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Integration of Lie algebroid comorphisms

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Abstract. We show that the path construction integration of Lie algebroids by Lie groupoids is an actual equivalence from the of integrable Lie algebroids and *complete* Lie algebroid comorphisms to the of source 1-connected Lie groupoids and Lie groupoid comorphisms. This allows us to construct an actual symplectization functor in Poisson geometry. We include examples to show that the integrability of comorphisms and Poisson maps may not hold in the absence of a completeness assumption.

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1. Introduction

A classical result in differential geometry is that any finite dimensional Lie algebra $\mathcal{G}$ can be integrated by a 1-connected Lie group $\Sigma(\mathcal{G})$ (conversely, any Lie group endows its tangent space at the unit with the structure of a Lie algebra). This bijective correspondence between finite dimensional Lie algebras and 1-connected finite dimensional Lie groups is actually the object component of an integration functor

$$\Sigma : \mathbf{LieAlg} \to \mathbf{LieGp},$$

which is an equivalence from the of finite dimensional Lie algebras to the of finite dimensional 1-connected Lie groups.

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There are several generalizations of finite dimensional Lie algebras: infinite-dimensional (e.g Banach) Lie algebras, Lie algebroids, Poisson manifolds, and L_∞-algebras, for instance. In each case, it is natural to ask whether there is a corresponding integration functor. For this, we need to find out the right notion for the objects integrating these generalized Lie algebras as well as the right notion of morphisms between them.

We will be concerned here with Lie algebroids and Poisson manifolds (see [14] for a general reference), each of whose have natural integrating objects: Lie groupoids for Lie algebroids and symplectic groupoids for Poisson manifolds.

However, unlike finite dimensional Lie algebras, not all Lie algebroids are integrable by Lie groupoids. In [7], Crainic and Fernandes worked out a criterion to select those that are. In this paper, we will construct an integration functor for the class of *integrable* Lie algebroids. To allow non-integrable Lie algebroids to be in the domain of an integration functor, one must consider integration by microgroupoids (i.e. germs of groupoids) in the spirit of [3] or by differentiable stacks as in [18].

We will focus on an integration functor for Lie algebroids that translates into an integration functor for Poisson manifolds (also called "symplectization functor" by Fernandes [10]). For this, the main ingredient is to replace the usual notion of morphisms between Lie groupoids (i.e. smooth functors between the underlying groupoids, see [11], [14]) and their corresponding infinitesimal version for Lie algebroids by that of *comorphisms*. Lie algebroid and Lie groupoid comorphisms were introduced by Higgins and Mackenzie in [12]. (This notion for Lie groupoids had been studied earlier under a different name and with a different, but equivalent, definition by Zakrzewski in [19] and Stachura in [17].) Already there, comorphisms were seen as the "correct" notion of morphisms between Lie algebroids to be used with applications to Poisson geometry in mind.

From the perspective of Poisson geometry, Lie algebroid and Lie groupoid morphisms are not very well-suited, since a Poisson map ϕ from X to Y induces a Lie algebroid morphism from T^*X to T^*Y (that integrates to a symplectic groupoid morphism) only when ϕ is a diffeomorphism. This prevents us from constructing an integration functor whose domain would contain all Poisson maps. On the other hand, the cotangent map $T^*\phi$ to a Poisson map ϕ is always a comorphism from T^*X to T^*Y.

There are also a number of facts independent of Poisson geometry that make comorphisms the "correct" notion of morphisms between Lie groupoids. For instance, a comorphism between Lie groupoids naturally induces a group morphism between the corresponding groups of bisections as well as a C^*-algebra morphism between the corresponding convolution C^*-algebras (see [17]). This gives functors from the of Lie groupoids and comorphisms to the of groups and to the of C^*-algebras (see [17]). Moreover, the graph of a Lie groupoid comorphism is a monoid map in the “ ” of differentiable relations between the monoid objects associated to the multiplication graph of the corresponding Lie groupoids (see [19]).

On the other hand, in contrast to Lie algebroid morphisms, Lie algebroid comorphisms do not always integrate to Lie groupoid comorphisms. The same holds in Poisson geometry, where completeness of Poisson maps insures integrability in terms of (symplectic) comorphisms. We will give an example of a non complete Lie algebroid comorphism that is also non integrable, and whose dual is a non complete and non integrable Poisson map.

Dazord in [9] already stated without proof that both *complete* Lie algebroid comorphisms and Poisson maps always do integrate to comorphisms. In [19], Zakrzewski proved that complete Poisson maps are integrable to what he called “morphisms of regular D*-algebras,” which turn out to be nothing but symplectic comorphisms.

More recently, Caseiro and Fernandes in [1] proved that a complete Poisson map ϕ from integrable Poisson manifolds X to Y always integrates to a natural left action of the symplectic groupoid $\Sigma(Y)$ on X with moment map ϕ. This action naturally induces an embedded lagrangian subgroupoid integrating the graph of the Poisson map. Their proof, at contrast with the one of Zakrzewski which uses the method of characteristics, is readily transposable to complete Lie algebroid comorphisms. They use the existence of lifting properties by complete Poisson maps for both admissible paths and their homotopies, which also holds for complete Lie algebroid comorphisms and makes them resemble “Serre fibrations” in topology.

These lifting properties (Proposition 4.2 and 4.3) will also be central to our main result (Theorem 5.1): namely, that the path construction of [4], [8], [7], which associates a source 1-connected Lie groupoid $\Sigma(A)$ to each integrable Lie algebroid A, is an actual equivalence from the of integrable Lie algebroids and complete Lie algebroid comorphisms to the of source 1-connected Lie groupoids and Lie groupoid comorphisms.

As a corollary, we obtain that Lie algebroid comorphisms are integrable if and only if they are complete, which strengthens Dazord’s statement. We show that this implies a corresponding theorem in Poisson geometry, where the path construction implements an equivalence between the of integrable Poisson manifolds and complete Poisson maps and the of source 1-connected symplectic groupoids and symplectic comorphisms. From this, we may conclude that Poisson maps are integrable if and only if they are complete, which was already shown by Zakrzewski in the language of regular D*-algebras.

Let us conclude this introduction by remarking that the composition of the path construction with the functor constructed by Stachura in [17] yields a sort of “prequantization” functor that takes an integrable Poisson manifold to the convolution C^*-algebra of its integrating symplectic groupoid and a complete Poisson map to a C^*-algebra morphism between these convolution C^*-algebras.

2. Morphisms and comorphisms

Much of what follows in this section may already be found in [5], [9], [11], [12]. We work in the smooth. Let $a : A \to X$ and $b : B \to Y$ be submersions, which we may think of as families of manifolds parametrized by X and Y.

A map ϕ from X to Y together with a map Φ to A from the pullback $\phi^! B = X \times_Y B$ will be called a *comorphism* from a to b; ϕ will be called the *core map* of the comorphism. When the families are vector bundles and Φ is linear on fibres, we call (ϕ, Φ) a vector bundle comorphism. It induces a dual vector bundle map Φ^* from $a^* : A^* \to X$ to $b^* : B^* \to Y$ covering ϕ and a pullback map $\Phi^\dagger$ to the space $\Gamma(A)$ of sections of A from $\Gamma(B)$.

On the other hand, a *morphism* from a to b is simply a map of fibrations, which we also denote by (ϕ, Φ), where the core map ϕ is the base map of the bundle map, and Φ is a collection of smooth maps Φ_x from the fibers A_x to $B_{\phi(x)}$. When a and b are vector bundles and Φ is linear on fibers, a morphism (ϕ, Φ) is a vector bundle map.

As observed in [12], the notions of morphisms and comorphisms for vector bundles are dual to each other in the sense that (ϕ, Φ) is a comorphism from a to b if and only if (ϕ, Φ^*) is a morphism from a^* to b^* (and conversely).

We now specialize the notion of morphisms and comorphisms to Lie algebroids and Lie groupoids, and we introduce corresponding Lie functors (see also [11], [12]).

2.1. Lie algebroids. If A and B are Lie algebroids, a vector bundle comorphism (ϕ, Φ) is called a *Lie algebroid comorphism* if Φ^* is a Poisson map for the natural Lie-Poisson structures on the dual Lie algebroids. Equivalently, $\Phi^\dagger$ is a homomorphism of Lie algebras, and

$$\phi_* \circ \rho_A \circ \Phi^\dagger = \rho_B, \tag{1}$$

where ρ_A and ρ_B are the anchor maps of respectively A and B. In terms of diagrams, we can represent a comorphism and relation (1) as follows:

$$\begin{array}{ccc} A & \xleftarrow{\ \Phi\ } & B \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow[\ \phi\ ]{} & Y \end{array} \qquad \begin{array}{ccc} \Gamma(A) & \xleftarrow{\ \Phi^\dagger\ } & \Gamma(B) \\ {\scriptstyle \rho_A}\big\downarrow & & \big\downarrow{\scriptstyle \rho_B} \\ \mathrm{vect}(X) & \xrightarrow[\ \phi_*\ ]{} & \mathrm{vect}(Y) \end{array}$$

On the other hand, a vector bundle morphism (ϕ, Φ) is called a *Lie algebroid morphism* if Φ^* induces a chain map from the Lie algebroid complexe $\Gamma(\wedge^\bullet B^*)$ to $\Gamma(\wedge^\bullet A^*)$ (see [11]).

We denote by $\mathbf{Algd}^+$ the category of Lie algebroids and Lie algebroid *morphisms* and $\mathbf{Algd}^-$ the of Lie algebroids and Lie algebroid *comorphisms*.

Observe that the graph of both a Lie algebroid morphism and a Lie algebroid comorphism is a Lie subalgebroid of the Lie algebroid product $A \times B$ and that morphisms and comorphisms coincide (up to the direction of arrows when the core map is a diffeomorphism.

Example 2.1. Let $\phi : X \to Y$ be a smooth map. The tangent map $T\phi$ is a Lie algebroid morphism from TX to TY (seen as algebroids with identity as anchor and the usual Lie bracket on vector fields), while the cotangent map $T^*\phi$ is a Lie algebroid comorphism from T^*X to T^*Y (seen as Lie algebroids with zero anchor and zero bracket). $T\phi$ and $T^*\phi$ are both, at the same time, Lie algebroid morphisms and comorphisms when ϕ is a diffeomorphism.

Example 2.2. If $A = TX$ and $B = TY$ carry the usual Lie algebroid structures (as in the previous example), then (ϕ, Φ) is a comorphism from TX to TY when ϕ is a submersion and Φ is the horizontal lift map of a flat Ehresmann connection over the open submanifold $\phi(X) \subseteq Y$.

Example 2.3. If A and B are Lie algebras, considered as Lie algebroids over a point, then a Lie algebroid comorphism from A to B is a Lie algebra morphism from B to A, while a Lie algebroid morphism from A to B is a Lie algebra morphism from A to B.

