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Variations of gwistor space

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Abstract. We study natural variations of the G_2 structure $\sigma_0 \in \Lambda^3_+$ existing on the unit tangent sphere bundle SM of any oriented Riemannian 4-manifold M. We find a circle of structures for which the induced metric is the usual one, the so-called Sasaki metric, and prove how the original structure has a preferred role in the theory. We deduce the equations of calibration and cocalibration, as well as those of W_3 pure type and nearly-parallel type.

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1. Introduction

In [6] it was shown how a natural $G_2 = \mathrm{Aut}\,\mathbb{O}$ structure is associated to the unit tangent sphere bundle $\pi : SM \to M$ of any given oriented Riemannian 4-manifold M. The techniques are twistorial, such as those learned by the author from [14], so we have chosen to give the name of G_2-twistors or simply *gwistors* to the new spaces.

The theory starts by a construction of the octonions inside TTM, restricted to the 3-sphere fibre bundle SM, which we take a moment to explain. Recall the Levi-Civita connection of the base induces a canonical splitting of the tangent bundle of TM. Both vertical and horizontal subbundles V, H become isometric to π^*TM with the pull-back metric. The direct-sum metric over TM is called the Sasaki metric of this manifold. Independently of the metric, V has a tautological section, denoted U and defined by $U_u = u$; hence also a vertical vector field on $SM = \{u \in TM : \|u\| = 1\}$. Now each point U_u is identified with the identity

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element, the generator of the real line in $T_uTM \simeq \mathbb{O}$. Then we use the volume form $\pi^* \mathrm{vol}_M$ coupled with U, to induce a cross-product on $u^\perp \subset V$. A conjugation map is equally trivial to define. Together these induce a quaternionic structure on V. Then, applying the well-known Cayley-Dickson process, we obtain the structure of $\mathbb{O}$ in $V \oplus H$.

The pull-back of TM also inherits a metric connection $\nabla^* = \pi^*\nabla$ and hence parallel identifications of horizontals with verticals, passing through π^*TM, cf. loc. cit. and [15]. The manifold SM is endowed with the induced metric from the canonical or Sasaki metric on TM. Clearly TSM coincides with $V_1 \oplus H$ where $V_1 = \{v \in V : \langle U_u, v\rangle = 0\}$ at each point u. Since u is pointing outwards, our space SM inherits a G_2-structure, for which it receives the name of gwistor space. Recall $\mathrm{G}_2 = \mathrm{Aut}\,\mathbb{O}$, but clearly the structure is the extension of an $\mathrm{SO}(3)$ structure. The connection induces a projection $\nabla^*_{\cdot} U : TSM \to V$ with kernel H and the identity on V.

By a Theorem of Y. Tashiro in [7] it is known that SM has a metric almost contact structure for a Riemannian base of arbitrary dimension. As these are rigid geometrical objects, the contact structure is bound to be K-contact if and only if M has constant sectional curvature 1. Then it turns out also to be Sasakian. Locally the space is the same as the Stiefel manifold $V_{5,2} = \mathrm{SO}(5)/\mathrm{SO}(3)$.

Now we leave aside the Cayley–Dickson process and concentrate on the five invariant 3-forms which are naturally defined on SM. Then we may try to find other interesting G_2 structures. This article is devoted to them, the *variations* of gwistor space, which may also be called g-natural G_2-structures on the unit tangent sphere bundle, in analogy with the terms for the metrics used by [1], [2] and many references therein. On the other hand, the terms deformation or perturbation are also used in similar context by other authors, so we made a choice.

We readily announce the support of some computer algebra software for the proof of Theorem 1.13 below. It is a polynomial computation of the 7th order in four variables which we believe anyone can reproduce easily.

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The author dedicates this work to Marta Barata.

1.1. The basic 3-forms. We start by abbreviating the notation and write $SM = \mathscr{G}$. There is, as we have seen, an isometry connecting H with V, which we denote by B. We extend it by 0 to V, thus defining an endomorphism B of TTM. Then the transpose tangent vector field B^tU generates a real line bundle, contained in $T\mathscr{G}$, and a 1-form $\theta = (B^t U)^\flat$. We may write a splitting, with $H_1 = B^t V_1$:

$$T^*\mathscr{G} = \mathbb{R}\theta \oplus H_1^* \oplus V_1^*. \tag{1}$$

We pass to the language of differential forms. The 1-form θ is the aforementioned almost contact structure, satisfying

$$\theta_u(v) = \langle u, \mathrm{d}\pi(v)\rangle \qquad \text{for all } u \in \mathscr{G},\ v \in T\mathscr{G}.$$

The usual pull-back (horizontal) of the volume form of M is also denoted by vol. The vertical pull-back of vol $\in \Omega^4(M)$ contracted with U is denoted by α; then we define analogously a 3-form $\alpha_3 = (B^t U) \lrcorner \mathrm{vol}$. Of course,

$$\theta \wedge \alpha_3 = \mathrm{vol}, \qquad \mathrm{vol} \wedge \alpha = \mathrm{Vol}_{\mathscr{G}}.$$

As shown in [4], it is possible to find locally an ‘adapted’ frame, i.e. an oriented orthonormal frame $e_0, e_1, \ldots, e_6$ respecting (1). In particular such that (with usual notation for the co-framing, $e^{ab\ldots c} = e^a \wedge e^b \wedge \cdots \wedge e^c$)

$$\theta = e^0, \qquad \alpha_3 = e^{123}, \qquad \alpha = e^{456}. \tag{2}$$

It is easy to compute that $\mathrm{d}\theta(v, w) = \langle (B^t - B)v, w\rangle$ for all $v, w \in T\mathscr{G}$, which restricts to a symplectic 2-form on the vector bundle $H_1 \oplus V_1$ and hence is written as $\mathrm{d}\theta = e^{41} + e^{52} + e^{63}$.

