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Multiple points, scheme rank and symmetric tensor rank

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Abstract. In this paper we prove some upper bounds for the symmetric tensor rank of a symmetric tensor (or a homogeneous polynomial) in terms of integers associated to any zero-dimensional scheme evincing the scheme rank of the homogeneous polynomial.

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1. Introduction

Fix an algebraically closed base field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For all positive integers m, d let $\mathbb{K}[x_0, \dots, x_m]_d$ denote the $\mathbb{K}$-vector space of all homogenous polynomials with degree d in the variables $x_0, \dots, x_m$. We have $\dim(\mathbb{K}[x_0, \dots, x_m]_d) = \binom{m+d}{m}$. For each $f \in \mathbb{K}[x_0, \dots, x_m]_d \setminus \{0\}$ its *symmetric tensor rank* $\mathrm{sr}(f)$ is the minimal integer $s > 0$ such that $f = \sum_{i=1}^{s} \ell_i^d$ for some $\ell_i \in \mathbb{K}[x_0, \dots, x_m]_1$ ([8], [3], [5], [11], [12], [7]). The definition of symmetric tensor rank of a homogeneous polynomial may be translated into the following language.

Set $r := \binom{m+d}{m} - 1$. Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^r$ denote the Veronese embedding of $\mathbb{P}^m$ induced by $\mathbb{K}[x_0, \dots, x_m]_d$. For any $P \in \mathbb{P}^r$ the *symmetric tensor rank* $\mathrm{sr}_{m,d}(P)$ of P is the minimal cardinality of some $A \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(A) \rangle$, where $\langle\ \rangle$ denote the linear span. Each $f \in \mathbb{K}[x_0, \dots, x_m]_d \setminus \{0\}$ corresponds to a unique $P \in \mathbb{P}^r$ and we have $\mathrm{sr}(f) = \mathrm{sr}_{m,d}(P)$. For a fixed f (or, equivalently, a fixed P) it is important to give upper bounds for its symmetric tensor rank in terms of m, d and invariants associated to P ([12], Corollary 5.2, [1]). A very interesting and useful invariant is the scheme rank (called scheme length in [10], p. 135), i.e. the minimal degree $z_{m,d}(P)$ of a zero-dimensional scheme $Z \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(Z) \rangle$ ([4], [2], [1]). The case $m = 1$ is completely known by a theorem of Sylvester ([6], [11], [12], Theorem 4.1, [3]). Hence from now on we assume $m \geq 2$.

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For each $P \in \mathbb{P}^m$ and any integer $k > 0$ let kP be the zero-dimensional subscheme of $\mathbb{P}^m$ with $(\mathscr{I}_P)^k$ as its ideal sheaf. We have $\dim(kP) = 0$, $(kP)_{red} = \{P\}$ and $\deg(kP) = \binom{m+k-1}{m}$. Every zero-dimensional scheme evincing the scheme rank $z_{m,d}(P)$ of some $P \in \mathbb{P}^r$ is Gorenstein ([4], Lemma 2.4). If $m \geq 2$ and $k \neq 1$, then kP is not Gorenstein. However, a connected Gorenstein scheme Z_i may be contained in some w_iP_i and if $Z = Z_1 \sqcup \cdots \sqcup Z_s$ with $Z_i \subseteq w_iP_i$, then $Z \subseteq w_1P_1 \sqcup \cdots \sqcup w_sP_s$. Let $Z \subset \mathbb{P}^m$ be a connected zero-dimensional scheme. The width $w(Z)$ of Z is the minimal integer $k > 0$ such that $Z \subseteq kP$, where $\{P\} := Z_{red}$. Let $Z \subset \mathbb{P}^m$ any zero-dimensional scheme. Call $Z_1, \ldots, Z_s$ the connected components of Z. The width $w(Z)$ of Z is the integer $\max\{w(Z_1), \ldots, w(Z_s)\}$; this integer is not a good estimate of the complexity of Z, unless we also prescribe the integer $s := \#(Z_{red})$. Now we fix an order $Z_1, \ldots, Z_s$ of the connected components of Z.

