal generating systems X_1 and X_2, respectively.

Lemma 4.7. *If $\varphi : S_1 \to S_2$ is a surjective homomorphism, then $X_2 \subseteq \varphi(X_1)$. In particular $|X_2| \le |X_1|$.*

Proof. Since φ is surjective, then $\varphi(X_1)$ generates S_2. Thus $X_2 \subseteq \varphi(X_1)$, by the minimality of X_2. Moreover, $|X_2| \le |\varphi(X_1)| \le |X_1|$, as required. $\square$

It follows immediately that:

Proposition 4.8. *Let $\varphi : S_1 \to S_2$ be an isomorphism. Then $\varphi(X_1) = X_2$. In particular $|X_1| = |X_2|$.*

Let S and T be (any) two semigroups and let $\langle X \mid R \rangle$ be a presentation of S.

Let $f : X \to T$ be a mapping and let $\phi : X^+ \to T$ be the (unique) homomorphism extending f (regarding X as a set of letters). If f *satisfies R*, i.e. $\phi(u) = \phi(v)$, for all $(u, v) \in R$, then the mapping $\varphi : S \to T$ defined by $\varphi(s) = \phi(w_s)$, where w_s is any (fixed) word of X^+ representing s, for all $s \in S$, is the unique homomorphism extending f (regarding X as a generating set of S). Moreover, $f(X)$ generates T if and only if $\varphi : S \to T$ is a surjective homomorphism. In particular, supposing that S is a finite semigroup, if $f(X)$ generates T and $|S| = |T|$, then $\varphi : S \to T$ is an isomorphism.

Conversely, let $\varphi : S \to T$ be a homomorphism and let $f : X \to T$ be the restriction of φ to X. Then, clearly, f must satisfy R.

In view of Proposition 4.8 and the above observations, we have the following interesting conclusion regarding isomorphisms of Rees quotient numerical semigroups.

Theorem 4.9. *Let S_1 and S_2 be two Rees quotient numerical semigroups with minimal generating systems X_1 and X_2, respectively. Let $\langle X_1 \mid R \rangle$ be a (fixed) presentation of S_1. If $|S_1| = |S_2|$ then the isomorphisms from S_1 to S_2 are precisely the homomorphisms $\varphi : S_1 \to S_2$ extending bijections $f : X_1 \to X_2$ satisfying R.*

For a numerical semigroup S with multiplicity m and embedding dimension e, we may compute a presentation with less than or equal to $\frac{(2m-e+1)(e-2)}{2} + 1$ relations [7]. Thus, if I is an ideal of S with minimal ideal generating system G, regarding Theorem 4.6, we may compute a presentation (with zero) for the `rqns` S/I over its minimal generating system with less than or equal to $\frac{(2m-e+1)(e-2)}{2} + |G| + 1$ relations. Therefore, Theorem 4.9 gives us a reasonable practical method to compute all the isomorphisms between two Rees quotient

numerical semigroups (at least for the ones having *small-size* minimal generating systems) and, in particular, to compute the automorphism group of a `rqns`:

Corollary 4.10. *Let S be a* `rqns` *with minimal generating system X and let* $\langle X \,|\, R \rangle$ *be a (fixed) presentation of S. Then, the automorphisms of S are the endomorphisms of S that extend permutations of X satisfying R.*

A recent paper by García-García and Moreno [5] treats problems on morphisms of commutative monoids which at first sight could be thought as similar to the ones treated in this subsection, but we have not discovered any strong connection.

5. Generators of the pseudovariety $\mathsf{N} \cap \mathsf{Com}$

Recall that $\mathsf{CN} = \{ \texttt{rqns}(\mathbb{N}, I_k) \,|\, k \in \mathbb{N} \}$. We show that this *small* class, consisting of easily described finite semigroups, forms a set of generators of the pseudovariety of finite nilpotent and commutative semigroups.

Theorem 5.1. *The class* CN *generates the pseudovariety* $\mathsf{N} \cap \mathsf{Com}$.

Proof. Let V be the pseudovariety generated by CN. Then, as $\mathsf{V} \subset \mathsf{Com}$, by [1, Theorem 6.2.6], V is admits a finite basis of pseudoidentities of the form

$$\Sigma \cup \{ xy = yx, \pi(x)x^{\omega} = x^{\omega} \},$$

where π is a pseudoword on one variable such that $\mathsf{V} \cap \mathsf{G}$ is defined by the pseudoidentities $\pi = 1$ and $xy = yx$ and Σ consists of pseudoidentities which are valid in V of the form $x_1^{\alpha_1} \ldots x_n^{\alpha_n} = x_1^{\beta_1} \ldots x_n^{\beta_n}$, with $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{N}_0 \cup \{\omega\}$ and variables $x_1, \ldots, x_n$ not necessarily distinct.