The endomorphism B allows one to construct two other 3-forms. We turn the reader to [4] for the invariant definition, i.e., to see these forms depend only of the metric on M and not of the choice of adapted frame. They are:

$$\alpha_1 = e^{156} + e^{264} + e^{345}$$

and

$$\alpha_2 = e^{126} + e^{234} + e^{315}.$$

One can prove the five 3-forms $\alpha, \alpha_1, \ldots, \alpha_3$, $\theta \wedge \mathrm{d}\theta$ correspond to a basis for the space of invariants in $\Lambda^3(\mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3)$ under the action of $\mathrm{SO}(3)$, the underlying structure group of $\mathscr{G}$, i.e., there are five irreducible 1-dimensional submodules[1].

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[1] The author acknowledges I. Agricola and Th. Friedrich for this computation.

The five 3-forms satisfy the ‘first structure equations’ for all $i = 1, 2, 3$:

$$\begin{gathered}
*\alpha = \theta\wedge\alpha_3 = \mathrm{vol} = \pi^*\,\mathrm{vol}_M, \qquad *\alpha_1 = -\theta\wedge\alpha_2, \qquad *\alpha_2 = \theta\wedge\alpha_1, \\
*\mathrm{d}\theta = \frac{1}{2}\theta\wedge(\mathrm{d}\theta)^2, \qquad *(\mathrm{d}\theta)^2 = 2\theta\wedge\mathrm{d}\theta, \qquad *(\mathrm{d}\theta)^3 = 6\theta, \\
\alpha_1\wedge\alpha_2 = 3\alpha_3\wedge\alpha = 3*\theta = \frac{1}{2}(\mathrm{d}\theta)^3, \qquad (3) \\
\mathrm{d}\theta\wedge\alpha_i = \mathrm{d}\theta\wedge*\alpha_i = \alpha_3\wedge\alpha_i = 0, \\
\mathrm{d}\theta\wedge\alpha = \mathrm{d}\theta\wedge*\alpha = \alpha\wedge\alpha_1 = \alpha\wedge\alpha_2 = 0.
\end{gathered}$$

The natural G_2 structure on $\mathscr{G}$ to which we have referred *is* given[2] by the 3-form

$$\sigma_0 = \alpha_2 - \alpha + \theta\wedge\mathrm{d}\theta.$$

This form gives the *canonical* representation theory without changing the canonical orientation of $\mathscr{G}$; namely it gives the usual G_2-modules Λ^2_7, Λ^2_{14} (which appeared from opposite highest weights in [4], [5], [6]).

The integrability of σ_0 was studied in the case of the Levi-Civita connection on M in the seminal article [6], and in the case of metric connections with torsion, which clearly allow the same construction, in [4]. For the first case we know that the G_2-twistor structure is cocalibrated, i.e., $\mathrm{d}*\sigma_0 = 0$, if and only if the base M is an Einstein manifold.

1.2. Variations of G_2 structures. Let us recall the definition of stable forms from the theory of G_2-manifolds, [8], [9].

Let σ denote a linear G_2 structure on a 7-dimensional oriented vector space V, i.e., some identification of V with the canonical $\mathbb{R}^7$ is assumed. A consequence of the study of the Lie group $\mathrm{G}_2 = \mathrm{Aut}\,\sigma \subset \mathrm{SO}(7)$ is that it is connected and 14-dimensional; henceforth, that the orbit of σ under $\mathrm{GL}(7,\mathbb{R})$ is an open set inside the module Λ^3V^*. This orbit is denoted $\Lambda^3_{\pm}$ and known as the space of stable G_2-structures on V. We detect the boundaries of such stability by the non-degeneracy of the induced Euclidean product. Indeed, the inner product $\langle\,,\rangle_\sigma$ is given by the clearly symmetric map $v\otimes w \mapsto v\lrcorner\sigma\wedge w\lrcorner\sigma\wedge\sigma$—with this image 7-form required to be, on the diagonal of V, a positive multiple of the chosen orientation. The given σ satisfies this condition by assumption. Allowing also σ to vary, we do have a $\mathrm{GL}(7,\mathbb{R})$-equivariant map

$$V\otimes V\otimes\Lambda^3V^* \to \Lambda^7V^*.$$

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[2] Actually the structure was given first by the opposite, $-\sigma_0$, but we take the opportunity here to make the change.

Since $\Lambda^7 V^*\backslash 0$ has two connected components, we conclude $\Lambda^3_\pm$ is the union of two open orbits under the action of the subgroup $\mathrm{GL}^+(7,\mathbb{R})$, identified bijectively by a $-$ sign because $(-1)^3 = (-1)^7$. Moreover, the orientation Vol_σ in V induced by the first map itself is preserved in each of *these* orbits. Next we shall be concerned only with the positive-definite side Λ^3_+ of $\Lambda^3_\pm$.

We further remark on the existence of split-octonionic structures, with automorphism group the non-compact dual form of G_2, this time inducing metrics of signature $(3,4)$ or $(4,3)$. In the following applications to gwistor space we shall not be worried with the parallelism with the split-octonionic structures, since such study may be much more easily undertaken later.