Definition 1.1. The width-vector $\underline{w}(Z)$ of Z is the s-ple $\big(w(Z_1), \ldots, w(Z_s)\big)$.

If $s > 1$ the width-vector of Z is well-defined only if we fix an ordering of the connected components of Z. We may alway find an ordering, say $Z_{i_1}, \ldots, Z_{i_s}$, of the connected components of Z such that $w(Z_{i_j}) \geq w(Z_{i_h})$ for all $i \leq h$. For this ordering the width-vector of Z, say $(w_1, \ldots, w_s)$, has non-decreasing entries, and the s-ple $(w_1, \ldots, w_s)$ is uniquely determined by Z.

Write $\{P_i\} := (Z_i)_{red}$ and set $W := \bigcup_{i=1}^{s} w_iP_i$, where $w_i := w(Z_i)$. W is the minimal fat-point scheme containing Z. If $P \in \langle \nu_d(Z) \rangle$, then $P \in \langle \nu_d(W) \rangle$. In this paper we prove the following upper bound for $\mathrm{sr}_{m,d}(P)$ in terms of m, d and the width-vector of Z.

Theorem 1.2. *Fix $P \in \mathbb{P}^r$, $r := \binom{d+m}{m} - 1$, and any zero-dimensional scheme $Z \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(Z) \rangle$. Let $\underline{w}(Z) = (w_1, \ldots, w_s)$ be the width-vector of Z with, say, $w_1 \geq \cdots \geq w_s$. Let x be the maximal integer $\leq s$ such that $w_x \geq 2$. Assume $w_1 \leq d + 1$. Then*

$$\mathrm{sr}_{m,d}(P) \leq s - 2x + \sum_{i=1}^{x} (d - w_i + 1)\binom{m + w_i - 2}{m - 1} + \sum_{i=1}^{x} \binom{m + w_i - 1}{m}.$$

As an easy corollary we state the following result.

Corollary 1.3. *Fix $P \in \mathbb{P}^r$, $r := \binom{d+m}{m} - 1$, and any zero-dimensional scheme $Z \subset \mathbb{P}^m$, say with width-vector $\underline{w}(Z) = (w_1, \ldots, w_s)$, such that $P \in \langle \nu_d(Z) \rangle$. Then* $\mathrm{sr}(P) \leq \sum_{i=1}^{s} d\binom{m+w_i-2}{m-1}$.

If $m \geq 2$ and w is not too small, then the integer $\deg(wP)$ is huge with respect to the degree of many Gorenstein subschemes of wP. Hence, when Theorem 1.2

may be applied it often gives upper bounds far better that the one, $\mathrm{sr}_{m,d}(P) \le (z_{m,d}(P) - 1)d + 2 - z_{m,d}(P)$, conjectured in [1], Conjecture 2.

In the case $m = 2$ we may prove the following stronger result.

Theorem 1.4. *Take $m = 2$ and fix $P \in \mathbb{P}^r$, $r := \binom{d+2}{2} - 1$, and any zero-dimensional scheme $Z \subset \mathbb{P}^2$ such that $P \in \langle v_d(Z) \rangle$. Let $\underline{w}(Z) = (w_1, \dots, w_s)$ be the width-vector of Z with, say, $w_1 \ge \cdots \ge w_s$. Let x be the maximal integer $\le s$ such that $w_x \ge 2$. Set $\theta := w_1 + \cdots + w_x - x + 1$ and assume $\theta \le d$. Then*

$$\mathrm{sr}_{2,d}(P) \le \binom{d+2}{2} - \binom{d - \theta + 2}{2} - 1 + s - x.$$