As already noted by Higgins and Mackenzie in [12], Lie algebroid comorphisms are closely related to Lie algebroid actions. Without entering into much details here, let us recall these relations briefly. First of all, a comorphism (ϕ, Φ) from A to B induces an action of B on the map $\phi : X \to Y$, which endows the vector bundle pullback $\phi^! B$ with the structure of a Lie algebroid. This algebroid is called the **action Lie algebroid**, and there is a Lie algebroid *morphism* from it to $A \times B$, whose image is the comorphism graph. This map establishes a Lie algebroid isomorphism between the comorphism seen as an algebroid over the graph of ϕ and the action algebroid.

In a reciprocal way, an action of a Lie algebroid B on a map $\phi : X \to Y$ induces a Lie algebroid comorphism from the tangent bundle TX to B. This remark together with the previous paragraph show that a comorphism (ϕ, Φ) from A to B always induces another comorphism from TX to B; the original comorphism can then be decomposed into this particular comorphism from TX and a base-fixing one as observed by Higgins and Mackenzie in [12].

Example 2.4. Infinitesimal action of a Lie algebra $\mathcal{G}$ on a manifold X (i.e. Lie algebra morphism from $\mathcal{G}$ to the Lie algebra of vector fields on X) are in one-to-one correspondence with comorphisms from TX to $\mathcal{G}$.

2.2. Lie groupoids. Now let $G \rightrightarrows X$ and $H \rightrightarrows Y$ be groupoids with target and source maps l_G, r_G, l_H and r_H. A comorphism (ϕ, Φ) from r_G to r_H is called *a comorphism of groupoids* if

(1) Φ takes unit elements to unit elements;

(2) it is compatible with the target maps in the sense that, for any (x, h) in the pullback $X \times_Y H$, $(\phi \circ l_G)\big(\Phi(x, h)\big) = l_H(h)$;

(3) it is multiplicative in the sense that $\Phi(y, h_1)\Phi(z, h_2) = \Phi(z, h_1 h_2)$ whenever the products are defined; i.e., when $\phi(y) = l_H(h_2)$.

A groupoid comorphism as above may be represented by its graph $\gamma_{(\phi,\Phi)}$, which is the smooth closed subgroupoid of $G \times H \rightrightarrows X \times Y$ consisting of those pairs (g, h) for which $g = \Phi\big(r_G(g), h\big)$. The objects of $\gamma_{(\phi,\Phi)}$ are just the points of the graph of ϕ, and the projection to H of the source fibre of $\gamma_{(\phi,\Phi)}$ over $\big(x, \phi(x)\big)$ is a diffeomorphism onto the source fibre of H over $\phi(x)$. These properties characterize those subgroupoids of $G \times H$ which are the graphs of comorphisms.

Remark 2.5. Zakrzewski in [19] introduced the notion of regular D^*-algebra and showed that it coincides with that of Lie groupoid, observing though that their natural morphisms do not correspond to Lie groupoid morphisms. D^*-algebra morphisms were further studied by Stachura in [17], who called them simply "groupoid morphisms." From Lemma 4.1 in [19] and Proposition 2.6 in [17], one sees that D^*-algebra morphisms are exactly the Lie groupoid comorphisms introduced later on by Higgins and Mackenzie in [12]. However, neither Zakrzewski nor Stachura discussed a corresponding notion for Lie algebroids.

As for Lie algebroid comorphisms, a Lie groupoid comorphism (ϕ, Φ) from G to H induces a groupoid action of B on the map $\phi : X \to Y$, which turns the pullback

$$\phi^! H = X_\phi \times_{r_H} H$$

into a smooth groupoid, the action groupoid (see [12]). There is also a groupoid morphism from the action groupoid to $G \times H$, whose image is precisely $\gamma_{(\phi,\Phi)}$, implementing a groupoid isomorphism between $\gamma_{(\phi,\Phi)}$ seen as a groupoid over $\operatorname{gr}\phi$ and the action groupoid. Conversely, a groupoid action of H on a map $\phi : X \to Y$ yields a comorphism from the fundamental groupoid $\pi(X)$ (or the pair groupoid $X \times X$) to H. Based on that fact, there is a decomposition of

Lie groupoid comorphisms similar to the decomposition of Lie algebroid comorphisms described above.

Example 2.6. There is a one-to-one correspondence between actions of a Lie group G on a manifold X and Lie groupoid comorphisms from the pair groupoid $X \times X$ (or the fundamental groupoid $\pi(X)$) to G seen as a groupoid over a point.

A *morphism of Lie groupoids* is a functor between the underlying groupoids, whose object and morphism components are smooth.

We denote by $\mathbf{Gpd}^+$ the of Lie groupoids and Lie groupoid *morphisms* and $\mathbf{Gpd}^-$ the of Lie groupoids and Lie groupoid *comorphisms*.

Correspondingly, there are two Lie functors

$$\mathbf{Lie} : \mathbf{Gpd}^{\pm} \to \mathbf{Algd}^{\pm},$$

as defined in [12], [14], which agree on objects (i.e. they both send a Lie groupoid to its associated Lie algebroid) but one sends morphisms to morphisms while the other sends comorphisms to comorphisms. Geometrically though, the morphism component of both functors can be defined the "same way," using the object component. Namely, the underlying graph $\gamma_{(\phi,\Phi)}$ of a groupoid morphism or a groupoid comorphism (which, in both cases, we denote by (ϕ, Φ)) from G to H is itself a groupoid (actually, a subgroupoid of $G \times H$). Then $\mathbf{Lie}(\gamma_{(\phi,\Phi)})$ is a subalgebroid of $\mathbf{Lie}(G) \times \mathbf{Lie}(H)$, which is the graph of a Lie algebroid morphism when (ϕ, Φ) is a Lie groupoid morphism and the graph of a Lie algebroid comorphism when (ϕ, Φ) is a Lie groupoid comorphism.

The two Lie functors are essentially the same on Lie algebras, since morphisms and comorphisms are the same in this case except for arrow direction.

2.3. Integrability and completeness. We say that a Lie algebroid comorphism between integrable Lie algebroids is *integrable* if it is in the image of the Lie functor. This means that the (possibly only immersed) Lie subgroupoid integrating the comorphism graph (which is a Lie subalgebroid) is, at the same time, a closed embedded Lie subgroupoid *and* a comorphism (the latter implying the former).

If A and B are the Lie algebroids of groupoids G and H, then every Lie algebroid comorphism from A to B may be integrated locally to a groupoid comorphism from G to H. In contrast with Lie algebroid morphisms, which are always integrable to Lie groupoid morphisms under a simple connectivity assumption (see [15], Appendix for instance), the global situation for Lie algebroid comorphisms is more complicated. The following example gives a Lie algebroid comorphism whose graph integrates, as a Lie algebroid, to an embedded Lie subgroupoid that is not the graph of a Lie groupoid comorphism.

Example 2.7. The inclusion i of an open subset X in a manifold Y yields the Lie algebroid comorphism (i, id) from TX to TY with the natural Lie algebroid structure. The integration of the Lie subalgebroid $\gamma_{(i,\mathrm{id})}$ is the embedded subgroupoid

$$\{(x, x, x, x) : x \in X\} \rightrightarrows X \times i(X)$$

of the groupoid product $(X \times X) \times (Y \times Y) \rightrightarrows X \times Y$. This is not the graph of a comorphism, although there are partially defined maps (namely the identity restricted to X) from $\{x\} \times Y$ to $\{x\} \times X$ for each $x \in X$, the union of whose graphs is the integrating subgroupoid.

Although "partially defined" Lie groupoid comorphisms as in the example above still compose, and thus form a, even worse situations can arise. In general, a Lie algebroid comorphism can be integrated only to what Dazord calls a "relation," and which we will call a *hypercomorphism*. A hypercomorphism from $G \rightrightarrows X$ to $H \rightrightarrows Y$ consists of a map $\phi : X \to Y$ and a groupoid R over the graph of ϕ along with a homomorphism to $G \times H$ which is an immersion such that the projection to H is étale between source fibres of R and H. It is a comorphism just when these maps between source fibres are diffeomorphisms. The image of the immersion $R \to G \times H$ is a subgroupoid which can sometimes be neither smooth nor closed, as we will see in the next section.

As we will show in Theorem 5.1, global integrability in terms of Lie groupoid comorphisms is guaranteed if the source fibres of H are 1-connected and the Lie algebroid comorphism is *complete*. This means that the pullback map on sections takes complete sections of B to complete sections of A, where a section of a Lie algebroid is called complete if the anchor maps it to a complete vector field.[1]

This result on global integrability was first announced in [9] without proof. To the best of our knowledge, such a proof has never appeared, although very close results have been achieved in the context of Poisson geometry for complete Poisson maps (which induce complete Lie algebroid comorphisms as we shall see in Section 6) by Caseiro and Fernandes in [1] and by Zakrzewski in [19]. We will give one in Section 5.1 using the path integration techniques developed recently in [4], [7], [8].

We can already see the following:

Proposition 2.8. *Let G and H be Lie groupoids over X and Y respectively, and let (ϕ, Φ) be a groupoid comorphism from G to H. Then $(\phi, T\Phi) = \mathbf{Lie}(\phi, \Phi)$ is a* complete *comorphism from $\mathbf{Lie}(G)$ to $\mathbf{Lie}(H)$, where*

$$(T\Phi)(x, v) := D_2\Phi\big(x, \phi(x)\big)v,$$

[1] If A is integrable to a Lie groupoid G, completeness of a section of A means that the section is the initial derivative of a 1-parameter group of bisections of G.

and D_2 denotes the derivative w.r.t. the second argument. In other words, integrable Lie algebroid comorphisms are complete.

Proof. To simplify notation, set $A = \mathbf{Lie}(G)$ and $B = \mathbf{Lie}(H)$. Let s be a complete section of B. It induces a *complete* right-invariant vector field ξ_H^s on the integrating groupoid H (see [13], Appendix), whose corresponding left-invariant flow Ψ_t^H exists thus for all t. Using (ϕ, Φ), we define a left-invariant flow on G, which also exists for all times:

$$\overline{\Psi}_t^G(g) = R_g\big(\Phi\big(r_G(g), \Psi_t^H\big(\phi\big(r_G(g)\big)\big)\big)\big),$$

where $R_g(g') = g'g$ is the right-translation in G (where it makes sense).