We return to the gwistor space $\mathscr{G} \to M$ and consider a variation of the standard structure σ_0. We let $f_0,\dots,f_4$ be scalar functions on $\mathscr{G}$ and define

$$\sigma = f_0\alpha + f_1\alpha_1 + f_2\alpha_2 + f_3\alpha_3 + f_4\theta\wedge \mathrm{d}\theta. \tag{4}$$

The original G_2 structure σ_0 is given by $-f_0 = f_2 = f_4 = 1$, $f_1 = f_3 = 0$. At least for sufficiently close values to the standard, we do obtain new G_2-structures. For the fixed orientation $\mathrm{Vol}_{\mathscr{G}} = e^{0\dots 6}$, induced by the Sasaki structure on TM and the vector field U, we have that on any two vectors v, w:

$$v\lrcorner\sigma\wedge w\lrcorner\sigma\wedge\sigma = 6\langle v,w\rangle_\sigma \mathrm{Vol}_\sigma = 6\langle v,w\rangle_\sigma m\,\mathrm{Vol}_{\mathscr{G}}.$$

The second identity defines $m > 0$ as a scalar function of σ, by linearity and because, as explained, σ determines both the metric and the volume form, given the orientation. $m : \mathscr{G} \to \mathbb{R}$ is already assumed to be positive—as we may without loss of regularity, if the f_i are smooth, or significant generality of the same set of functions.

Lengthy but easy computations yield the result which we present next.

Lemma 1.1. *The metric matrix of $\langle\cdot,\cdot\rangle_\sigma$ with respect to the adapted frame is*

$$[\langle e_i, e_j\rangle_\sigma] = t\begin{bmatrix} f_4^2 &&&&&& \\ & x &&& z && \\ && x &&& z & \\ &&& x &&& z \\ & z &&& y && \\ && z &&& y & \\ &&& z &&& y \end{bmatrix},$$

where we have simplified notation by writing

$$t = \frac{f_4}{m},\qquad x = f_2^2 - f_1 f_3,\qquad y = f_1^2 - f_0 f_2,\qquad z = f_1 f_2 - f_0 f_3. \tag{5}$$

Notice that σ_0 corresponds to the identity 1_7. Computing determinants, the metric is positive-definite if $f_4 > 0$, $x > 0$ and $xy - z^2 > 0$. This proves the following result.

Theorem 1.2. *If a set of scalar functions* $f_0, \ldots, f_4$ *induces a* G_2 *structure on* $\mathscr{G}$, *then it satisfies* $f_4 > 0$, $f_2^2 - f_1 f_3 > 0$ *and*

$$3f_0f_1f_2f_3 - f_0f_2^3 - f_0^2f_3^2 - f_3f_1^3 > 0. \tag{6}$$

Remark 1.3. The homogeneous fourth degree polynomial is irreducible and has no critical values in the domain.

Remark 1.4. The metrics obtained are all natural metrics in the sense of [1], [2] and other references therein.

Using the Gram–Schmidt process on the new metric, we obtain the oriented orthonormal frame, for $i = 1, 2, 3$,

$$\tilde{e}_0 = \frac{1}{f_4\sqrt{t}}e_0, \qquad \tilde{e}_i = \frac{1}{\sqrt{tx}}e_i, \qquad \tilde{e}_{i+3} = \sqrt{\frac{x}{th}}\left(e_{i+3} - \frac{z}{x}e_i\right), \tag{7}$$

where h is the polynomial in (6):

$$h = xy - z^2. \tag{8}$$

A dual co-frame is then

$$\tilde{e}^0 = f_4\sqrt{t}e^0, \qquad \tilde{e}^i = \sqrt{tx}e^i + z\sqrt{\frac{t}{x}}e^{i+3}, \qquad \tilde{e}^{i+3} = \sqrt{\frac{th}{x}}e^{i+3}. \tag{9}$$

We obtain also the useful formulas

$$e^0 = \frac{1}{f_4\sqrt{t}}\tilde{e}^0, \qquad e^i = \frac{1}{\sqrt{txh}}(\sqrt{h}\tilde{e}^i - z\tilde{e}^{i+3}), \qquad e^{i+3} = \sqrt{\frac{x}{th}}\tilde{e}^{i+3}. \tag{10}$$

Indeed the frame (7) is oriented, i.e., $\tilde{e}^{0123456} = me^{0123456}$ is a positive multiple of the chosen orientation. Immediately through (5) and (9) we find that

$$m = f_4h^{1/3}. \tag{11}$$

1.3. G_2-structures σ compatible with the Sasaki metric. Let σ be a variation of σ_0.

Proposition 1.5. *The metric induced by σ coincides with the Sasaki metric on $\mathcal{G}$ if and only if*

$$f_0^2+f_1^2=1, \qquad f_2=-f_0, \qquad f_3=-f_1, \qquad f_4=1.$$

Under the action of $\mathrm{SO}(7)$ *the orbit of* 3*-forms which can be written in the form* (4) *is a circle* S^1.