2. The proofs

Recall that $r := \binom{m+d}{m} - 1$. For any reduced projective set $Y \subset \mathbb{P}^n$ and any $P \in \langle Y \rangle$ let $r_Y(P)$ be the minimal cardinality of a finite set $S \subset Y$ such that $P \in \langle S \rangle$. If $Y \subseteq W \subset \mathbb{P}^n$, then $r_W(P) \le r_Y(P)$. The definition of symmetric tensor rank gives $\mathrm{sr}_{m,d}(P) = r_{v_d(\mathbb{P}^m)}(P)$ for all $P \in \mathbb{P}^r$. For all schemes $E \subset W \subseteq \mathbb{P}^n$ let $\mathscr{I}_{E,W}$ be the ideal sheaf of E in W. For any $t \in \mathbb{Z}$ let $H^i(W, \mathscr{I}_{E,W}(t))$ be the i-th cohomology group of the sheaf $\mathscr{I}_{E,W}(t)$. Set $h^i(W, \mathscr{I}_{E,W}(t)) := \dim_{\mathbb{K}} H^i(W, \mathscr{I}_{E,W}(t))$. If $W = \mathbb{P}^n$ we often write $\mathscr{I}_E$, $H^i(\mathscr{I}_E(t))$ and $h^i(\mathscr{I}_E(t))$ instead of $\mathscr{I}_{E,W}$, $H^i(W, \mathscr{I}_{E,W}(t))$ and $h^i(W, \mathscr{I}_{E,W}(t))$.

We recall that the one-dimensional case of [12], Proposition 5.1, holds for any connected curve, not just for integral ones (see [5], Lemma 8.1). Hence we will use it in the following form.

Lemma 2.1. *Let $Y \subset \mathbb{P}^n$ be a reduced and connected curve spanning $\mathbb{P}^n$. Then $r_Y(P) \le n$ for all $P \in \mathbb{P}^n$.*

Lemma 2.2. *Fix integers $d \ge w - 1 \ge 0$ and $m \ge 2$. Let $H \subset \mathbb{P}^m$ be a hyperplane. Fix $O \in \mathbb{P}^m \backslash H$ and a set $E \subset H$ such that $\#(E) = \binom{m+w-2}{m-1}$ and $h^0(H, \mathscr{I}_{E,H}(w-1)) = 0$ (e.g., take as E a general subset of H with cardinality $\binom{m+w-2}{m-1}$). Let $D \subset \mathbb{P}^m$ be the union of all lines spanned by P and a point of E. Then* $\dim(\langle v_d(D) \rangle) = -1 + (d - w + 1)\binom{m+w-2}{m-1} + \binom{m+w-1}{m}$.

Proof. The cone D is the one considered in [9], Proposition 4. The cone D is reduced and connected. Let $p_a(D)$ be the arithmetic genus of D, i.e. $p_a(D) = h^1(D, \mathscr{O}_D)$. We have $\deg(v_d(D)) = d \cdot \#(E) = d\binom{m+w-2}{m-1}$. Since $\dim(D) = 1$, the exact sequence

$$0 \to \mathscr{I}_D(t) \to \mathscr{O}_{\mathbb{P}^m}(t) \to \mathscr{O}_D(t) \to 0$$

gives $h^1(D, \mathcal{O}_D(t)) = h^2(\mathcal{I}_D(t))$ for all $t \geq 0$. Since $\#(E) = \binom{m+w-2}{m-1}$ and $h^0(H, \mathcal{I}_E(w-1)) = 0$, we have $h^1(H, \mathcal{I}_{E,H}(w-1)) = 0$. Castelnuovo-Mumford's lemma gives $h^1(H, \mathcal{I}_{E,H}(t)) = 0$ for all $t \geq w$. Since E is zero-dimensional, we have $h^2(H, \mathcal{I}_{E,H}(t)) = 0$ for all $t \geq 0$. Since $H \cap D = E$, for each integer t we have an exact sequence

$$0 \to \mathcal{I}_D(t-1) \to \mathcal{I}_D(t) \to \mathcal{I}_{E,H}(t) \to 0 \tag{1}$$