On the other hand, the section $(T\Phi)^\dagger s$ induces a right-invariant vector field $\xi_G^{(T\Phi)^\dagger s}$ on G, whose flow Ψ_t^G projects on the flow Ψ_t^X on X of $\rho_A(T\Phi)^\dagger s$, that is,

$$\Psi_t^X\big(l_G(g)\big) = l_G\big(\Psi_t^G(g)\big).$$

What remains to be proven is that Ψ_t^G coincides with $\overline{\Psi}_t^G$: Since the latter exists for all t, this would imply that Ψ_t^X exists for all t and, thus, that the image $(T\Phi)^\dagger s$ of a complete section s by $(\phi, T\Phi)$ is complete. To see this, let us check that both flows are flows of the same vector field. Namely, since $\overline{\Psi}_t^G$ is left-invariant, we have that

$$\begin{aligned}
\frac{d}{dt_{|t=0}}\overline{\Psi}_t^G(g) &= DR_g\big(r_G(g)\big)D_2\Phi\big(r_G(g), \phi\big(r_G(g)\big)\big)\frac{d}{dt_{|t=0}}\Psi_t^H\big(\phi\big(r_G(g)\big)\big),\\
&= DR_g\big(r_G(g)\big)(T\Phi)\big(r_G(g), s\big(\phi\big(r_G(g)\big)\big)\big),\\
&= DR_g\big(r_G(g)\big)\big((T\phi)^\dagger s\big)\big(r_G(g)\big),
\end{aligned}$$

which, by definition, coincides with $\frac{d}{dt_{|t=0}}\Psi_t^G(g)$. □

Example 2.9. Let (ϕ, Φ) be a comorphism between tangent bundle Lie algebroids TX and TY. As noted above, this corresponds to a flat Ehresmann connection, i.e. a "horizontal" foliation of X for which the projection of each leaf to Y is étale. The comorphism is complete when the connection is complete in the sense that these projections are all covering maps, i.e. when each path $\sigma : [0,1] \to Y$ has a horizontal lift starting at any point in $\phi^{-1}\big(\sigma(0)\big)$.

To integrate this comorphism to a hypercomorphism between the fundamental groupoids $\pi(X)$ and $\pi(Y)$ integrating TX and TY respectively, we let R be the leafwise fundamental groupoid of the foliation of X. This is a (1-connected, but possibly non-Hausdorff) Lie groupoid over X and may hence be considered as a groupoid over the graph of $\phi : X \to Y$. An element of R is a homotopy class of paths with fixed endpoints and contained in a single leaf of the foliation. Let us call these "foliated paths". Mapping each such class of foliated paths to the homotopy class of paths in X (without the "leafwise" restriction) in which it is contained, and to the class of the image in Y, is a groupoid morphism from R to $G \times H$. Restricting this morphism to a source fibre of R and projecting to $\pi(Y)$ takes the homotopy classes of foliated paths beginning at some $x \in X$ to the homotopy classes of paths in Y beginning at $\phi(x)$. A neighborhood, in a source fibre of R, of the class of a foliated path σ may be identified with a neighborhood of $\sigma(1)$ in its leaf, and a neighborhood of the class of the projected path may be identified with a neighborhood of $\phi(\sigma(1))$ in Y. The projection from the first neighborhood to the second is étale by the definition of a flat Ehresmann connection, so the requirements for R to be a hypercomorphism are met. We may describe the relation R in rough terms by saying that it takes a point $x \in X$ and a path ρ in Y beginning at $\phi(x)$ to its horizontal lift through x. But this horizontal lift may not exist if (ϕ, Φ) is not complete, and it might not be unique since a homotopy of paths in Y may not have a horizontal lift in the absence of completeness.

3. An example

We give in this section an example of an Ehresmann connection, the graph of whose integration is neither closed nor embedded.

Let Y be $\mathbb{R}^2$ with cartesian coordinates (x, y) and polar coordinates (r, θ). X will be an open subset of $\mathbb{R}^2 \times \mathbb{C} \times \mathbb{R}$ with polar coordinates (r, θ) on the first factor, a complex coordinate $z = Re^{i\Theta}$ on the second, and real coordinate h on the third one. As the notation suggests, ϕ will be the projection on the first factor. X and Y will be 1-connected, so the source 1-connected groupoids integrating TX and TY will be $X \times X$ and $Y \times Y$. Since Y is 1-connected, the leaves of any foliation defined by a complete Ehresmann connection over Y are simply connected and therefore have trivial holonomy.

We define X as $\mathbb{R}^2 \times \mathbb{C} \times \mathbb{R} \backslash J$, where J is the three-dimensional slab

$$\{(r, \theta, R, \Theta, h) \mid r = 0, -1 \leq h \leq 1\}.$$

Although J is of codimension 2, the restriction on h leaves X simply connected, so that its fundamental groupoid is still $X \times X$. Nevertheless, we can construct an interesting Ehresmann connection for the submersion $\phi : X \to Y$. For $-1 \leq h \leq 1$, the horizontal subspaces of the connection are spanned by the vector fields $\partial/\partial r$ and $\partial/\partial\theta + v(h)\partial/\partial\Theta$, where v is a smooth function, not identically zero, supported in the interval $-\frac{1}{2} \le h \le \frac{1}{2}$. This makes sense since r is not zero for these values of h. Outside the support of v, the vector fields $\partial/\partial r$ and $\partial/\partial\theta$ may be replaced by the cartesian coordinate vector fields $\partial/\partial x$ and $\partial/\partial y$, which extend to the entire (x, y) plane.

We will think of our Ehresmann connection as a family, parametrized by h, of unitary connections on the trivial complex Lie bundle over Y with fibre coordinate z. The connection form in this description is $iv(h)\,d\theta$, and the holonomy around a loop encircling the origin in the (x, y) plane is multiplication by $e^{2\pi i v(h)}$. In the region where $-1 \le h \le 1$ (and so r is not zero), each leaf lies in a fixed level of R and h and is a covering of the punctured (r, θ) plane. The covering is a diffeomorphism if $R = 0$. For positive R, the covering has k sheets when $v(h)$ has order k as an element of $\mathbb{R}/\mathbb{Z}$; this includes the possibilities $k = 1$ and $k = \infty$. Over the region where $-1 < h < 1$, the leafwise fundamental groupoid may be parametrized by

$$\Gamma = (\mathbb{R}^+ \times S^1) \times (\mathbb{R}^+ \times \mathbb{R}) \times \mathbb{C} \times (-1, 1).$$

The element

$$\gamma = (r, \theta, r', \tau, z, h)$$

of Γ corresponds to the homotopy class of the horizontal path

$$t \mapsto \big(r + (r' - r)t, \theta + \tau t, e^{iv(h)\tau t}z, h\big), \quad 0 \le t \le 1.$$

Thus, the source map is

$$(r, \theta, r', \tau, z, h) \mapsto (r, \theta, z, h),$$

and the target is

$$(r, \theta, r', \tau, z, h) \mapsto (r', \theta + \tau, e^{iv(h)\tau}z, h).$$

The unit elements of the groupoid are defined by the conditions $r = r'$ and $\tau = 0$, while the isotropy groups are defined by $r = r'$, $\tau \in 2\pi\mathbb{Z}$, and $v(h)\tau \in 2\pi\mathbb{Z}$.

We now look at the leafwise fundamental groupoid as the integration of the Lie algebroid comorphism given by the flat Ehresmann connection. The Lie algebroid sits inside $TX \times TY$; since X and Y are simply connected, the integrating subgroupoid S should sit inside $X \times X \times Y \times Y$; it is the image of Γ under the target-source map

$$(r, \theta, r', \tau, z, h) \mapsto \big((r, \theta, z, h), (r', \theta + \tau, e^{iv(h)\tau}z, h), (r, \theta), (r', \theta + \tau)\big).$$

To study the immersion of Γ into $X \times X \times Y \times Y$, we can forget about the last two factors, since they are redundant (namely, the image of Γ lies in the graph of $\phi \times \phi$, which can be identified with $X \times X$). This image consists of all 8-tuples $(r, \theta, z, h, r', \theta', z', h')$ for which there exists τ such that $\theta' = \theta + \tau$, $z' = e^{iv(h)\tau} z$, and $h' = h$. The two conditions involving τ can be combined, with the elimination of τ, to give $z' = e^{iv(h)(\theta' - \theta)} z$. Where z is nonzero, these define, for each h, a hypersurface in the 4-torus with coordinates $(\theta, \arg z, \theta', \arg z')$. The subgroupoid $S \subset X \times X \times Y \times Y$ sits as a family of these hypersurfaces inside the 8-dimensional submanifold defined by $|z'| = |z|$ and $h' = h$, which is a bundle of these 4-tori over the space parametrized by (r, z, h).

From this description, we see immediately that S is not closed. In fact, when $v(h)$ is irrational, each of our hypersurfaces is dense but not closed in its 4-torus. To see that S has nontrivial self-intersections, we must look at the section $z = 0$ of our complex line bundle, since otherwise we are simply dealing with flat hypersurfaces in tori. In fact, when $z = 0$, adding an integer multiple of 2π to τ does not change the value of the target-source map, but it does change the image of the derivative as long as $v(h)$ is not an integer. This results in the sought-for nontrivial self-intersections.

4. Path construction

In this section, we start by briefly recalling the integration of Lie algebroids by Lie groupoids in terms of quotients of certain admissible path sets by homotopies, as in [4], [7]. We explain how this path construction allows us to integrate comorphisms between Lie algebroids to comorphisms between Lie groupoids. Then we show that a complete Lie algebroid comorphism from A to B allows us to lift admissible paths and homotopies in B to A (Proposition 4.2 and 4.3). These lifting properties, which make complete comorphisms resemble "Serre fibrations," will be the main ingredients in the proof of Theorem 5.1 in the next section.

Similar lifting properties in the context of Poisson map integration have already been considered by Caseiro and Fernandes in [1].

4.1. Lie algebroid integration. All the source 1-connected Lie groupoids integrating an integrable Lie algebroid $A \to X$ are isomorphic to the following construction in terms of homotopy classes of paths [4], [7]. Consider the space $\mathscr{P}(A)$ of admissible paths; i.e., the set of paths $g : [0, 1] \to A$, $g(t) = (x(t), \eta(t))$, where $x(t) \in X$ and $\eta(t)$ lies in the fiber of A over $x(t)$, such that

$$\frac{dx(t)}{dt} = \rho(x(t))\eta(t),$$

where ρ is the anchor map of A. The source 1-connected Lie groupoid integrating A can be realized as the quotient of $\mathscr{P}(A)$ by a homotopy relation $\sim$ that fixes the endpoints of the base component of the admissible path (see [4], [7]). More precisely, $\big(x_1(t), \eta_1(t)\big)$ is homotopic to $\big(x_2(t), \eta_2(t)\big)$ iff there is a family

$$\big(x_1(t), \eta_1(t)\big) \overset{s=0}{\longleftarrow} \big(x(t,s), \eta(t,s)\big) \overset{s=1}{\longrightarrow} \big(x_2(t), \eta_2(t)\big)$$

of admissible paths parametrized by $s \in [0,1]$ that satisfies the following condition: There exists a section β of A defined along $x(t,s)$ that vanishes for $t = 0, 1$, such that, locally,

$$\frac{\partial x^i(t,s)}{\partial s} = \rho_a^i\big(x(t,s)\big)\beta^a(t,s), \tag{2}$$

$$\frac{\partial \eta^c(t,s)}{\partial s} = \frac{\partial \beta^c(t,s)}{\partial t} + f_{ab}^c\big(x(t,s)\big)\eta^a(t,s)\beta^b(t,s), \tag{3}$$

where $\rho_a^i(x) : U \to \mathbb{R}$ and $f_{ab}^c(x) : U \to \mathbb{R}$ are the structure functions of respectively the anchor map and the Lie bracket on the sections of A expressed in terms of a system of trivializing sections $e_a : U \to A_{|U}$ (where a ranges from 1 to the dimension of the fibers in A) over the local patch with coordinates x^i.