Proof. By hypothesis, we have $tf_4^2=tx=ty=1$ and $z=0$. Hence $f_4^3=f_4x=f_4y=m$ and $h=xy=f_4^4$. Knowing m must equal 1 or equating through (11) we get all these equal to 1, except for z. Now solving the system (5) we deduce the equivalence in the first part of the result. The second follows from the first (as the metric is preserved) and the analysis of the orbit of $\sigma_0=\alpha_2-\alpha+\theta\wedge\mathrm{d}\theta$ through known methods. So, we note that already $\mathrm{U}(3)\subset\mathrm{SO}(7)$ acts as a real group, fixing e_0, on the vector space $E=H_1\oplus V_1$, which has a natural complex structure. Moreover,

$$\begin{aligned}(e^1+\sqrt{-1}e^4)\wedge(e^2+\sqrt{-1}e^5)\wedge(e^3+\sqrt{-1}e^6)\\ =\alpha_3-\alpha_1+\sqrt{-1}(\alpha_2-\alpha)=:\eta\in\Lambda^3E^{(1,0)^*}.\end{aligned}$$

Since $\mathrm{SU}(3)\subset\mathrm{G}_2$, we have only to consider maps g such that $g_{|E}=\mathrm{e}^{is}1_E$ for some $s\in\mathbb{R}$. One finds easily the role of g as a real map. Immediately we deduce g fixes the 3-form $\theta\wedge\mathrm{d}\theta=e^{041}+e^{052}+e^{063}$. On the other hand $g\cdot\eta=g^3\eta$. Letting g be such that $g^3=f_0+\sqrt{-1}f_1\in S^1$ we find that this real map solves ($\Im$ denotes imaginary part)

$$\begin{aligned}g\cdot\sigma_0=g\cdot(\Im\eta+\theta\wedge\mathrm{d}\theta)=\Im(g^3\eta)+\theta\wedge\mathrm{d}\theta\\ =-f_0\alpha-f_1\alpha_1+f_0\alpha_2+f_1\alpha_3+\theta\wedge\mathrm{d}\theta.\end{aligned}$$

The result follows (notice the space $\mathrm{SO}(7)/\mathrm{G}_2$ is 7-dimensional so we have to restrict our statement to the specific forms). ☐

For the following computations we apply formulas which have been deduced in [4], [6]. We start by the particular case found above, when the Sasaki metric is preserved.

Theorem 1.6. *Suppose the Riemannian manifold M is connected. Let σ be a variation of gwistor space satisfying the condition that the induced metric coincides* *with the Sasaki metric on* $\mathscr{G}$, *that is,* $\sigma = -f_0\alpha - f_1\alpha_1 + f_0\alpha_2 + f_1\alpha_3 + \theta\wedge \mathrm{d}\theta$ *with* $(f_0, f_1) : \mathscr{G} \to S^1$ *a smooth function. Then we have:*

1. $\mathrm{d}\sigma \neq 0$.
2. *If* $(f_0, f_1) \neq (\pm 1, 0)$, *then* $\mathrm{d}*\sigma = 0$ *if and only if the functions* f_0, f_1 *are constant and the Riemannian base M has constant sectional curvature.*
3. *If* $(f_0, f_1) = (\pm 1, 0)$, *then* $\mathrm{d}*\sigma = 0$ *if and only if M is Einstein.*

The proof follows by recalling the list of derivatives of the fundamental 3-forms in (19), which were deduced in [4], Proposition 2.3. Result (1) is the particular case of Theorem 1.12 (below). For (2) we may easily compute $\mathrm{d}*\sigma$. If it is to vanish, then we deduce a curvature equation $R_{0123} = 0$, which implies constant sectional curvature on the base, and that $f_0\mathrm{d}f_0 = -f_1\mathrm{d}f_1$ is a multiple of θ, which implies (f_0, f_1) is constant. Finally, if the base metric has constant sectional curvature k, then another curvature term appearing satisfies $\mathscr{R}^U\alpha = -k\theta\wedge\alpha_1$, and we find this is the solution required in case $f_1 \neq 0$.

Theorem 1.6 shows that the original gwistor space structure we found, the standard σ_0, is indeed preferred; it has greater interest than the others on the circle (of course, besides the antipodal of σ_0, a duality which as explained in section 1.2 we shall not explore here).

We shall now see a result concerning the type of $\mathrm{d}\sigma$ with respect to the G_2-decomposition of $\Lambda^4T^*\mathscr{G}$. We follow the description by [10] also found in several good references such as [3], [8], [9]. A structure is said to be of pure type W_3 if $\mathrm{d}\sigma = *\tau_3$ with τ_3 the W_3 part, that is satisfying $\tau_3\wedge\sigma = \tau_3\wedge *\sigma = 0$.

Theorem 1.7. *The gwistor space* $(\mathscr{G}, \sigma)$ *of a constant sectional curvature k manifold with* σ *given as before and* f_0, f_1 *constant, is of pure type* W_3 *if and only if* $k = -2$.

Proof. Our invoked Riemann tensor satisfies $R_{ijpq} = k(\delta_i^q\delta_j^p - \delta_i^p\delta_j^q)$ for a constant sectional curvature metric (this is not a sign convention; it is a compatibility condition between required tensors on $\mathscr{G}$ and tensors on the base manifold). By definitions in (20), (21), seen below but known from [4], we have $\mathscr{R}^U\alpha = -k\theta\wedge\alpha_1$, $\mathscr{R}^U\alpha_1 = -2k\theta\wedge\alpha_2$.

Now, since the metric is Einstein we have $\mathrm{d}*\sigma = 0$ by Theorem 1.6 and thence $\mathrm{d}\sigma = \lambda*\sigma + *\tau_3$ (in other words, cf. [9], we have $\tau_1 = \tau_2 = 0$). The condition of pure type W_3, equivalently $\lambda = 0 \in \mathbb{R}$, corresponds by a simple argument to $(\mathrm{d}\sigma)\wedge\sigma = 0$.