As N is defined by the pseudoidentity $x^{\omega} = 0$ and $\mathsf{V} \subset \mathsf{N}$, we may replace $\pi(x)x^{\omega} = x^{\omega}$ simply by $x^{\omega} = 0$.

Next, suppose that V is strictly contained in $\mathsf{N} \cap \mathsf{Com}$. Hence V must satisfy a non-trivial pseudoidentity, which is not satisfied by $\mathsf{N} \cap \mathsf{Com}$, of the form $x_1^{\alpha_1} \ldots x_n^{\alpha_n} = x_1^{\beta_1} \ldots x_n^{\beta_n}$, with $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{N}_0 \cup \{\omega\}$ and $x_1, \ldots, x_n$ not necessarily distinct. As V also satisfies $x^{\omega} = 0$, it follows that this pseudoidentity must be an identity of the form $x_1^{\alpha_1} \ldots x_n^{\alpha_n} = 0$, with $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ or of the form $x_1^{\alpha_1} \ldots x_n^{\alpha_n} = x_1^{\beta_1} \ldots x_n^{\beta_n}$, with $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{N}_0$. Moreover, in both cases, we may assume that the variables $x_1, \ldots, x_n$ are distinct.

Let r be a positive integer and consider the semigroup $Q_r = \texttt{rqns}(\mathbb{N}, I_r)$. If $r > \alpha_1 + \cdots + \alpha_n$, then we have $\alpha_1 + \cdots + \alpha_n \neq \infty$ in Q_r, whence Q_r does not

satisfy the identity $x_1^{\alpha_1} \ldots x_n^{\alpha_n} = 0$ and so the taken pseudoidentity cannot be of this form. Now, let $r > \max\{\alpha_1 + \cdots + \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_n \,|\, 2 \le i \le n\}$. Since Q_r must satisfy the identity $x_1^{\alpha_1} \ldots x_n^{\alpha_n} = x_1^{\beta_1} \ldots x_n^{\beta_n}$, on one hand, we have

$$\infty \neq \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i + \alpha_{i+1} + \cdots + \alpha_n$$
$$= \beta_1 + \cdots + \beta_{i-1} + \beta_i + \beta_{i+1} + \cdots + \beta_n \tag{3}$$

in Q_r and, on the other hand, for $2 \le i \le n$ (by considering $x_j = 1$, for $j \neq i$, and $x_i = 2$), we have

$$\infty \neq \alpha_1 + \cdots + \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_n$$
$$= \beta_1 + \cdots + \beta_{i-1} + 2\beta_i + \beta_{i+1} + \cdots + \beta_n \tag{4}$$

in Q_r. Hence, for each $2 \le i \le n$, by combining (3) with (4), we deduce that $\alpha_i = \beta_i$. Then, by (3) it follows also that $\alpha_1 = \beta_1$. Therefore, the identity $x_1^{\alpha_1} \ldots x_n^{\alpha_n} = x_1^{\beta_1} \ldots x_n^{\beta_n}$ is trivial, which is a contradiction.

Thus $\mathsf{V} = \mathsf{N} \cap \mathsf{Com}$, as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6. Open problems

In this section we state some natural questions which we have not yet been able to answer, opening this way a line of research.

We divide the section into subsections, one of which contains a table where the numerical semigroups with Frobenius number up to 10 are listed. An analysis of this table (and some other computations of the same kind) leads to several questions and conjectures. We state a decidability question in the last subsection.

6.1. Some more terminology. Next we grasp some more terminology from our references.

A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it.

An irreducible numerical semigroup is either *symmetric*, if its Frobenius number is odd, or is *pseudo-symmetric*, if its Frobenius number is even.

The elements of a numerical semigroup that are not greater than the conductor are called *small elements*. (The name is taken from the *numericalsgps* GAP package [3], although the current version of the package always includes the 0 as a small element of a semigroup.)

As the set of small elements of a numerical semigroup completely determines it, we can represent the numerical semigroup through its small elements.

6.2. A table of semigroups with small Frobenius numbers. Table 1 lists the numerical semigroups whose conductor is not greater than 11. The use of gray tones will be explained in the next subsection.

The table has been constructed with the help of the *numericalsgps* GAP package [3]. For instance, the data in the following example was used to produce the line corresponding to the semigroups with Frobenius number 8.