We denote by $\Sigma(A)$ the quotient of $\mathscr{P}(A)$ by this homotopy relation and by $[g]$ the homotopy class of g. Since A is assumed to be integrable, $\Sigma(A)$ is a Lie groupoid over X, whose source and target maps $r, l : \Sigma(A) \rightrightarrows X$ are given by the endpoints of the path projection on the base: $r([g]) = x(0)$ and $l([g]) = x(1)$. The groupoid product is given by concatenation of paths $[g][g'] = [gg']$, where $g \in [g]$ and $g' \in [g']$ are two representatives whose ends $r([g]) = l([g'])$ match smoothly, and where

$$(gg')(t) = \begin{cases} 2g(2t), & 0 \le t \le \frac{1}{2}, \\ 2g'(2t-1), & \frac{1}{2} < t \le 1. \end{cases}$$

From now on, we will reserve the notation $\Sigma(A) \rightrightarrows X$ for the source 1-connected Lie groupoid integrating A coming from the construction above.

Note that $\Sigma(A)$ exists as a groupoid but not as a manifold if A is not integrable: for non-integrable Lie algebroids, $\Sigma(A)$ can be realized as a smooth stack (see [18]).

4.2. Comorphism integration. Suppose that $A' \to X'$ is a subalgebroid of an integrable Lie algebroid $A \to X$ with integrating 1-connected Lie groupoid $\Sigma(A) \rightrightarrows X$. Then A' is automatically integrable as a Lie algebroid, and we can take its integrating Lie groupoid to be the one obtained by the path construction; namely, $\Sigma(A') \rightrightarrows X'$. An admissible path in A' is by definition also an admissible path in A. Moreover, if two admissible paths are homotopic in A', they also are homotopic in A. Therefore, we have a natural immersion

$$\imath : \Sigma(A') \to \Sigma(A),$$

which is a groupoid morphism. However, this map is, in general, not an embedding nor is its image a closed submanifold. When the Lie subalgebroid A' is *over the same base* as A, then Moerdijk and Mrčun in [16] gave a necessary and sufficient condition for $\imath$ to be a closed embedding.

The situation is similar for the integration of a comorphism (ϕ, Φ) from a Lie algebroid $A \to X$ to a Lie algebroid $B \to Y$, since a comorphism graph $\gamma_{(\phi,\Phi)}$ is a Lie subalgebroid (over the graph of ϕ) of the direct product of the Lie algebroids A and B. If A and B are integrable, we can thus realize the hypercomorphism between the Lie groupoids $\Sigma(A)$ and $\Sigma(B)$ in terms of the path construction as the groupoid immersion

$$\imath : \Sigma(\gamma_{(\phi,\Phi)}) \to \Sigma(A) \times \Sigma(B).$$

Section 3 gave an explicit example of a comorphism for which $\imath$ is not an embedding and its image is not a closed submanifold.

Let us now describe $\Sigma(\gamma_{(\phi,\Phi)})$ in more explicit terms. It can be realized as the set of homotopy classes $[\gamma]$ of paths $\gamma(t) = \big(g(t), h(t)\big)$, where $g(t)$ is an admissible path in the Lie algebroid $A \to X$ and $h(t)$ is an admissible path in the Lie algebroid $B \to Y$ of the form

$$g(t) = \big(x(t), \Phi\big(x(t), \xi(t)\big)\big), \tag{4}$$
$$h(t) = \big(\phi\big(x(t)\big), \xi(t)\big). \tag{5}$$

In other words, γ is an admissible path in the Lie algebroid $\gamma_{(\phi,\Phi)} \to \operatorname{gr}\phi$ which satisfies the following equations

$$\dot{x}(t) = \rho_A\big(x(t)\big)\Phi\big(x(t), \xi(t)\big),$$
$$\phi_*\big(\dot{x}(t)\big) = \rho_B\big(\phi\big(x(t)\big)\big)\xi(t),$$

where ρ_A and ρ_B are, respectively, the anchor maps of A and B. The immersion $\imath$ from $\Sigma(\gamma_{(\phi,\Phi)})$ into $\Sigma(A) \times \Sigma(B)$ is the groupoid morphism given explicitly by

$$\imath : [(g,h)] \to ([g],[h]), \tag{6}$$

where $[(g,h)]$ is the class of admissible paths up to homotopy in $\gamma_{(\phi,\Phi)}$, while $([g],[h])$ is the corresponding pair of admissible paths up to homotopy in A and B, respectively.

4.3. Homotopy lifting property. In this section, we prove some lifting properties for admissible paths and homotopies via complete comorphisms. Let us start by stating two simple facts concerning complete Lie algebroid sections and comorphisms:

- Let s_t^A and s_t^B be (time-dependent) complete sections of Lie algebroids $A \to X$ and $B \to Y$, respectively. Then, $\tilde{s}_t = s_t^A \times s_t^B$ is a complete section of the Lie algebroid product $A \times B \to X \times Y$.
- For $i = 1, 2$, let (ϕ_i, Φ_i) be complete comorphisms from $A_i \to X_i$ to $B_i \to Y_i$. Then $(\phi_1 \times \phi_2, \Phi_1 \times \Phi_2)$ is a complete comorphism from $A_1 \times A_2 \to X_1 \times X_2$ to $B_1 \times B_2 \to Y_1 \times Y_2$.

Lemma 4.1. *Let (ϕ, Φ) be a complete Lie algebroid comorphism from $A \to X$ to $B \to Y$ and let $s_t : Y \to B$ be a complete (time-dependent) section of B. Then*

$$(\Phi^\dagger s_t)(x) = \Phi\big(x, s_t(\phi(x))\big)$$

is a complete (time-dependent) section of A.

Proof. To remove the time-dependency, we can lift s_t to the Lie algebroid $B \times T\mathbb{R} \to Y \times \mathbb{R}$ by considering the section

$$\tilde{s}(y, t) = s_t(y) + \partial_t,$$

which remains complete but which is now time-independent. The product $\tilde{\Phi} = (\phi \times \mathrm{id}_{\mathbb{R}}, \Phi \times \mathrm{id}_{T\mathbb{R}})$ is a complete comorphism from $A \times T\mathbb{R}$ to $B \times T\mathbb{R}$, since both factors are complete comorphisms. Thus, the lift

$$(\tilde{\Phi}^\dagger \tilde{s})(x, t) = (\Phi^\dagger s_t)(x) + \partial_t$$

is a complete section of $A \times T\mathbb{R}$, and the induced flow on $X \times \mathbb{R}$

$$\tilde{\Psi}_t(x, t) = \big(\Psi_t(x), t\big)$$

exists for all $x \in X$ and all times $t \in \mathbb{R}$. Since Ψ_t is the flow generated by the section $\Phi^\dagger s_t$, this implies that $\Phi^\dagger s_t$ is complete. □

Proposition 4.2 (Path lifting). *Let (ϕ, Φ) be a complete comorphism from the Lie algebroid $A \to X$ to the Lie algebroid $B \to Y$, and let $g(t) = \big(y(t), \xi(t)\big)$* be an admissible path in $\mathscr{P}(B)$. *Then, through any point $x \in \phi^{-1}\big(y(0)\big)$, there exists a smooth curve $x(t)$ starting at x, which projects onto $y(t)$ via ϕ, and such that*

$$\tilde{g}(t) := \big(x(t), \Phi\big(x(t), \xi(t)\big)\big)$$

is an admissible path in $\mathscr{P}(A)$.

Proof. It is enough to show that we can lift the admissible path g piecewise in coordinate patches. In a local chart, we can regard the base component $y(t)$ of a admissible path $g(t) = \big(y(t), \xi(t)\big)$ as being the integral curve of a time-dependent vector field; namely,

$$X_t(y) = \rho_B(y)\big(\chi(y)\tilde{\xi}(t)\big), \tag{7}$$

where we consider $s_t(y) := \chi(y)\tilde{\xi}(t)$ to be a local (time-dependent) section of B. In (7), χ is a cutoff function that vanishes outside a compact containing the image of the curve $y(t)$ and that is equal to 1 on a smaller compact containing it, and $\tilde{\xi}$ is a smooth extension of ξ to $\mathbb{R}$ that coincides with ξ on $[0,1]$.

The idea is to pullback (7) to a vector field on X and to obtain the lift of our admissible path as an integral curve of this new vector field.

Because of the cutoff function, X_t is compactly supported and thus complete. By Lemma 4.1, we obtain that $\Phi^\dagger s_t$ is complete. Thus the integral curve $x(t)$ of $\rho_A \Phi^\dagger s_t$ starting at the point

$$x(0) \in \phi^{-1}\big(y(0)\big),$$

exists for all t, and, in particular, for all $t \in [0,1]$. On this interval, we have that

$$\dot{x}(t) = \rho_A\big(x(t)\big)\Phi\big(x(t), \xi(t)\big),$$

which shows that

$$\big(x(t), \Phi\big(x(t), \xi(t)\big)\big)$$

is an admissible path that lifts the one we started with. □

Now we can apply Proposition 4.2 to homotopies

$$g(t,s) = \big(y(t,s), \xi(t,s)\big)$$

between admissible paths in $\mathscr{P}(B)$. By definition of homotopy, the path $g_s : t \mapsto g(t,s)$ is an admissible path in $\mathscr{P}(B)$ for each fixed value $s \in [0,1]$ of the homotopy parameter. Then, given a complete comorphism (ϕ, Φ) from A to B and a starting point $x \in \phi^{-1}\big(y(0,0)\big)$, Proposition 4.2 gives us a family of admissible paths,

$$\tilde{g}_s(t) = \big(x(t,s), \Phi\big(x(t,s), \xi(t,s)\big)\big), \tag{8}$$

in $\mathscr{P}(A)$ indexed by $s \in [0,1]$, and such that $\phi\big(x(t,s)\big) = y(t,s)$ for all t and s.

Proposition 4.3 (Homotopy lifting). *The family $\tilde{g}(t,s) := \tilde{g}_s(t)$ as above is a homotopy between admissible paths in $\mathscr{P}(A)$.*

Proof. Consider the Lie algebroid product $\tilde{A} := A \times TI \times TJ$ (resp. $\tilde{B} := B \times TI \times TJ$). We denote by t the variable in $I = [0,1]$ and by s the variable in $J = [0,1]$. We introduce the following local sections of $\tilde{B}$:

$$s_\xi(y,t,s) = \xi(t,s) + \partial_t,$$
$$s_\beta(y,t,s) = \beta(t,s) + \partial_s,$$

where $\xi(t,s)$ is the fiber component of the homotopy $g(t,s)$ and where $\beta(t,s)$ is the local expression of the associated section β_t, restricted to $y(t,s)$. Now the Lie algebroid bracket between these sections is

$$[s_\xi, s_\beta]_{\tilde{B}} = [\xi(s,t), \beta(s,t)] + [\xi(s,t), \partial_s] + [\partial_t, \beta(s,t)],$$

which in components yields

$$[s_\xi, s_\beta]^c_{\tilde{B}} = f^c_{ab}\xi^a\beta^b - \partial_s\xi(s,t)^c + \partial_t\beta(s,t)^c,$$

and thus vanishes, since g is a homotopy. Moreover, for each fixed s the curve $t \mapsto \big(y(t,s), t, s\big)$ is an integral curve of $\tilde{\rho}_B s_\xi$, while, for each fixed t, the map $s \mapsto \big(y(t,s), t, s\big)$ is an integral curve of $\tilde{\rho}_B s_\beta$, where $y(t,s)$ is the base component of the homotopy $g(t,s)$.