With $\sigma = -f_0\alpha - f_1\alpha_1 + f_0\alpha_2 + f_1\alpha_3 + \theta\wedge\mathrm{d}\theta$, we get the following formula:

$$\mathrm{d}\sigma = \theta\wedge(-3f_1\alpha + f_0(k+2)\alpha_1 + f_1(2k+1)\alpha_2 - 3f_0k\alpha_3) + (\mathrm{d}\theta)^2. \qquad (12)$$

Using the 'first structure equations' from (3) or [4], Proposition 2.1, and $f_0^2 + f_1^2 = 1$, we have

$$\begin{aligned}\mathrm{d}\sigma\wedge\sigma &= \big(3f_1^2 + 3f_0^2(k+2) + 3f_1^2(2k+1) + 3f_0^2k + 6\big)\,\mathrm{Vol}_{\mathscr{G}}\\ &= \big(6f_1^2 + 6f_0^2 + 6(f_1^2+f_0^2)k + 6\big)\,\mathrm{Vol}_{\mathscr{G}}\\ &= 6(2+k)\,\mathrm{Vol}_{\mathscr{G}}.\end{aligned}$$

Hence the result. ☐

We recover, in particular, the result in [4], Corollary 3.1, for the preferred $\sigma_0 = \alpha_2 - \alpha + \theta\wedge\mathrm{d}\theta$ on hyperbolic space of sectional curvature -2. Notice however the independency from the pair $(f_0, f_1) \in S^1$. The same is true with the following quite noticeable formula.

Proposition 1.8. *Assuming the above conditions,* $\|\mathrm{d}\sigma\|^2 = 12(k^2+k+2)$. *In particular,* $\|\mathrm{d}\sigma\|^2 = 48$ *if and only if* $k=-2$ *or* $k=1$.

Proof. Immediate from (12). ☐

1.4. Properties of the general case. Let us consider some metric problems related with the variations of gwistor space.

Suppose $(f_0,\dots,f_4):\mathscr{G}\to\mathbb{R}^5$ is a function satisfying the conditions in Theorem 1.2. We study those 3-forms

$$\sigma = f_0\alpha + f_1\alpha_1 + f_2\alpha_2 + f_3\alpha_3 + f_4\theta\wedge\mathrm{d}\theta \tag{13}$$

which define G_2-structures on $\mathscr{G}\to M$.

Remark 1.9. 1. Recall a metric almost contact structure is said to be K-contact if the characteristic vector field is Killing. In the case of the Sasaki metric, $(\mathscr{G}, \theta, B^tU)$ is K-contact if and only if M is locally isometric to S^4 of radius 1, a result due to Y. Tashiro. In general, our metrics $\langle\cdot,\cdot\rangle_\sigma$ induced from σ turned out to be 'g-natural' contact metrics in the sense of e.g. [1] (in particular the immediate question of $\langle\cdot,\cdot\rangle_\sigma$ being K-contact is solved in the same reference).

2. Another feature of gwistor theory is that σ seems to be never preserved by the vector field B^tU. This is known both as the geodesic spray or the geodesic flow vector field, cf. [13], [15]. Indeed, computations for constant f_i have shown that the equation $\mathscr{L}_{B^tU}\sigma = 0$ has no solution $\sigma\in\Lambda^3_+$. For any f_i defined on $\mathscr{G}$, or even just the pull-back of functions on M, one may write interesting differential equations.

Now we shall compute the exterior derivatives of the G_2-structures. From the formulas in (10) we deduce

$$\begin{aligned}
\theta &= \frac{1}{f_4 t^{1/2}}\tilde{\theta}, \qquad \mathrm{d}\theta = \frac{1}{th^{1/2}}\widetilde{\mathrm{d}\theta}, \qquad \alpha = \frac{x^{3/2}}{(th)^{3/2}}\tilde{\alpha},\\
\alpha_1 &= \frac{x^{1/2}}{t^{3/2}h}\left(\tilde{\alpha}_1 - \frac{z}{h^{1/2}}\tilde{\alpha}\right),\\
\alpha_2 &= \frac{1}{x^{1/2}(th)^{3/2}}(h\tilde{\alpha}_2 - 2h^{1/2}z\tilde{\alpha}_1 + 3z^2\tilde{\alpha}),\\
\alpha_3 &= \frac{1}{(txh)^{3/2}}(h^{3/2}\tilde{\alpha}_3 - hz\tilde{\alpha}_2 + h^{1/2}z^2\tilde{\alpha}_1 - z^3\tilde{\alpha}).
\end{aligned}$$

The forms with a tilde are defined algebraically using the orthonormal basis for σ, formally introduced as the respective $\theta, \mathrm{d}\theta, \alpha, \ldots, \alpha_3$. For instance $\tilde{\theta} = \tilde{e}^0$, $\widetilde{\mathrm{d}\theta} = \tilde{e}^{41} + \tilde{e}^{52} + \tilde{e}^{63}$, cf. (2). In particular, we note that we may use the already mentioned 'first structure equations' from (3) but with a tilde!