Example 6.1. The following lines correspond to a GAP session where the irreducible numerical semigroups with Frobenius number 8 are determined.

```
gap> LoadPackage("numericalsgps");
true
gap> n := 8;;
gap> irrn := IrreducibleNumericalSemigroupsWithFrobeniusNumber(n);;
gap> List(irrn,s->SmallElementsOfNumericalSemigroup(s));
[ [ 0, 5, 6, 7, 9 ], [ 0, 3, 6, 7, 9 ] ]
```

In the same GAP session, the non irreducibles can be determined as follows:

```
gap> fn:=NumericalSemigroupsWithFrobeniusNumber(n);;
gap> nirrn:=Filtered(fn,s->not IsIrreducibleNumericalSemi group(s));;
gap> List(nirrn,s->SmallElementsOfNumericalSemigroup(s));
[ [ 0, 5, 7, 9 ], [ 0, 6, 7, 9 ], [ 0, 7, 9 ], [ 0, 5, 6, 9 ],
  [ 0, 3, 6, 9 ], [ 0, 5, 9 ], [ 0, 6, 9 ], [ 0, 9 ] ]
```

The numbers in the first column of Table 1 indicate the Frobenius numbers of the numerical semigroups on the right cells. The second column contains the irreducible semigroups and the third contains the non irreducible ones.

6.3. Analysis of the table. Let S be a numerical semigroup. We observe that the size of the Rees quotient $\mathtt{rqns}(S, I_{c(S)})$ is precisely the number of small elements of S.

The set of small elements can also be used to represent the $\mathtt{rqns}$ where the conductor has been taken as cutting point. Notice that, contrary to what happens when using this form of representing a numerical semigroup, several sets of integers can represent the same quotient semigroup (since we do not make any distinction between isomorphic objects).

In this way, Table 1 represents also the Rees quotient numerical semigroups obtained from the numerical semigroups with Frobenius number less than 11 through the use of the conductor as cutting point. There are many repetitions, since many of the numerical semigroups in the table lead to isomorphic quotients. Quotients of the same size (which, as observed, are obtained from numerical

Table 1. Numerical semigroups with conductor up to 11

F	Irreducibles	Non irreducibles
1	$\{2^>\}$	
2	$\{3^>\}$	
3	$\{2,4^>\}$	$\{4^>\}$
4	$\{3,5^>\}$	$\{5^>\}$
5	$\{3,4,6^>\}$, $\{2,4,6^>\}$	$\{3,6^>\}$, $\{4,6^>\}$, $\{6^>\}$
6	$\{4,5,7^>\}$	$\{5,7^>\}$, $\{4,7^>\}$, $\{7^>\}$
7	$\{3,5,6,8^>\}$, $\{4,5,6,8^>\}$, $\{2,4,6,8^>\}$	$\{4,5,8^>\}$, $\{5,6,8^>\}$, $\{3,6,8^>\}$, $\{5,8^>\}$, $\{4,8^>\}$, $\{4,6,8^>\}$, $\{6,8^>\}$, $\{8^>\}$
8	$\{5,6,7,9^>\}$, $\{3,6,7,9^>\}$	$\{5,7,9^>\}$, $\{6,7,9^>\}$, $\{7,9^>\}$, $\{5,6,9^>\}$, $\{3,6,9^>\}$, $\{5,9^>\}$, $\{6,9^>\}$, $\{9^>\}$
9	$\{5,6,7,8,10^>\}$, $\{4,6,7,8,10^>\}$, $\{2,4,6,8,10^>\}$	$\{5,6,7,10^>\}$, $\{6,7,8,10^>\}$, $\{5,7,8,10^>\}$, $\{4,7,8,10^>\}$, $\{5,6,8,10^>\}$, $\{6,7,10^>\}$, $\{5,7,10^>\}$, $\{5,6,10^>\}$, $\{7,8,10^>\}$, $\{4,6,8,10^>\}$, $\{6,8,10^>\}$, $\{5,8,10^>\}$, $\{4,8,10^>\}$, $\{7,10^>\}$, $\{6,10^>\}$, $\{5,10^>\}$, $\{8,10^>\}$, $\{10^>\}$
10	$\{4,7,8,9,11^>\}$, $\{6,7,8,9,11^>\}$, $\{3,6,8,9,11^>\}$	$\{7,8,9,11^>\}$, $\{4,8,9,11^>\}$, $\{6,7,9,11^>\}$, $\{7,9,11^>\}$, $\{3,6,9,11^>\}$, $\{6,7,8,11^>\}$, $\{4,7,8,11^>\}$, $\{6,7,11^>\}$, $\{6,8,9,11^>\}$, $\{8,9,11^>\}$, $\{6,9,11^>\}$, $\{9,11^>\}$, $\{7,8,11^>\}$, $\{6,8,11^>\}$, $\{4,8,11^>\}$, $\{7,11^>\}$, $\{6,11^>\}$, $\{8,11^>\}$, $\{11^>\}$

semigroups with the same number of small elements) that are isomorphic are represented using the same gray tone.

