

Relations between minimal usco and minimal cusco maps

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Abstract. In our paper we give a characterization of (set-valued) maps which are minimal usco and minimal cusco simultaneously. Let X be a topological space and Y be a Banach space. We show that there is a bijection between the space $\text{MU}(X, Y)$ of minimal usco maps from X to Y and the space $\text{MC}(X, Y)$ of minimal cusco maps from X to Y , and we study this bijection with respect to various topologies on underlying spaces. Let X be a Baire space and Y be a Banach space. Then $(\text{MU}(X, Y), \tau_U)$ and $(\text{MC}(X, Y), \tau_U)$ are homeomorphic, where τ_U is the topology of uniform convergence.

Mathematics Subject Classification (2010). 54C60; 54B20.

Keywords. Minimal cusco map, minimal usco map, quasicontinuous function, subcontinuous function, set-valued mapping, selection, extreme function.

1. Introduction

The acronym usco (cusco) stands for a (convex) upper semicontinuous non-empty compact-valued set-valued map. Such set-valued maps are interesting because they describe common features of maximal monotone operators, of the convex subdifferential and of Clarke generalized gradient. Examination of cuscoss and uscoss leads to serious insights into the underlying topological properties of the convex subdifferential and the Clarke generalized gradient. (It is known that the Clarke subdifferential of a locally Lipschitz function and, in particular, the subdifferential of a convex continuous functions are weak* cuscoss.) (see [BZ1])

In our paper we are interested in minimal usco and minimal cusco maps. We give a characterization of such maps which are minimal usco and minimal cusco simultaneously. We also show that there is a bijection between the space of minimal usco maps and the space of minimal cusco maps and we study this bijection under the topologies of pointwise convergence, uniform convergence on compacta, uniform convergence and under the Vietoris topology.

*Both authors would like to thank to grant APVV-0269-11 and L'. Holá also to grant Vega 2/0018/13.

Minimal usco and minimal cusco maps are used in many papers (see [BZ1], [BZ2], [DL], [GM], [HH1], [HH3], [HoM], [Wa]). Historically, minimal usco maps seem to have appeared first in complex analysis (in the second half of the 19th century), in the form of a bounded holomorphic function and its “cluster sets”, see e.g. [CL]. Minimal usco maps are a very convenient tool in the theory of games (see [Ch]) or in functional analysis (see [BM]), where a differentiability property of single-valued functions is characterized by their Clarke subdifferentials being minimal cuscus. Minimal uscous/cuscous also appear in:

- (i) optimization, [CK], [CKR], [KR];
- (ii) the study of weak Asplund spaces, [Fa], [Ka], [MS2], [PPN], [St];
- (iii) selection theorems, [MS1];
- (iv) the study of differentiability of Lipschitz functions, [Bo], [Mo], [Za].

2. Minimal usco and minimal cusco maps

In what follows let X, Y be Hausdorff topological spaces, \mathbb{R} be the space of real numbers with the usual metric and Z^+ be the set of positive integers. Also, for $x \in X$, $\mathcal{U}(x)$ is always used to denote a base of open neighborhoods of x in X . The symbol \bar{A} and $\text{Int } A$ will stand for the closure and interior of the set A in a topological space.

A set-valued map, or a multifunction, from X to Y is a function that assigns to each element of X a subset of Y . If F is a set-valued map from X to Y , then its graph is the set $\{(x, y) \in X \times Y : y \in F(x)\}$. Conversely, if F is a subset of $X \times Y$ and $x \in X$, define $F(x) = \{y \in Y : (x, y) \in F\}$. Then we can assign to each subset F of $X \times Y$ a set-valued map which takes the value $F(x)$ at each point $x \in X$ and which graph is F . In this way, we identify set-valued maps with their graphs. Following [DL] the term map is reserved for a set-valued map.

Notice that if $f : X \rightarrow Y$ is a single-valued function, we will use the symbol f also for the graph of f .

Given two maps $F, G : X \rightarrow Y$, we write $G \subset F$ and say that G is contained in F if $G(x) \subset F(x)$ for every $x \in X$.