We also need the inverse formulas of the above:

$$\begin{aligned}
\widetilde{\theta\wedge\mathrm{d}\theta} &= f_4 t^{3/2}h^{1/2}\theta\wedge\mathrm{d}\theta, \qquad \tilde{\alpha} = \frac{(th)^{3/2}}{x^{3/2}}\alpha,\\
\tilde{\alpha}_1 &= \frac{ht^{3/2}}{x^{3/2}}(x\alpha_1 + 3z\alpha),\\
\tilde{\alpha}_2 &= \frac{h^{1/2}t^{3/2}}{x^{3/2}}(x^2\alpha_2 + 2xz\alpha_1 + 3z^2\alpha),\\
\tilde{\alpha}_3 &= \frac{t^{3/2}}{x^{3/2}}(x^3\alpha_3 + x^2z\alpha_2 + xz^2\alpha_1 + z^3\alpha).
\end{aligned}$$

Using the 'first structure equations' for the Hodge operator of the metric and orientation induced by σ, and writing back in terms of the usual frame, we obtain the following result.

Theorem 1.10.

$$*_\sigma(\theta\wedge\mathrm{d}\theta) = \frac{t^{1/2}h^{1/2}}{2f_4}(\mathrm{d}\theta)^2, \tag{14}$$

$$*_\sigma\alpha = \frac{f_4 t^{1/2}}{h^{3/2}}\theta\wedge(x^3\alpha_3 + x^2z\alpha_2 + xz^2\alpha_1 + z^3\alpha), \tag{15}$$

$$\begin{aligned}
*_\sigma\alpha_1 = -\frac{f_4 t^{1/2}}{xh^{3/2}}\theta\wedge\big(&3x^3z\alpha_3 + x^2(h+3z^2)\alpha_2\\
&+ x(2hz + 3z^3)\alpha_1 + (3hz^2 + 3z^4)\alpha\big),
\end{aligned} \tag{16}$$

$$\begin{aligned}*_\sigma\alpha_2 &= \frac{f_4t^{1/2}}{x^2h^{3/2}}\theta\wedge\big(3x^3z^2\alpha_3+x^2(2hz+3z^3)\alpha_2\\&\quad+x(h^2+4hz^2+3z^4)\alpha_1+(3h^2z+6hz^3+3z^5)\alpha\big),\qquad(17)\\ *_\sigma\alpha_3 &= -\frac{f_4t^{1/2}}{x^3h^{3/2}}\theta\wedge\big(x^3z^3\alpha_3+x^2(hz^2+z^4)\alpha_2\\&\quad+x(h^2z+2hz^3+z^5)\alpha_1+(h^3+3h^2z^2+3hz^4+z^6)\alpha\big).\qquad(18)\end{aligned}$$

Corollary 1.11. *The Hodge* $*$ *operator is homogeneous of degree* $\frac{1}{3}$ *on* 3*-forms viewed as a map* $\sigma\rightsquigarrow *_\sigma$.

Proof. From definitions, we see x, y, z have degree 2 and thence h has degree 4; then $m=f_4h^{1/3}$ and Vol_σ have degree $\frac{7}{3}$ and finally $t=f_4/m$ has degree $-\frac{4}{3}$. Finally, observing (14) the result follows (though quite easily seen as a corollary from the above, this result also follows from the definition of $*_\sigma$). □

Now we recall the formulas from [4], Proposition 2.3:

$$\begin{aligned}\mathrm{d}\alpha&=\mathscr{R}^U\alpha,\\ \mathrm{d}\alpha_1&=3\theta\wedge\alpha+\mathscr{R}^U\alpha_1,\\ \mathrm{d}\alpha_2&=2\theta\wedge\alpha_1-\underline{r}\,\mathrm{vol},\\ \mathrm{d}\alpha_3&=\theta\wedge\alpha_2.\end{aligned}\qquad(19)$$

$\mathscr{R}^U\alpha$, $\mathscr{R}^U\alpha_1$ are linearly independent forms depending on the curvature R of M, and $\underline{r}$ is a scalar function on $\mathscr{G}$ defined by $\underline{r}(u)=r(u,u)$, with R and r the usual Riemann and Ricci curvature tensors. Concretely, cf. [4], formulas 25 and 26,

$$\mathscr{R}^U\alpha=\sum_{0\le i<j\le 3}R_{ij01}e^{ij56}+R_{ij02}e^{ij64}+R_{ij03}e^{ij45},\qquad(20)$$

$$\mathscr{R}^U\alpha_1=\sum_{0\le i<j\le 3}R_{ij01}(e^{ij26}+e^{ij53})+R_{ij02}(e^{ij61}+e^{ij34})+R_{ij03}(e^{ij15}+e^{ij42}).\qquad(21)$$

In particular $\theta\wedge\mathscr{R}^U\alpha_1=-\rho\wedge\mathrm{vol}$, where $\rho=\sum_{i=1}^3 r(e_i,e_0)e^{i+3}$.

Theorem 1.12. *For any functions* $f_0,\dots,f_4$, *we have* $\mathrm{d}\sigma\neq 0$.

Proof. Indeed, since $\mathrm{d}\theta\wedge\alpha_i=0$ for all $i=0,1,2,3$, $\alpha_0=\alpha$, we have by the Bianchi identity

$$\begin{aligned}\theta\wedge\mathrm{d}\theta\wedge\mathrm{d}\sigma&=\theta\wedge\mathrm{d}\theta\wedge\Big(f_4(\mathrm{d}\theta)^2+\sum\mathrm{d}f_i\wedge\alpha_i+f_i\,\mathrm{d}\alpha_i\Big)\\&=\big(6f_4+f_0(R_{2301}+R_{3102}+R_{1203})\big)\,\mathrm{Vol}_{\mathscr{G}}=6f_4\,\mathrm{Vol}_{\mathscr{G}}.\end{aligned}$$

However, we saw f_4 must be positive. □

From now on we assume the functions $f_0,\ldots,f_4$ are constant.