Since D is a cone with E as a basis, the cone with vertex O of any hypersurface of H containing E is a hypersurface of $\mathbb{P}^m$ containing D. Hence for every $t \in \mathbb{Z}$ the restriction map $H^0(\mathcal{I}_D(t)) \to H^0(H, \mathcal{I}_{E,H}(t))$ is surjective. From (1) we get that $h^1(\mathcal{I}_D(t-1)) \leq h^1(\mathcal{I}_D(t))$ for all $t \in \mathbb{Z}$. Since $h^1(\mathcal{I}_D(t)) = 0$ if $t \gg 0$, we get $h^1(\mathcal{I}_D(t)) = 0$ for all $t \in \mathbb{Z}$. From (1) we get $h^2(\mathcal{I}_D(t-1)) \leq h^2(\mathcal{I}_D(t))$ for all $t \geq w-1$. Since $h^2(\mathcal{I}_D(t)) = 0$ if $t \gg 0$, we get $h^2(\mathcal{I}_D(w-2)) = 0$. Since $h^1(D, \mathcal{O}_D(w-2)) = 0$, we have $h^1(D, \mathcal{O}_D(w-1)) = 0$. Since $d \geq w-2$, we get $h^1(D, \mathcal{O}_D(d)) = 0$. Hence $h^0(D, \mathcal{O}_D(d)) = d\binom{m+w-2}{m-1} + 1 - p_a(D)$ and $h^0(D, \mathcal{O}_D(w-1)) = (w-1)\binom{m+w-2}{m-1} + 1 - p_a(D)$ (Riemann-Roch). Since $h^0(\mathcal{I}_D(w-1)) = h^1(\mathcal{I}_D(w-1)) = 0$, we have $h^0(D, \mathcal{O}_D(w-1)) = \binom{m+w-1}{m}$. Hence $1 - p_a(D) = \binom{m+w-1}{m} - (w-1)\binom{m+w-2}{m-1}$. Hence $h^0(D, \mathcal{O}_D(d)) = (d-w+1)\binom{m+w-2}{m-1} + \binom{m+w-1}{m}$. Since $h^1(\mathcal{I}_D(d)) = 0$, we get $\dim(\langle \nu_d(D) \rangle) = -1 + (d-w+1)\binom{m+w-2}{m-1} + \binom{m+w-1}{m}$. □

Lemma 2.3. *Fix $O \in \mathbb{P}^m$ and an integer $w > 0$. Let $D \subset \mathbb{P}^m$ be a reduced union of finitely many lines, each of them containing O. Fix a hyperplane $H \subset \mathbb{P}^m$ such that $O \notin H$ and set $E := H \cap D$. We have $h^0(H, \mathcal{I}_{E,H}(w-1)) = 0$ if and only if $wO \subset D$.*

Proof. The algebraic set D is the scheme-theoretic intersection of cones with vertex containing O. For any such cone T we have $H \nsubseteq T$ and T contains D if and only if $E \subseteq T \cap H$. Hence $h^0(H, \mathcal{I}_{E,H}(w-1)) = 0$ if and only if every cone with vertex O containing D has multiplicity at least w at O, i.e. if and only if $wO \subset D$. □

Lemma 2.4. *Fix $O \in \mathbb{P}^2$ and an integer $w > 0$. Let $H \subset \mathbb{P}^2$ be a line such that $O \notin H$. Let $D \subset \mathbb{P}^2$ be a union of finitely many lines through O. Set $t := \deg(D)$ and assume $t \leq d$. Then $\mathrm{sr}_{2,d}(P) \leq \binom{d+2}{2} - \binom{d-t+2}{2} - 1$ for any $P \in \langle \nu_d(D) \rangle$.*

Proof. Since $h^1(\mathcal{O}_{\mathbb{P}^2}(d-t)) = 0$, we have $\dim(\langle \nu_d(D) \rangle) = \binom{d+2}{2} - \binom{d-t+2}{2} - 1$. Since D is connected, it is sufficient to apply Lemma 2.1. □