We can now lift the local sections s_ξ and s_β of $\tilde{B}$ to local sections $\tilde{\Phi}^\dagger s_\xi$ and $\tilde{\Phi}^\dagger s_\beta$ of $\tilde{A}$ via the comorphism $(\tilde{\phi}, \tilde{\Phi})$ from $\tilde{A}$ to $\tilde{B}$ defined by

$$\tilde{\phi} = \phi \times \mathrm{id}_I \times \mathrm{id}_J, \qquad \tilde{\Phi} = \Phi \times \mathrm{id}_{TI} \times \mathrm{id}_{TJ}.$$

Because $[s_\xi, s_\beta]_{\tilde{B}} = 0$ and because $\tilde{\Phi}^\dagger$ and $\tilde{\rho}_A$ are Lie morphisms, we obtain that

$$[\tilde{\Phi}^\dagger s_\xi, \tilde{\Phi}^\dagger s_\beta] = 0 \quad \text{and} \quad [\tilde{\rho}_A\tilde{\Phi}^\dagger s_\xi, \tilde{\rho}_A\tilde{\Phi}^\dagger s_\beta] = 0. \tag{9}$$

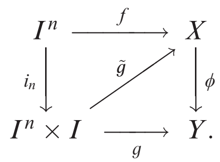
Now consider the family of curves

$$\gamma(t,s) := \big(x(t,s), t, s\big),$$

where $x(t,s)$ is the base component of the lift $\tilde{g}(t,s)$ in (8). A straightforward computation shows that the curve $\gamma(\cdot,s) : t \mapsto \gamma(t,s)$ is an integral curve of $\tilde{\rho}_A\tilde{\Phi}^\dagger s_\xi$ for each $s \in [0,1]$ (namely, we obtained these curves as lifts of admissible paths in B for each s, and thus they are admissible paths in A for each s). Similarly, a direct computation gives that the curve $\gamma(0,\cdot) : s \mapsto \gamma(0,s)$ is an integral curve of $\tilde{\rho}_A\tilde{\Phi}^\dagger s_\beta$ (this relies mostly on the fact that $\beta_0 = 0$ and that $x(0,s) = x$ is constant). Since the vector fields $\tilde{\rho}_A \tilde{\Phi}^\dagger s_\xi$ and $\tilde{\rho}_A \tilde{\Phi}^\dagger s_\beta$ commute, the family of integral curves of $\tilde{\rho}_A \tilde{\Phi}^\dagger s_\beta$ starting at $\gamma(t,0)$ for $t \in [0,1]$ coincide with the family $\gamma(t,\cdot) : s \mapsto \gamma(t,s)$, which implies, in particular, that $x(t,s)$ satisfies equation (2). Now the vanishing in (9) implies the second homotopy equation (3) by direct computation. $\square$

Corollary 4.4. *Let $A \to X$ and $B \to Y$ be two integrable Lie algebroids, and let (ϕ, Φ) be a complete comorphism from A to B. For all $g \in \mathcal{P}(B)$ with source $r_B(g)$ in the image of ϕ, we have that $(\tilde{g}, g) \in \mathcal{P}(\gamma_{(\phi,\Phi)})$, where $\tilde{g}$ is a lift of g through $x \in \phi^{-1}(r_B(g))$. Moreover, if $h \sim g$, then $(\tilde{h}, h) \sim (\tilde{g}, g)$.*

4.4. Analogy with Serre fibrations. There is a certain similarity between complete comorphisms and "Serre fibrations" in topology. Namely, a Serre fibration is a continuous map $\phi : X \to Y$ between topological spaces (more precisely CW-complexes) such that for all $n \geq 0$, $f : I^n \to X$ and $g : I^n \times I \to Y$ satisfying $\phi \circ f = g \circ i_n$, where $i_n : I^n \to I^n \times I$ is the inclusion given by $i_n(\vec{t}) = (\vec{t}, 0)$, there exists $\tilde{g}$ that makes the following diagram commute:

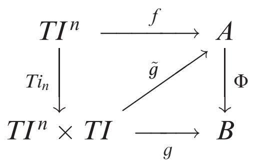


When $n = 0$, $I^0 = \{\star\}$, and we obtain the path lifting property for ϕ; when $n = 1$, we obtain the homotopy lifting property for homotopies between paths in Y.

The analogy comes from the following facts:

- An admissible path in the algebroid $A \to X$ is the same thing as a Lie algebroid morphism from TI to A;
- A homotopy between admissible paths is a Lie algebroid morphism from $TI \times TI$ to A;
- The tangent map to the inclusion i_n is a Lie algebroid morphism from TI^n to $TI^n \times TI$.

With this in mind, Propositions 4.2 and 4.3 can be summarized diagrammatically (for $n = 0, 1$) as follows:



where Φ is a complete algebroid comorphism from Lie algebroids $A \to X$ to $B \to Y$. For $n = 0$, g is an admissible path, and, for $n = 1$, g is a homotopy between admissible paths.

The problem with the diagram above is that its arrows do not belong to the same, since it involves morphisms and comorphisms of Lie algebroids. Going beyond a mere analogy would require a whose objects are the Lie algebroids and whose morphisms comprise both morphisms and comorphisms of Lie algebroids.

5. The integration functor

In [9], Dazord announced, without proof ([9], Thm. 4.1), that a complete comorphism between integrable Lie algebroids always integrates to a unique comorphism between the integrating Lie groupoids. We will prove this result here using the path construction, which, together with Proposition 2.8, yields an improvement of Dazord's Theorem: namely, that a Lie algebroid comorphism is integrable *if and only if* it is complete.

Actually, we will show that the classical integration functor for Lie algebras generalizes to integrable Lie algebroids and complete comorphisms:

Theorem 5.1. *The path construction Σ is a functor from the of integrable Lie algebroids and complete comorphisms to the of source* 1*-connected Lie groupoids and comorphisms. It is an inverse to the Lie functor* **Lie***, and, thus, implements an equivalence between these two categories.*

As corollary of Theorem 5.1, Proposition 2.8, and Corollary 5.11 (for the uniqueness part), we obtain Dazord's statement:

Corollary 5.2 (Dazord [9]). *Let $A \to X$ and $B \to Y$ be two integrable Lie algebroids with source* 1*-connected integrating Lie groupoids G and H. Then a Lie algebroid comorphism from A to B integrates to a* (*unique*) *Lie groupoid comorphism from G to H if and only if it is complete.*

The rest of this section is devoted to the proof of Theorem 5.1. In Paragraph 5.1, we show that Σ takes a complete Lie algebroid comorphism to a Lie groupoid comorphism. In Paragraph 5.2, we show that Σ is functorial, and in Paragraph 5.3 we show that it is a homotopy inverse to the Lie functor.

Remark 5.3. that a complete Lie algebroid comorphism integrates to a Lie groupoid comorphism would be to adapt the corresponding proof for complete Poisson maps of Caseiro and Fernandes (Prop. 4.8 an Prop. 4.9 in [1]) to comorphisms and to show that the resulting embedded subgroupoid is the graph of a Lie groupoid comorphism. One could also use Corollary 7 in [8], where it is stated that any complete action of a Lie algebroid $\Sigma(A)$ on $\mu : S \to M$ determines an action of the groupoid $\Sigma(A)$ on S and $\mu^*\big(\Sigma(A)\big) \simeq \Sigma(\mu^* A)$ as groupoids. Since, given a comorphism (ϕ, Φ) from $A \to X$ to $B \to Y$ induces an action of B on ϕ, one could then follow the lines of [1]. We thank an anonymous referee for this remark. We give in Paragraph 5.1 a different proof, which, however, relies also on the same kind of lifting properties as in [1] and [8].

5.1. Embeddability. Recall that any source 1-connected Lie groupoid integrating a Lie algebroid $A \to X$ is isomorphic to the Lie groupoid $\Sigma(A)$ obtained by the path construction. Therefore, in order to prove the first part of the theorem, it is enough to show that the immersion

$$\begin{aligned} \imath : \Sigma(\gamma_{(\phi,\Phi)}) &\to \Sigma(A) \times \Sigma(B), \\ [(g,h)] &\mapsto ([g],[h]), \end{aligned}$$

defined in (6) (i.e., the hypercomorphism integrating the comorphism (ϕ, Φ) from A to B) is a closed embedding whose image is the graph of a comorphism (ϕ, Ψ) from $\Sigma(A)$ to $\Sigma(B)$, when (ϕ, Φ) is complete.

For that consider the diagram

$$\begin{array}{ccc} \Sigma(\gamma_{(\phi,\Phi)}) & \xrightarrow{\ \imath\ } & \Sigma(A) \times \Sigma(B) \\ & \searrow^{K} & \Big\downarrow {\scriptstyle r_A \times \mathrm{id}_{\Sigma(B)}} \\ & & \phi^! \Sigma(B), \end{array}$$

where $\phi^!\Sigma(B)$ is the pullback $X_\phi \times_{r_B} \Sigma(B)$ in the category of smooth manifolds; since r_A is a submersion, this pullback is a closed submanifold of $X \times \Sigma(B)$. Observe that the composition $(r_A \times \mathrm{id}_{\Sigma(B)}) \circ \imath$ has its image in this pullback; namely,

$$(r_A \times \mathrm{id}_{\Sigma(B)}) \circ \imath : \big[\big(x(t), \Phi\big(x(t), \xi(t)\big)\big), \big(\phi\big(x(t)\big), \xi(t)\big)\big] \mapsto \big[\big(x(0), \big(\phi\big(x(t)\big), \xi(t)\big)\big)\big],$$

defining thus the smooth map K. Now the homotopy lifting properties for complete comporphisms in the form of Corollary 4.4 imply that K is invertible. In turn, this means that $r_A \times \mathrm{id}_{\Sigma(B)}$ is a diffeomorphism from the image of $\imath$ to the closed submanifold $\phi^!\Sigma(B)$. Therefore, the image of $\imath$ is also a closed submanifold of the product $\Sigma(A) \times \Sigma(B)$, and $\imath$ itself is an embedding. This yields that the hypercomorphism $\imath$ integrating the Lie algebroid comorphism (ϕ, Φ) is actually a Lie groupoid comorphism, when the Lie algebroid comorphism is complete.