Returning to the Hodge duals of Theorem 1.10, then we have by simple reasons

$$\mathrm{d}\big(*_\sigma(\theta\wedge\mathrm{d}\theta)\big)=0, \tag{22}$$

$$\mathrm{d}(*_\sigma\alpha)=-\frac{f_4t^{1/2}}{h^{3/2}}\theta\wedge(xz^2\mathscr{R}^U\alpha_1+z^3\mathscr{R}^U\alpha), \tag{23}$$

$$\mathrm{d}(*_\sigma\alpha_1)=\frac{f_4t^{1/2}}{xh^{3/2}}\theta\wedge\big(x(2hz+3z^3)\mathscr{R}^U\alpha_1+(3hz^2+3z^4)\mathscr{R}^U\alpha\big), \tag{24}$$

$$\begin{aligned}\mathrm{d}(*_\sigma\alpha_2)=&-\frac{f_4t^{1/2}}{x^2h^{3/2}}\theta\wedge\big(x(h^2+4hz^2+3z^4)\mathscr{R}^U\alpha_1\\&+(3h^2z+6hz^3+3z^5)\mathscr{R}^U\alpha\big),\end{aligned} \tag{25}$$

$$\begin{aligned}\mathrm{d}(*_\sigma\alpha_3)=&\frac{f_4t^{1/2}}{x^3h^{3/2}}\theta\wedge\big(x(h^2z+2hz^3+z^5)\mathscr{R}^U\alpha_1\\&+(h^3+3h^2z^2+3hz^4+z^6)\mathscr{R}^U\alpha\big).\end{aligned} \tag{26}$$

Adding up the above with the respective coefficients from (4), we find the vanishing of the two polynomials

$$\begin{aligned}\mathfrak{p}_1=&-f_0x^3z^2+f_1x^2(2hz+3z^3)-f_2x(h^2+4hz^2+3z^4)\\&+f_3(h^2z+2hz^3+z^5),\end{aligned} \tag{27}$$

$$\begin{aligned}\mathfrak{p}_2=&f_0x^3z^3-f_1x^2(3hz^2+3z^4)+f_2x(3h^2z+6hz^3+3z^5)\\&-f_3(h^3+3h^2z^2+3hz^4+z^6)\end{aligned} \tag{28}$$

is a sufficient condition for the vanishing of $\mathrm{d}(*_\sigma\sigma)$:

$$\mathrm{d}(*_\sigma\sigma)=\frac{f_4t^{1/2}}{x^3h^{3/2}}\theta\wedge(x\mathfrak{p}_1\mathscr{R}^U\alpha_1-\mathfrak{p}_2\mathscr{R}^U\alpha).$$

Also the reader understands now why we chose constant coefficients. If $z\neq0$, we may multiply the first polynomial by z, add to the second and factor out a $h(>0)$ from the result to obtain

$$-f_1x^2z^2+2f_2xhz+2f_2z^3x-f_3h^2-2f_3hz^2-f_3z^4. \tag{29}$$

Finally, introducing equations (5), (8) and resorting to some computer algebra software, we are able to find two independent expressions in the original parameters $f_0, \dots, f_3$:

$$\begin{aligned} \mathfrak{p}_1 &= -f_0(f_1^2 - f_0 f_2)(-f_2^2 + f_1 f_3)^2 && (30)\\ \mathfrak{p}_2 &= (f_2^2 - f_1 f_3)^3(-2f_0 f_1^3 f_2^3 + 3f_0^2 f_1 f_2^4 - f_1^6 f_3 + 6f_0 f_1^4 f_2 f_3 - 6f_0^2 f_1^2 f_2^2 f_3 \\ &\quad - 2f_0^3 f_2^3 f_3 - 3f_0^2 f_1^3 f_3^2 + 6f_0^3 f_1 f_2 f_3^2 - f_0^4 f_3^3) && (31) \end{aligned}$$

Notice they are homogeneous, as expected, and notice the factor $y = f_1^2 - f_0 f_2$ in the second polynomial and the common factor $x = f_2^2 - f_1 f_3$, which must both be positive by hypothesis. From equivalence we get the simple expression

$$(f_1^3 - 2f_0 f_1 f_2 + f_0^2 f_3)(f_2^2 - f_1 f_3)^3 \qquad (= (29)). \tag{32}$$

Theorem 1.13. *A* 3*-form* σ *as above defining a* G_2*-structure, with* $f_0, \dots, f_4$ *constant, satisfies* $\mathrm{d} *_\sigma \sigma = 0$ *if and only if any one of the following occurs:*

(i) *The polynomial* $\mathfrak{p}_2$ *from* (31) *vanishes and* M *is Einstein.*

(ii) M *has constant sectional curvature.*

Proof. Notice first that $\mathrm{d} *_\sigma \sigma = 0$ if and only if both $\theta \wedge \mathfrak{p}_1 \mathscr{R}^U \alpha_1$ and $\theta \wedge \mathfrak{p}_2 \mathscr{R}^U \alpha$ vanish. Also we note that, if $f_0 = 0$, then neither f_1 or f_3 can vanish (otherwise we would get $y = 0$ or $h = 0$ from definition). So the two main polynomials cannot vanish simultaneously, as we see directly, or from the implied equation (32).