Proof of Theorem 1.2. First assume $s = x$. Write $Z = Z_1 \sqcup \cdots \sqcup Z_s$ with each Z_i connected. Set $\{P_i\} := (Z_i)_{red}$, $W_i := w_i P_i$ and $W := \bigsqcup_{i=1}^{s} W_i$. Since $Z \subseteq W$, we have $P \in \langle \nu_d(W) \rangle$. Hence there is $O_i \in \langle W_i \rangle$ such that $P \in \langle \{O_1, \dots, O_s\} \rangle$. Hence it is sufficient to prove that $\mathrm{sr}(O_i) \le (d - w_i + 1)\binom{m+w_i-2}{m-1} + \binom{m+w_i-1}{m} - 1$ for all i. Apply Lemma 2.2 to the integer $w := w_i$ to get a union, D_i, of lines through P_i. Lemma 2.4 gives $w_i P_i \subset D_i$. Then apply Lemma 2.1 to the connected curve D_i. Notice that this construction works even if $D_i \cap D_j$ contains a line for some $i \ne j$, because we apply Lemmas 2.1 and 2.2 separately to each P_i.

Now assume $s > x$. Hence $Z_i = \{P_i\}$ for all $i > x$. Set $A := \{P_{x+1}, \dots, P_s\}$ and $Z' := Z_1 \sqcup \cdots \sqcup Z_x$. Since $Z = Z' \sqcup A$ and $P \in \langle \nu_d(Z) \rangle$, there is $O \in \langle \nu_d(Z') \rangle$ such that $P \in \langle \nu_d(A) \cup \{O\} \rangle$. The case $s = x$ just proved gives the existence of a set $B \subset \mathbb{P}^m$ such that $\#(B) \le -x + \sum_{i=1}^{x} (d - w_i + 1)\binom{m+w_i-2}{m-1} + \sum_{i=1}^{x} \binom{m+w_i-1}{m}$ and $O \in \langle \nu_d(A) \rangle$. Since $\#(A) = s - x$ and $P \in \langle \nu_d(A \cup B) \rangle$, we have $\mathrm{sr}_{m,d}(P) \le s - 2x + \sum_{i=1}^{x} (d - w_i + 1)\binom{m+w_i-2}{m-1} + \sum_{i=1}^{x} \binom{m+w_i-1}{m}$. □

Proof of Corollary 1.4. Take D as in Lemma 2.2. Since D is a reduced and connected curve, we have $h^0(D, \mathcal{O}_D(d)) \le 1 + \deg(\mathcal{O}_D(d)) = 1 + d\binom{m+w-2}{m-1}$. Apply this weaker inequality instead of Lemma 2.2 to the curves D_i constructed in the proof of Theorem 1.2. □

Proof of Theorem 1.4. As in the second part of the proof of Theorem 1.2 we reduce to the case $x = s$, i.e. to the case in which $w_i \ge 2$ for all i. Hence from now on we assume $w_i \ge 2$ for all i. Set $\{P_1, \dots, P_s\} := Z_{red}$. In the case $s = 1$ we take a union of w_1 distinct lines through P_1. Then we apply Lemmas 2.2 and 2.1.