Observe that we also obtain a very explicit description of the integrating comorphism (ϕ, Ψ) from $\Sigma(A)$ to $\Sigma(B)$; namely, the fiber maps are given by

$$\Psi_x([\gamma]) = [\tilde{\gamma}],$$

where $[\tilde{\gamma}]$ is the (unique) homotopy class of the lift of γ by the complete Lie algebroid comorphism through the point $x \in X$.

5.2. Functoriality. As we explained in Section 4.2, the path construction Σ associates a source 1-connected Lie groupoid $\Sigma(A)$ with an *integrable* Lie algebroid A. In the previous paragraph, we showed that Σ associates the comorphism $\Sigma(\gamma_{(\phi,\Phi)})$ from $\Sigma(A)$ to $\Sigma(B)$ with a *complete* comorphism from A to B.

We want to show that Σ is a functor from the of integrable Lie algebroids and complete comorphisms to the of source 1-connected Lie groupoids and comorphisms. For this, we need to show that

$$\Sigma(R_2) \circ \Sigma(R_1) = \Sigma(R_3),$$

where R_1 is the graph of a comorphism (ϕ_1, Φ_1) from A to B, R_2 is the graph of a comorphism (ϕ_2, Φ_2) from B to C, and R_3 is the graph of the composition of (ϕ_1, Φ_1) with (ϕ_2, Φ_2). The bases of the integrable Lie algebroids A, B, and C are, respectively, X, Y, and Z.

Recall that the composition of comorphisms between Lie algebroids, Lie groupoids, or, more generally, between fibrations $r_A : A \to X$, $r_B : B \to Y$ and $r_C : C \to Z$ is given by

$$(\phi_2, \Psi_2) \circ (\phi_1, \Psi_1) = (\phi_2 \circ \phi_1, \Psi_1 \star \Psi_2),$$
$$(\Psi_1 \star \Psi_2)(x, c) = \Psi_1\big(x, \Psi_2(\phi_1(x), c)\big),$$

for $x \in X$ and $c \in r_C^{-1}\big(\phi_2 \circ \phi_1(x)\big)$. This composition translates in terms of the comorphism graphs R_1 and R_2 into the composition of the underlying binary relations: i.e., the graph of the comorphism composition is the relation $R_2 \circ R_1$ in $A \times C$ obtained by projecting the image of

$$(R_1 \times R_2) \cap (A \times \Delta_B \times C),$$

where Δ_B is the diagonal in $B \times B$, to $A \times C$. The fact that these relations come from comorphism graphs guarantees that the result of the composition is a closed submanifold of $A \times C$.

Remark 5.4. There are three ways of looking at a Lie algebroid comorphism from A to B: (1) as the pair (ϕ, Φ); (2) as the underlying relation $\gamma_{(\phi,\psi)} \subset A \times B$;

(3) as the corresponding Lie algebroid $\gamma_{(\phi,\Phi)} \to \operatorname{gr}\phi$. Similarly, there are three ways of looking at a Lie groupoid comorphism from G to H: (1) as the pair (ϕ,Φ); (2) as the underlying relation $R_{(\phi,\Phi)} \subset G \times H$; (3) as the Lie groupoid $R_{(\phi,\Phi)} \rightrightarrows \operatorname{gr}\phi$.

Lemma 5.5. $\Sigma(R_2) \circ \Sigma(R_1)$ *contains* $\Sigma(R_2 \circ R_1)$.

Proof. Given $[\gamma] \in \Sigma(R_2 \circ R_1)$, we will exhibit an element $[\gamma_1] \times [\gamma_2] \in \Sigma(R_1) \times \Sigma(R_2)$, whose image by the projection

$$\Sigma(A) \times \Delta_{\Sigma(B)} \times \Sigma(C) \to \Sigma(A) \times \Sigma(C) \tag{10}$$

is precisely $[\gamma]$. Namely, a representative of $[\gamma] \in \Sigma(R_2 \circ R_1)$ is of the form

$$\gamma : t \mapsto \big(x(t), (\Phi_1 \star \Phi_2)\big(x(t), \xi(t)\big), (\phi_2 \circ \phi_1)\big(x(t)\big), \xi(t)\big)$$

for some path $t \mapsto \big(x(t), \xi(t)\big)$. We set

$$y(t) := \phi_1\big(x(t)\big),$$
$$\eta(t) := \Phi_2\big(y(t), \xi(t)\big).$$

This gives us two representatives of paths,

$$\gamma_1 = (g_1, h_1) \quad \text{in } \Sigma(R_1),$$
$$\gamma_2 = (g_2, h_2) \quad \text{in } \Sigma(R_2),$$

respectively given by

$$\gamma_1 : t \mapsto \big(x(t), \Phi_1\big(x(t), \eta(t)\big), \phi_1\big(x(t)\big), \eta(t)\big),$$
$$\gamma_2 : t \mapsto \big(y(t), \Phi_2\big(y(t), \xi(t)\big), \phi_2\big(y(t)\big), \xi(t)\big).$$

Since $h_1 = g_2$ by definition, we obtain that

$$[\gamma_1] \times [\gamma_2] \in \Sigma(A) \times \Delta_{\Sigma(B)} \times \Sigma(C),$$

and thus $[\gamma_1] \times [\gamma_2]$ projects via (10) on the equivalence class of the path

$$t \mapsto \big(x(t), \Phi_1\big(x(t), \eta(t)\big), \phi_2\big(y(t)\big), \xi(t)\big),$$

which we recognize to be precisely γ since

$$(\Phi_1 \star \Phi_2)\big(x(t), \xi(t)\big) = \Phi_1\big(x(t), \Phi_2\big(\phi_1\big(x(t)\big), \xi(t)\big)\big). \qquad \square$$

Lemma 5.6. $\Sigma(R_2 \circ R_1)$ *contains* $\Sigma(R_2) \circ \Sigma(R_1)$.

Proof. For this, consider

$$[\gamma] = [(\gamma_A, \gamma_B)] \in \Sigma(R_1),$$
$$[\delta] = [(\delta_B, \delta_C)] \in \Sigma(R_2)$$

such that

$$([\gamma], [\delta]) \in \Sigma(A) \times \Delta_{\Sigma(B)} \times \Sigma(C).$$

This means that $[\gamma_B]$ and $[\delta_B]$ define the same homotopy class of paths in $\mathscr{P}(B)$. Thus, there is a homotopy

$$\gamma_B \overset{\nu_B}{\rightsquigarrow} \delta_B, \qquad \nu_B(t,s) := \big(y(t,s), \eta(t,s)\big),$$

from γ_B to δ_B. The homotopy lifting property (Proposition 4.3) tells us that we can lift ν_B to a homotopy μ_A (among the admissible paths in $\mathscr{P}(A)$) of the form

$$\gamma_A \overset{\mu_A}{\rightsquigarrow} \delta_A, \qquad \mu_A(t,s) := \big(x(t,s), \Phi_1\big(x(t,s), \eta(t,s)\big)\big)$$

such that

$$\mu_A(t,0) = \gamma_A(t),$$
$$\phi_1\big(x(t,s)\big) = y(t,s),$$

and where we have set $\delta_A := \mu_A(t,1)$. Now putting ν_B and μ_A together, we obtain with Corollary 4.4 the homotopy

$$(\gamma_A, \gamma_B) \overset{\theta}{\rightsquigarrow} (\delta_A, \delta_B), \qquad \theta := (\mu_A, \nu_B),$$

among the paths in $\mathscr{P}(R_1)$. Thus, we can take (δ_A, δ_B) as a representative of $[\gamma] \in \Sigma(R_1)$. Finally, we see that the representative

$$([\gamma], [\delta]) := \big([(\delta_A, \delta_B)], [(\delta_B, \delta_C)]\big)$$

projects, after reduction, to $[(\delta_A, \delta_C)]$, which belongs to $\Sigma(R_2 \circ R_1)$. □

5.3. Equivalence. We now prove that the functor Σ is a homotopy inverse to the Lie functor **Lie**.

Since, by construction, $\mathbf{Lie}\,\Sigma(A) = A$ for any integrable Lie algebroid A, we have that $\mathbf{Lie} \circ \Sigma$ is the identity functor. We need to show that $\Sigma \circ \mathbf{Lie}$ is also homotopic to the identity functor.

Given a source 1-connected Lie groupoid G, it is well-known that $\Sigma(\mathbf{Lie}(G))$ is isomorphic, as groupoid, to G (see [7] for instance). Moreover for each such G, there is a canonical groupoid isomorphism $\alpha_G : \Sigma(\mathbf{Lie}(G)) \to G$, which is constructed as follows:

To begin with, consider the monodromy groupoid $\hat{G}$ of G, whose elements are end-point-fixing homotopy classes $[\gamma]$ of G-paths, that is, paths $\gamma : [0,1] \to G$ such that $\gamma(0)$ is a groupoid unit x and $\gamma(t)$ stays in the source-fiber of x (i.e. $r_G(\gamma(t)) = x$). When G is source 1-connected, the map

$$\mathbf{ev} : \hat{G} \to G : [\gamma] \mapsto \gamma(1)$$

is a groupoid isomorphism. There is also a groupoid isomorphism D from the monodromy groupoid $\hat{G}$ to $\Sigma(\mathbf{Lie}(G))$. Namely, given a G-path γ,

$$D(\gamma)(t) = T_{l_G(\gamma(t))} L_{\gamma(t)^{-1}} \dot{\gamma}(t),$$

where $L_g(h) = gh$ is the groupoid left-translation, is an admissible path in $\mathbf{Lie}(G)$. It turns out that D preserves the homotopy classes of G-paths and A-paths and defines a groupoid isomorphism with inverse $D^{-1} : \Sigma(\mathbf{Lie}(G)) \to \hat{G}$. We refer the reader to [7] for details and proofs.

For each source 1-connected groupoid, we can now define the groupoid isomorphism

$$\alpha_G := \mathbf{ev} \circ D^{-1} : \Sigma(\mathbf{Lie}(G)) \to G.$$

Since isomorphisms in $\mathbf{Gpd}^+$ and $\mathbf{Gpd}^-$ coincide, α_G is also a comorphism.

Let us show now that the α_G's are the components of a natural isomorphism between the identity functor and $\Sigma \circ \mathbf{Lie}$; in other words, that, for any Lie groupoid comorphism (ϕ, Φ) from G to H, the following diagram commutes:

$$\begin{array}{ccc} \Sigma \circ \mathbf{Lie}(G) & \xrightarrow{\alpha_G} & G \\ {\scriptstyle \Sigma\circ\mathbf{Lie}(\phi,\Phi)}\downarrow & & \downarrow{\scriptstyle (\phi,\Phi)} \\ \Sigma \circ \mathbf{Lie}(H) & \xrightarrow[\alpha_H]{} & H. \end{array}$$

This follows from the following lemmas.