The reader can check without any difficulty that the isomorphism classes of the quotient semigroups in the table are completely determined by the size and the nilpotency class of the semigroups. In fact, in all cases of semigroups of the same size and nilpotency class, it is straightforward to construct an isomorphism. (We observe that these arguments would need to be greatly refined in order to deal with numerical semigroups with larger Frobenius numbers.)

The following seems to be true, but we still have no proof:

Conjecture 6.2. *Different symmetric numerical semigroups correspond to non-isomorphic quotient numerical semigroups.*

A quick look at the first part of the table could lead to ask whether one could obtain all the Rees quotient numerical semigroups determined by a cutting point by using the symmetric numerical semigroups and the conductors as cutting points. The last part of the table shows that this is not the case, since the Rees quotient of the pseudo-symmetric numerical semigroup $\{3,6,8,9,11^>\}$ obtained by cutting through its conductor is not isomorphic to any Rees quotient of a symmetric numerical semigroup with the same number of small elements cutting by its conductor. (Notice that the nilpotency class of $\{3,6,8,9,\infty\}$ is 4, while the nilpotency classes of the quotients obtained from symmetric numerical semigroups with Frobenius number 9 are 2, 3 or 5.)

One could now make the same question replacing symmetric by irreducible, although the unicity would in this case be out of question. More precisely, being N a numerical semigroup and k a cutting point, does there exist an irreducible numerical semigroup M such that $\mathtt{rqns}(N/I_k) \simeq \mathtt{rqns}(M, I_{c(S)})$? The answer is "No", as follows from the remaining part of this subsection.

Consider the semigroup $\langle 4, 11, 13, 18 \rangle$ and take the conductor, 15, as cutting point. The quotient is a nilpotent semigroup of size 6 which has 3 minimal generators and nilpotency class 4, as the following GAP session may help to confirm.

```
gap> N := NumericalSemigroup(4,11,13,18);;
gap> SmallElementsOfNumericalSemigroup(N);
[ 0, 4, 8, 11, 12, 13, 15 ]
```

From basic results on irreducible numerical semigroups (see [7]) it follows that an irreducible numerical semigroup with 6 small elements must have Frobenius number 11 or 12. These can be computed as follows:

Example 6.3. ```gap> n := 11;;
gap> irrn := IrreducibleNumericalSemigroupsWithFrobeniusNumber(n);;
gap> List(irrn,s->SmallElementsOfNumericalSemigroup(s));
[[0, 5, 7, 8, 9, 10, 12], [0, 4, 5, 8, 9, 10, 12],
  [0, 3, 6, 7, 9, 10, 12], [0, 6, 7, 8, 9, 10, 12],
  [0, 2, 4, 6, 8, 10, 12], [0, 4, 6, 8, 9, 10, 12]]
gap> n := 12;;
gap> irrn := IrreducibleNumericalSemigroupsWithFrobeniusNumber(n);;
gap> List(irrn,s->SmallElementsOfNumericalSemigroup(s));
[[0, 7, 8, 9, 10, 11, 13], [0, 5, 8, 9, 10, 11, 13]]
```

The only one with nilpotency class 4 is obtained from $\{3,6,7,9,10,12^>\}$, but it only has 2 minimal generators, therefore is not isomorphic to $\{4,8,11,12,13,\infty\}$.

**6.4. Notable elements in quotients.** It would be interesting to be able to find a reasonable correspondence between the definitions of notable elements in numerical semigroups given in Subsection 2.2 and similar notions to be defined in Rees quotient numerical semigroups. For instance, one should be able to give a reasonable definition for the Frobenius number of a rqns. At some point of our research we had the hope that one could do this via some irreducible numerical semigroup in the same isomorphism class, but the examples in the previous subsection show that something different must be tried.

**6.5. Decidability.**

**Question 6.4.** *Let Q be a finite commutative nilpotent semigroup. Give an effective way to construct a numerical semigroup S and an ideal $I_S$ such that Q and $S/I_S$ are isomorphic, if such numerical semigroup and ideal exist.*

As usual, we say that a class of finite semigroups is decidable if there is an algorithm that having as input a finite semigroup, it outputs whether the semigroup belongs to the class given. The following question, to which Section 5 is related and which would have a positive answer if Question 6.4 had a positive answer, appears naturally:

**Question 6.5.** *Is* RQNS *decidable?*

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