Hence we may assume $s \ge 2$. Write $Z = Z_1 \sqcup \cdots \sqcup Z_s$ with Z_i connected and $w(Z_i) = w_1$. Set $\{P_i\} := (Z_i)_{red}$, $W_i := w_i P_i$ and $W := W_1 \sqcup \cdots \sqcup W_s$. Since $Z \subseteq W$, we have $P \in \langle \nu_d(W) \rangle$. Let $D_1 \subset \mathbb{P}^2$ be any union of w_1 lines through P_1 with the only restriction that $D_1 \supset \langle \{P_1, P_2\} \rangle$ and either each line of D_1 contains at least one point of $\{P_2, \dots, P_s\}$ or $\{P_2, \dots, P_s\} \subset D_1$. Fix an integer $i \in \{2, \dots, s\}$ and suppose to have defined the reduced union D_j, $1 \le j \le i - 1$, of finitely many lines through P_j (we allow the case $D_j = \emptyset$ for some j) so that $D_j \cap D_h$ contains no line if $j \ne h$. For each $h \in \{i, \dots, s\}$ let $e_h \ge 0$ be the number of lines of $D_1 \cup \cdots \cup D_{i-1}$ containing P_i. If $e_i \ge w_i$, then we set $D_i = \emptyset$. Now assume $0 \le e_i < w_i$. We will take as D_i a union of $w_i - e_i$ distinct lines though P_i with the following further restrictions. Set $E := \{P_j \in \{P_{i+1}, \dots, P_s\} : w_j P_j \not\subseteq D_1 \cup \cdots \cup D_{i-1}$ and $\langle \{P_i, P_j\} \rangle \not\subseteq D_1 \cup \cdots \cup D_{i-1}\}$. If $E = \emptyset$ (this is always the case if $i = s$), then we take as D_i any $w_i - e_i$ lines through P_i, but different from the lines in $D_1 \cup \cdots \cup D_{i-1}$. Now assume $E \ne \emptyset$. We take $\langle \{P_i, P_j\} \rangle$ as the first line of D_i. If $w_i - e_i = 1$, then we set $D_i := \langle \{P_i, P_j\} \rangle$. Notice that the line $\langle \{P_i, P_j\} \rangle$ may contain some P_h with $j < h \le s$. Now assume $w_i - e_i \ge 2$. Set $E_1 := \{P_h \in \{P_{j+1}, \dots, P_s\} : e_h < w_h$ and $\langle \{P_i, P_h\} \rangle \notin D_1 \cup \cdots \cup D_{i-1} \cup \langle \{P_i, P_j\} \rangle\}$. If $E_1 = \emptyset$, then take as D_i the union of $\langle \{P_i, P_j\} \rangle$ and any $w_i - e_i - 1$ lines through P_i different from the lines of $D_1 \cup \cdots \cup D_{i-1} \cup \langle \{P_i, P_j\} \rangle$. Now assume $E_1 \ne \emptyset$ and let k be the minimal integer such that

$P_k \in E_1$. We take $\langle\{P_i, P_j\}\rangle \cup \langle\{P_i, P_k\}\rangle \subseteq D_i$, with equality if $w_i - e_i = 2$. Now assume $w_i - e_i \geq 3$. Set $E_2 := \{P_h \in \{P_{k+1}, \ldots, P_s\} : e_h < w_h$ and $\langle\{P_i, P_h\}\rangle \notin D_1 \cup \cdots \cup D_{i-1} \cup \langle\{P_i, P_j\}\rangle \cup \langle\{P_i, P_k\}\rangle\}$. If $E_2 = \emptyset$, then we take as D_i the union of $\langle\{P_i, P_j\}\rangle \cup \langle\{P_i, P_k\}\rangle$ and $w_i - e_i - 2$ lines though P_i and different from the lines of $D_1 \cup \cdots \cup D_{i-1} \cup \langle\{P_i, P_j\}\rangle \cup \langle\{P_i, P_k\}\rangle$. If $E_2 \neq \emptyset$, then we work as above. And so on (defining if necessary $E_3, \ldots$). We point out that at each step $i - 1 \Rightarrow i$ we make this construction, so that for all $j \in \{2, \ldots, s\}$ the curve D_j satisfies all the properties obtained in the construction of D_i starting with any given $D_1 \cup \cdots \cup D_{i-1}$.

Claim. For each $i \in \{1, \ldots, s\}$ we have $\deg(D_1 \cup \cdots \cup D_i) \leq w_1 + \cdots + w_i - i + 1$.