Lemma 5.7. *The diagram above commutes if and only if*

$$(\alpha_G \times \alpha_H)(\gamma_{\Sigma\circ\mathbf{Lie}(\phi,\Phi)}) = \gamma_{(\phi,\Phi)}, \tag{11}$$

where $\gamma_{\Sigma\circ\mathbf{Lie}(\phi,\Phi)}$ *and* $\gamma_{(\phi,\Phi)}$ *are the graphs of the corresponding comorphisms.*

Proof. This comes from the fact that composition of comorphisms is the same as the composition of their underlying graphs as binary relations (i.e. $\gamma_{(\phi_1,\Phi_1)} \circ \gamma_{(\phi_2,\Phi_2)} = \gamma_{(\phi_1,\Phi_1)\circ(\phi_2,\Phi_2)}$). Namely, one checks that for sets G, H, G', H', bijections $\alpha_G : G' \to G$ and $\alpha_H : H' \to H$, and binary relations $R \subset G \times H$ and $R' \subset G' \times H'$, we have $R \circ \operatorname{gr} \alpha_G = \operatorname{gr} \alpha_H \circ R'$ (where the composition is the composition of binary relations) if and only if $(\alpha_G \times \alpha_H)(R') = R$. □

Lemma 5.8. *Let $G \rightrightarrows X$ and $H \rightrightarrows Y$ be Lie groupoids, with H source 1-connected. Then the graph $\gamma_{(\phi,\Phi)}$ of a comorphism (ϕ,Φ) from G to H (seen as a subgroupoid of $G \times H$) is source 1-connected.*

Proof. We need to show that the source fiber $s^{-1}(x,\phi(x))$ in $\gamma_{(\phi,\Phi)}$ is 1-connected for all $x \in X$. Since $\gamma_{(\phi,\Phi)}$ is a subgroupoid of $G \times H$, we have that

$$s^{-1}(x,\phi(x)) = (s_G \times s_H)^{-1}(x,\phi(x)) \cap \gamma_{(\phi,\Phi)} = \operatorname{gr} \Phi_x.$$

Because the domain $s_H^{-1}(\Phi(x))$ of Φ_x is 1-connected by assumption, so is its graph. □

Lemma 5.9. *We have that $\alpha_{G\times H} = \alpha_G \times \alpha_H$, for G and H source 1-connected Lie groupoids. Moreover, the restriction of α_G to a source 1-connected Lie subgroupoid H coincides with α_H; in other words, the following diagram commutes:*

$$\begin{array}{ccc} \Sigma \circ \mathbf{Lie}(H) & \xrightarrow{k} & \Sigma \circ \mathbf{Lie}(G) \\ \downarrow{\scriptstyle \alpha_H} & & \downarrow{\scriptstyle \alpha_G} \\ H & \xrightarrow[i]{} & G \end{array}$$

where k sends a homotopy class $[\gamma]_H$ of admissible paths in $\mathbf{Lie}(H)$ to the homotopy class of path $[\gamma]_G$ in $\mathbf{Lie}(G)$, and i is the inclusion of G in H.

Proof. The first statement follows from the facts that

$$\begin{aligned} \Sigma \circ \mathbf{Lie}(G \times H) &= \Sigma \circ \mathbf{Lie}(G) \times \Sigma \circ \mathbf{Lie}(H), \\ \widehat{G \times H} &= \hat{G} \times \hat{H}, \\ \mathbf{ev}_{G\times H} &= \mathbf{ev}_G \times \mathbf{ev}_H, \\ D_{G\times H} &= D_G \times D_H. \end{aligned}$$

As for the second statement, observe first that k is well-defined, since an admissible path in $\mathbf{Lie}(H)$ is also an admissible path in $\mathbf{Lie}(G)$, and homotopic

Lie(H)-paths are also homotopic as **Lie**(G)-paths (since **Lie**(H)-homotopies are also **Lie**(G)-homotopies). Moreover, an admissible **Lie**(H)-path γ integrates to a H-path, which, considered as a G-path, is the same as the one γ integrates to when considered as a **Lie**(G)-path. We can then conclude by chasing in the diagram above, starting with the representative γ. □

Lemma 5.10. *The α's defined above are the components of a natural transformation.*

Proof. By Lemma 5.7, we only need to show that the Lie groupoid comorphisms $(\alpha_G \times \alpha_H)(\gamma_{\Sigma\circ\mathbf{Lie}(\phi,\Phi)})$ and $\gamma_{(\phi,\Phi)}$ coincide. Since $\gamma_{(\phi,\Phi)}$ is a source 1-connected groupoid by Lemma 5.8, we have that $\alpha_{\gamma_{(\phi,\Phi)}}$ is an isomorphism from $\gamma_{\Sigma\circ\mathbf{Lie}(\phi,\Phi)}$ to $\gamma_{(\phi,\Phi)}$. From Lemma 5.9, we can conclude that the restriction of $\alpha_G \times \alpha_H = \alpha_{G\times H}$ to the subgroupoid $\gamma_{\Sigma\circ\mathbf{Lie}(\phi,\Phi)}$ coincides with $\alpha_{\gamma_{(\phi,\Phi)}}$, whose image is exactly $\gamma_{(\phi,\Phi)}$. □

Consequently, Σ is a homotopy inverse to the Lie functor, implementing thus an equivalence of categories. As corollary, we have that

Corollary 5.11. *The Lie functor is faithful. In other words, the Lie groupoid comorphism integrating a complete Lie algebroid comorphism is unique.*

6. The symplectization functor

There is an immediate application of Theorem 5.1 in Poisson geometry. Namely, this theorem implies, as we will see below, that the integration of Poisson manifolds by symplectic groupoids using the path construction is an actual functor from the of integrable Poisson manifolds and *complete* Poisson maps to the **SGpd** of source 1-connected symplectic groupoids and *symplectic comorphisms*.

A *symplectic comorphism* from symplectic groupoids $G \rightrightarrows X$ to $H \rightrightarrows Y$ is a comorphism (ϕ, Φ) whose underlying graph $\gamma_{(\phi,\Phi)}$ is a canonical relation from G to H. In contrast with general canonical relations, symplectic comorphisms always compose well (because they are comorphisms in the first place), and thus form a. Observe that the graph $\gamma_{(\phi,\Phi)}$ of a symplectic comorphism is a lagrangian subgroupoid of $\bar{G} \times H$.

A *complete Poisson map* ϕ from X to Y is a Poisson map with the property that the hamiltonian vector field $\xi_{\phi^* f}$ on X with hamiltonian $\phi^* f$ is complete if the hamiltonian vector field ξ_f on Y with hamiltonian $f \in C^\infty(Y)$ is complete.

Fernandes in [10] studied constructions in Poisson geometry involving the integration of Poisson manifolds seen as a functor, which he called the "symplectization functor." However, in [10] the domain of this functor comprises all Poisson maps and its range has for morphisms from G to H all the lagrangian subgroupoids of $\bar{G} \times H$, instead of only those that are graphs of symplectic comorphisms. This choice has as a consequence that the range is not an honest (the compositions are not always well-defined). Moreover, if we drop the completeness condition, there are Poisson maps that do not integrate to symplectic comorphisms as illustrated in the example below. Hence, the symplectization functor is not a true functor with this choice of domain and range.

Example 6.1. Consider the non-complete and non-integrable Lie algebroid comorphism (ϕ, Φ) from TX to TY of Section 3. Its dual Φ^* is a non-complete Poisson map from cotangent bundles T^*X to T^*Y endowed with their canonical Poisson structure. Let us see that Φ^* is also non-integrable. Since the graph $\operatorname{gr}\phi^*$ is a coisotropic submanifold of the *symplectic* manifold $\overline{T^*X} \times T^*Y$ (with symplectic form $\Omega = -\omega + \omega$), the (immersed) lagrangian subgroupoid integrating Φ^* can be identified with the leafwise fundamental groupoid of the characteristic foliation $\tilde{\mathscr{F}}$ of $\operatorname{gr}\phi^*$ (whose associated distribution we denote by $\tilde{\Delta}$). For two vectors in the tangent space to $\operatorname{gr}\phi^*$ (which we identify with vectors $\underline{v} = v \oplus \theta_v$ in $TT^*X \simeq TX \oplus T^*X$), we have that

$$\Omega(\underline{v}, \underline{w}) = -\langle (\mathrm{id} - \Phi \circ T\phi)v, \theta_w \rangle + \langle (\mathrm{id} - \Phi \circ T\phi)w, \theta_v \rangle.$$

From this last equation, we see that $\Omega(\underline{v}, \underline{w}) = 0$ for all vectors $\underline{v}$ tangent to $\operatorname{gr}\phi^*$ iff $\underline{w} \in \operatorname{Im}\Phi \oplus (\operatorname{Ker} T\phi)^0$, where $\operatorname{Im}\Phi$ is the distribution of the foliation given by the flat Ehresmann connection associated with (ϕ, Φ) and $(\operatorname{Ker} T\phi)^0$ is the annihilator of vertical distribution associated with the submersion ϕ. Hence,

$$\tilde{\Delta} = \operatorname{Im}\Phi \oplus (\operatorname{Ker} T\phi)^0,$$

and the leafwise fundamental groupoid of $\tilde{\mathscr{F}}$ may be parametrized by $\tilde{\Gamma} = T^*\Gamma$, where

$$\Gamma = (\mathbb{R}^+ \times S^1) \times (\mathbb{R}^+ \times \mathbb{R}) \times \mathbb{C} \times (-1, 1)$$

parametrizes the leafwise fundamental groupoid of the Ehresmann connection on X as described in Section 3. The element

$$\tilde{\gamma} = (r, \theta, r', \tau, z, h, \xi_r, \xi_\theta, \xi_r', \xi_\tau, \xi_z, \xi_h)$$

of $\tilde{\Gamma}$ corresponds to the homotopy class of the leafwise path

$$t \mapsto \big(r + (r' - r)t, \theta + \tau t, e^{iv(h)\tau t}z, h, \xi_r + (\xi_r' - \xi_r)t, \xi_\theta + \xi_\tau t, \xi_z, \xi_h\big),$$

with $0 \leq t \leq 1$. We see that we obtain the same non-trivial self-intersections for the immersion

$$\tilde{\Gamma} \to \overline{T^*X} \times T^*X \times \overline{T^*Y} \times T^*Y$$

at $z = 0$ for the exact same reasons as in Section 3.

As with Lie algebroids, there is a Lie functor **Lie** from **SGpd** to the of Poisson manifolds and Poisson maps. It takes a symplectic groupoid $G \rightrightarrows X$ to the Poisson manifold (X, Π_X), where Π_X is the unique Poisson structure turning r_G into a Poisson map (and l_G into an anti-Poisson map). Since the graph of a symplectic comorphism (ϕ, Φ) from $G \rightrightarrows X$ to $H \rightrightarrows Y$ is a lagrangian subgroupoid $\gamma_{(\phi,\Phi)} \rightrightarrows \operatorname{gr}\phi$, this implies that the graph of ϕ is a coisotropic submanifold (see [2]), and, hence, that ϕ is a Poisson map. The Lie functor on morphisms is thus defined as $\mathbf{Lie}(\phi, \Phi) = \phi$.