Now, if $\mathfrak{p}_2$ vanishes, then we may conclude that $f_0 \neq 0$, i.e., the first polynomial $\mathfrak{p}_1$ does not vanish. So the cocalibration equation becomes equivalent to the vanishing of $\theta \wedge \mathscr{R}^U \alpha_1 = -\rho \wedge \mathrm{vol}$, which happens if and only if M is Einstein. Conversely, if the polynomial $\mathfrak{p}_2$ does not vanish, then the equation relies on a metric such that $\theta \wedge \mathscr{R}^U \alpha = 0$; equivalently, $R_{1201} = R_{2301} = 0$, etc. This is the same as M having constant sectional curvature. In particular, M being Einstein. □

For example, if $f_0 = 0$, then we are certainly bound to the second case.

Noteworthy is the case when $f_1 f_2 = f_0 f_3$ (or $z = 0$), which generalizes Proposition 1.6. By formulas (22)–(26) we see that

$$\mathrm{d} *_\sigma \sigma = f_3 \frac{f_4 t^{1/2}}{x^3 h^{3/2}} h^3 \theta \wedge \mathscr{R}^U \alpha = \frac{f_3 f_4 t^{1/2} h^{3/2}}{x^3} \theta \wedge \mathscr{R}^U \alpha.$$

A question put to the author by colleagues was: if we could always find, invariant of the metric on M, a natural G_2 structure which would be co-closed.

The answer is no, because the two polynomials do not vanish simultaneously. By the contrary we stress the relevance of G_2 cocalibration goes much beyond the known cases and examples.

1.5. Nearly-parallel G_2-structures. Nearly-parallel G_2-structures on 7-dimensional manifolds are defined by $\mathrm{d}*_\sigma\sigma=0$ and $\mathrm{d}\sigma=c*_\sigma\sigma$ for some constant c. Clearly, if $c\neq0$, the condition is simply the latter equation.

We consider a variation of the G_2 structure on $\mathscr{G}$, as in (13). In order to find a nearly-parallel structure σ, we may assume already that it is cocalibrated ($c\neq0$). Recall the Hodge $*$ operator is homogeneous of degree $1/3$ on 3-forms viewed as a map $\sigma\rightsquigarrow*_\sigma$. Hence if we find a solution to the above in our subspace of $\sigma\in\Lambda^3_+$, we find a line of solutions:

$$\mathrm{d}(s\sigma)=cs*_\sigma\sigma=cs^{-1/3}*_{s\sigma}s\sigma,\qquad s\in\mathbb{R}^+.$$

We restrict here to the case $z=f_1f_2-f_0f_3=0$, the less ‘prohibitive’ condition. And continue to assume the coefficients are constants.

Theorem 1.14. *Under the previous condition, the only metric on an oriented Riemannian* 4*-manifold M for which a $(\mathscr{G},\sigma)$ is nearly-parallel is the constant sectional curvature* 1 *metric. Then there are two classes of solutions, represented by the following two* G_2*-structures:*

$$\sigma_\pm=\pm\frac{\sqrt{2}}{2}(\alpha_2-\alpha+\alpha_3-\alpha_1)+\sqrt{\frac{3}{2}}\theta\wedge\mathrm{d}\theta,$$

both satisfying $\mathrm{d}\sigma=\sqrt{6}*_\sigma\sigma$.

Proof. Since we assume $z=0$ and this is maintained on the line $\mathbb{R}^+\sigma$, there exists a positive multiple of σ such that (f_0,f_1) is in the unit circle. Then we easily deduce $x=y=1$ and $f_2=-f_0$, $f_3=-f_1$. Hence $h=1=t$ and $m=f_4$, cf. (11).

From formulas (14)–(18) and the hypothesis of σ being nearly-parallel, we see the 4-form $\mathrm{d}\sigma$ is again SO(3)-invariant. Then we easily deduce the curvature restriction: it must be of the constant kind. The equation $\mathrm{d}\sigma=c*_\sigma\sigma$ is solved using those same formulas. Looking at components, we find a system (k is the sectional curvature)

$$\begin{cases}c=2f_4,\\ f_0f_1-kf_0^2=0,\\ 2f_0f_1k+f_0f_1-3f_1^2=0,\\ 3f_1-2f_0f_4^2=0,\\ 2f_0+kf_0-2f_0f_4^2=0.\end{cases}$$

This yields $f_0 = f_1$, which occurs twice in the circle; and $k = 1$, $f_4 = \sqrt{3/2}$, $c = \sqrt{6}$. The given 3-forms satisfy the equation and are genuine G_2-structures. $\square$

Notice that the metric on $\mathscr{G}$ is the same on both solutions. Now we recall the classification of nearly-parallel G_2 structures in [12]. The ones we got correspond to the Stiefel manifold $V_{5,2} = \mathrm{SO}(5)/\mathrm{SO}(3)$ in their Table 2, which is of course the unit tangent sphere bundle of S^4. The G_2 structure is constructed as a $\mathrm{U}(1)$-bundle over the complex quadric $\mathrm{G}_{5,2}$, the Grassmannian of 2-planes, with a Kähler-Einstein metric. The resulting nearly-parallel G_2 structure is said to be Einstein-Sasakian for *some* homogeneous $\mathrm{SO}(5)$-invariant metric. We have thus found more detail of this case. It is also most interesting to see that our result gives a metric coinciding precisely with the Einstein metric on $V_{5,2}$ deduced in [2], Theorem 4. It has Riemannian scalar curvature $\frac{63}{4}$, by a formula there.

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