Proof of the Claim. We have $\deg(D_1) = w_1$. It is easy to check that $\deg(D_2) = w_2 - 1$. Hence we may assume that $i \geq 3$, that $\deg(D_1 \cup \cdots \cup D_{i-2}) \leq w_1 + \cdots + w_{i-2} - i + 3$ and that $\deg(D_1 \cup \cdots \cup D_{i-1}) \leq w_1 + \cdots + w_{i-1} - i + 2$. The last inequality shows that the Claim is true for the integer i if $\deg(D_i) \leq w_i - 1$. If $D_{i-1} \neq \emptyset$, then $\deg(D_i) \leq w_i - 1$, because by construction either $D_1 \cup \cdots \cup D_{i-2}$ contains a line through P_i or D_{i-1} contains the line $\langle\{P_{i-1}, P_i\}\rangle$. Now assume $D_{i-1} = \emptyset$. Since $\deg(D_i) \leq w_i$, we have $\deg(D_1 \cup \cdots \cup D_i) \leq w_1 + \cdots + w_{i-2} + w_i - i + 3$. Since $w_{i-1} \geq 2$, the Claim is proved even in this case.

By the Claim there is a union D of $t \leq w_1 + \cdots + w_s - s + 1 = \theta$ lines such that $W \subset D$. Since $Z \subseteq D$ we have $P \in \langle v_d(D)\rangle$. Lemma 2.4 gives $\dim(\langle v_d(D)\rangle) = \binom{d+2}{2} - \binom{d-t+2}{2} - 1 \leq \binom{d+2}{2} - \binom{d-\theta+2}{2} - 1$. Apply Lemma 2.1 to the reduced and connected curve D. □

Example 2.5. Fix integers $s \geq 2$, $w_i \geq 2$, $1 \leq i \leq s$, and s collinear points $P_i \in \mathbb{P}^2$, $1 \leq i \leq s$. Let $L := \langle\{P_1, P_2\}\rangle$ be the line containing each P_i. Let t be the minimal degree of a finite union $D \subset \mathbb{P}^2$ of distinct lines such that $w_i P_i \subset D$ for all i. It is easy to check that $t = w_1 + \cdots + w_s - s + 1$ and that any D with that degree is the union of L and, for each $i \in \{1, \ldots, s\}$, $w_i - 1$ lines through P_i and $\neq L$.

3. A mild generalization

For any connected zero-dimensional scheme $Z \subset \mathbb{P}^m$ set $\varepsilon(Z) := \dim(\langle Z\rangle)$, i.e. let $\varepsilon(Z)$ be the dimension of the minimal linear subspace of $\mathbb{P}^m$ spanned by Z. If Z has s connected components and we fix an ordering $Z_1, \ldots, Z_s$ of them, set $\underline{\varepsilon}(Z) := (\varepsilon(Z_1), \ldots, \varepsilon(Z_s))$ (the ε-vector of Z). When $\varepsilon(Z) \neq (m, \ldots, m)$ we may improve Theorem 1.2 and prove the following result.

Theorem 3.1. *Fix $P \in \mathbb{P}^r$, $r := \binom{d+m}{m} - 1$, and any zero-dimensional scheme $Z \subset \mathbb{P}^m$, say with width-vector $\underline{w}(Z) = (w_1, \ldots, w_s)$ and ε-vector $\underline{\varepsilon}(Z) = (e_1, \ldots,$*

e_s), such that $P \in \langle v_d(Z) \rangle$ with, say, $e_i \geq 2$ if and only if $1 \leq i \leq y$ and $e_i = 1$ if and only if $y+1 \leq i \leq x$. Then

$$\mathrm{sr}(P) \leq s - x - y + d(x-y) + \sum_{i=1}^{y} (d - w_i + 1) \binom{e_i + w_i - 2}{e_i - 1} + \sum_{i=1}^{y} \binom{e_i + w_i - 1}{e_i}.$$

Proof. Use the proof of Theorem 1.2 with the following modifications. First assume $e_i \geq 2$. The union of lines D_i through P_i is contained in an e_i-dimensional linear subspace M_i of $\mathbb{P}^m$. Apply Lemma 2.2 to M_i instead of $\mathbb{P}^m$. In the case $e_i = 1$ just use that $r_{D_i}(O_i) \leq d$ by Lemma 2.1. □

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