The path construction can also be extended in a functorial way to the Poisson realm. Namely, integrating a Poisson manifold X is equivalent to integrating its associated Lie algebroid $T^*X \to X$, whose bracket on sections is the Koszul bracket and whose anchor map is the map $\Pi^\# : T^*X \to TX$ associated with the Poisson bivector field $\Pi \in \Gamma(\wedge^2 TX)$. The path construction applied to this Lie algebroid yields a symplectic groupoid $\Sigma(T^*X) \rightrightarrows X$ (when the Poisson manifold is integrable as seen in [4]), that is, a groupoid whose total space is symplectic and whose multiplication graph is a lagrangian subgroupoid of $\overline{\Sigma(T^*X)} \times \overline{\Sigma(T^*X)} \times \Sigma(T^*X)$. (The bar on a Poisson manifold denotes the same Poisson manifold but with opposite Poisson structure.)

To extend the path construction to Poisson maps, we need the following proposition, which can already be (partly) found in [5] and in [12]:

Proposition 6.2. *Let (X, Π_X) and (Y, Π_Y) be two Poisson manifolds. A smooth map $\phi : X \to Y$ is a Poisson map if and only if its cotangent map $T^*\phi$ is a Lie algebroid comorphism from T^*X to T^*Y (with the Lie algebroid structure described above). Moreover, ϕ is complete if and only if $T^*\phi$ is.*

Proof. $T^*\phi$ is a comorphism if and only if its dual, the tangent map $T\phi$ from TX to TY, is a Poisson map with respect to the Poisson structure on TX and TY inherited from being duals of Lie algebroids. Thus we only need to show that ϕ is Poisson if and only if $T\phi$ is. Now a smooth map between two Poisson manifolds is Poisson if and only if its graph is a coisotropic submanifold of the product of the two Poisson manifolds. Hence, the problem reduces to showing that a submanifold C of a Poisson manifold X is coisotropic if and only if TC is coisotropic in TX.

Recall that the Poisson structure on TX can be described locally in terms of the matrix

$$\tilde{\Pi}_X(x,v) = \begin{pmatrix} 0 & \Pi_X(x) \\ -\Pi_X(x) & \partial_k \Pi_X(x) v^k \end{pmatrix}$$

and that we can identify a vector in $T_{(x,v)}TC$ with $\delta x \oplus \delta v \in T_x C \oplus T_x C$. This further gives the identification of $N^*_{(x,v)}TC$ with $N^*_x C \oplus N^*_x C$. Now TC is coisotropic iff for all $\theta \oplus \nu \in N^*_{(x,v)}TC$, we have that

$$\tilde{\Pi}^{\#}_X(x,v)(\theta \oplus \nu) = \Pi^{\#}_X(x)\nu \oplus \left(-\Pi^{\#}_X(x)\theta + \partial_k \Pi^{\#}_X(x) v^k \nu\right) \in T_x C \oplus T_x C,$$

which is equivalent to C being coisotropic, since $\partial_k \Pi^{\#}_X(x) v^k \nu$ is always in $T_x C$ provided that C is coisotropic (to see this, consider the derivative at 0 of the curve $\left(x(t), \Pi_X\left(x(t)\right)\nu\right)$ in TC such that $x(0) = x$, $\dot{x}(0) = v$, and ν is a section of N^*C).

This shows that ϕ is Poisson if and only if $T^*\phi$ is a Lie algebroid comorphism. Let us check now that ϕ is complete whenever $T^*\phi$ is.

First of all, a direct computation shows that the hamiltonian vector field $\xi_{\phi^* f}$ where $f \in C^\infty(Y)$ coincides with $\Pi^{\#}_X(T^*\phi)^\dagger\, df$. Moreover, by definition, ξ_f is complete if and only if the section df is complete.

Suppose now that the comorphism $(\phi, T^*\phi)$ is complete. Take a complete hamiltonian vector field ξ_f. Then $(T^*\phi)^\dagger\, df$ is complete (since df is complete) which implies thus that $\xi_{\phi^* f}$ is also complete. Therefore, the Poisson map ϕ is complete.

We prove the converse by contradiction. Suppose that ϕ is complete and that $s \in \Gamma(B)$ is a complete section with $(T^*\phi)^\dagger s$ non-complete. This means that there is an integral curve $x(t)$ of $\Pi^{\#}_X(T^*\phi)^\dagger s$ starting at $x(0) = x$ that does not exist beyond a certain time $\tilde{t}$. Consider the integral curve $y(t)$ of $\Pi^{\#}_Y s$ that starts at $y = \phi(x)$. As long as $x(t)$ exists, we have that $\phi\left(x(t)\right) = y(t)$. Now, since s is complete, the integral curve $y(t)$ exists for all times, including (and beyond) $\tilde{t}$. Since $y(t)$ is contained in a symplectic leaf of X, there is a sufficiently small $\varepsilon > 0$ and an open set U in Y containing $y([\tilde{t} - \varepsilon, \tilde{t} + \varepsilon])$ together with a function $f : M \to \mathbb{R}$ with compact support contained in U, whose hamiltonian vector field ξ_f coincides with $\Pi^{\#}_Y s$ on the curve $y(t)$ (but not necessarily on the whole U). (One can see this by taking local Darboux coordinates turning $y(t)$ into a straight line and considering a linear hamiltonian.) The Poisson map ϕ being complete, the integral curve $\bar{x}(t)$ of $\xi_{\phi^* f}$ starting at $\bar{x}(\tilde{t} - \varepsilon) = x(\tilde{t} - \varepsilon)$ exists for all t (including and beyon $\tilde{t}$). Since ξ_f and $\Pi^{\#}_X s$ coincide on $y(t)$ (for all $t \in \mathbb{R}$), their lifts $\xi_{\phi^* f}$ and $(T^*\phi)^\dagger s$ coincide on $x(t)$ for $t \in [\tilde{t} - \varepsilon, \tilde{t})$; therefore the integral curves $x(t)$ and $\bar{x}(t)$ coincide on $[\tilde{t} - \varepsilon, \tilde{t})$. But now, for $t \in [\tilde{t}, \tilde{t} + \varepsilon)$, we have that $\xi_{\phi^* f}\left(\bar{x}(t)\right)$ coincides with $\Pi^{\#}_X(T^*\phi)^\dagger s\left(\bar{x}(t)\right)$ because ξ_f and $\Pi^{\#}_Y s$ coincide on $y(t)$; thus, the integral curve $\bar{x}(t)$ is also an integral curve of $\Pi_X^{\#}(T^*\phi)^{\dagger}s$ that extends $x(t)$ beyond $\tilde{t}$. This contradicts our assumption that $x(t)$ can not be extended beyond $\tilde{t}$. □

The graph of a Poisson map ϕ from X to Y is a coisotropic submanifold of the Poisson manifold product $\overline{X} \times Y$. Thus, the conormal bundle $N^* \operatorname{gr} \phi$ to this graph is a subalgebroid of the algebroid product $\overline{T^*X} \times T^*Y$. Proposition 6.2 tells us that this conormal bundle is actually the graph of the Lie algebroid comorphism $(\phi, T^*\phi)$ from T^*X to T^*Y. Applying the path construction to this comorphism yields a hypercomorphism

$$\imath : \Sigma(N^* \operatorname{gr} \phi) \to \overline{\Sigma(T^*X)} \times \Sigma(T^*Y)$$

between the integrating symplectic groupoids, which happens to be in general only a lagrangian immersion (see [2] for instance).

Using Theorem 5.1 and Proposition 6.2, we obtain that, for a complete Poisson map, the lagrangian immersion $\imath$ is a closed lagrangian embedding, and its image, which we denote by $\Sigma(\phi)$, is a closed lagrangian submanifold of $\overline{\Sigma(T^*X)} \times \Sigma(T^*Y)$. In other words, for complete Poisson maps, $\Sigma(\phi)$ is, at the same time, a canonical relation from $\Sigma(T^*X)$ to $\Sigma(T^*Y)$, a lagrangian subgroupoid over the graph of ϕ, and a comorphism from $\Sigma(T^*X)$ to $\Sigma(T^*Y)$.

In complete analogy with the Lie algebroid case, we can summarize the discussion above by the following statements, some of which can already be found in [1], [9], [19]:

Proposition 6.3 (Zakrzewski [19]). *Let G and H be symplectic groupoids over X and Y respectively, and let (ϕ, Φ) be a symplectic comorphism from G to H. Then $\phi = \mathbf{Lie}(\phi, \Phi)$ is a* complete *Poisson map from X to Y.*

Proof. The lagrangian subgroupoid $\gamma_{(\phi,\Phi)} \rightrightarrows \operatorname{gr} \phi$ integrates the coisotropic submanifold $\operatorname{gr} \phi$, and, thus, integrates the corresponding Lie subalgebroid $N^* \operatorname{gr} \phi$ (see [2] for instance), which is nothing but the graph of the comorphism $T^*\phi$ from T^*X to T^*Y. By Proposition 2.8, we have then that $T^*\phi$ is complete because integrable, and, hence, that ϕ is complete by Proposition 6.2. □

Theorem 6.4. *The path construction Σ is a functor from the of integrable Poisson manifolds and complete Poisson maps to the of source* 1*-connected symplectic groupoids and symplectic comorphisms. It is an inverse to the Lie functor* **Lie**, *and, thus, implements an equivalence between these two categories.*

Corollary 6.5 (Caseiro-Fernandes [1], Dazord [9], Zakrzewski [19]). *Let X and Y be two integrable Poisson manifolds with source* 1*-connected integrating symplectic groupoids G and H. Then a Poisson map ϕ from X to Y integrates to a* (*unique*) *symplectic comorphism (ϕ, Φ) from G to H if and only if it is complete.*

Theorem 6.4 provides a rigorous foundation for constructions in Poisson geometry involving Σ in the spirit of Fernandes in [10] where Σ (which is called the "symplectization functor") is only considered heuristically.

Remark 6.6. A weaker version of Corollary 6.5 was already stated without proof by Dazord in [9], where only the implication from complete to integrable was considered. Recently, Caseiro and Fernandes in [1] proved that the object integrating a complete Poisson map from integrable Poisson manifolds X to Y is an embedded lagrangian subgroupoid of the symplectic groupoid product $\overline{\Sigma(X)} \times \Sigma(Y)$, using the path construction and similar lifting properties as described in Section 4.3. However, one can find versions and proofs of both this Corollary and Proposition 6.3 in Zakrzewski's paper [19], which was written even before the notion of comorphisms was formally introduced by Higgins and Mackenzie in [12]. In [19], Zakrzewski integrates complete Poisson maps not to symplectic comorphisms but to what he called "morphisms of S^*-algebras," which are defined as special canonical relations satisfying certain algebraic relations. One can shows that Zakrzewski's morphisms of S^*-algebras are nothing but symplectic comorphisms. His proof relies mostly on the method of characteristics for coisotropic submanifolds.

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