A map $F : X \rightarrow Y$ is upper semicontinuous at a point $x \in X$ if for every open set V containing $F(x)$, there exists $U \in \mathcal{U}(x)$ such that

$$F(U) = \bigcup \{F(u) : u \in U\} \subset V.$$

F is upper semicontinuous if it is upper semicontinuous at each point of X . Following Christensen [Ch] we say, that a map F is usco if it is upper semicontinuous and takes nonempty compact values. A map F from a topological space X

to a linear topological space Y is cusco if it is usco and $F(x)$ is convex for every $x \in X$.

Finally, a map F from a topological space X to a topological (linear topological space) Y is said to be minimal usco (minimal cusco) if it is a minimal element in the family of all usco (cusco) maps (with domain X and range Y); that is, if it is usco (cusco) and does not contain properly any other usco (cusco) map from X into Y . By an easy application of the Kuratowski-Zorn principle we can guarantee that every usco (cusco) map from X to Y contains a minimal usco (cusco) map from X to Y (see [BZ1], [BZ2], [DL]).

Other approach to minimality of set-valued maps can be found in [CM], [Ma] and [KKM].

In the paper [HH1] we can find an interesting characterization of minimal usco maps using quasicontinuous and subcontinuous selections, which will be also useful for our analysis.

A function $f : X \rightarrow Y$ is quasicontinuous at $x \in X$ [Ne] if for every neighborhood V of $f(x)$ and every $U \in \mathcal{U}(x)$ there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. If f is quasicontinuous at every point of X , we say that f is quasicontinuous.

The notion of quasicontinuity was perhaps the first time used by R. Baire in [Ba] in the study of points of separately continuous functions. As Baire indicated in his paper [Ba] the condition of quasicontinuity has been suggested by Vito Volterra. There is a rich literature concerning the study of quasicontinuity, see for example [Ba], [HP], [Ke], [KKM], [Ne]. A condition under which the pointwise limit of a sequence of quasicontinuous functions is quasicontinuous was studied in [HH2].

A function $f : X \rightarrow Y$ is subcontinuous at $x \in X$ [Fu] if for every net (x_i) convergent to x , there is a convergent subnet of $(f(x_i))$. If f is subcontinuous at every $x \in X$, we say that f is subcontinuous.

Let $F : X \rightarrow Y$ be a set-valued map. Then a function $f : X \rightarrow Y$ is called a selection of F if $f(x) \in F(x)$ for every $x \in X$.

It is well known that every selection of a usco map is subcontinuous ([HH1], [HN]).

Theorem 2.1 ([HH1], Theorem 2.5). *Let X, Y be topological spaces and Y be a T_1 regular space. Let F be a map from X to Y . The following are equivalent:*

- (1) F is a minimal usco map;
- (2) There exist a quasicontinuous and subcontinuous selection f of F such that $\bar{f} = F$;
- (3) Every selection f of F is quasicontinuous, subcontinuous and $\bar{f} = F$.

Let Y be a linear topological space and $B \subset Y$ is a set. By $\overline{\text{co}} B$ we denote the closed convex hull of the set B (see [AB]).

The proof of the following Lemma is a folklore result, but a reasonably old proof of this appears in [Ph], Lemma 7.12.

Lemma 2.1. *Let X be a topological space and Y be a Hausdorff locally convex linear topological space. Let G be a usco map from X to Y and $\overline{\text{co}} G(x)$ is compact for every $x \in X$. Then the map F defined by $F(x) = \overline{\text{co}} G(x)$ for every $x \in X$ is a cusco map.*

Remark 2.1. There are three important cases when the closed convex hull of a compact set is compact. The first is when the compact set is a finite union of compact convex sets. The second is when the space is completely metrizable and locally convex. This includes the case of all Banach spaces with their norm topologies. The third case is a compact set in the weak topology on a Banach space (see [AB]).

Let B be a subset of a linear topological space. By $\mathcal{E}(B)$ we denote the set of all extreme points of B . Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let $F : X \rightarrow Y$ be a map with nonempty compact values. Then a selection f of F such that $f(x) \in \mathcal{E}(F(x))$ for every $x \in X$ is called an extreme function of F .

Notice that all known characterizations of minimal cusco maps are given in the class of cusco maps (see [GM], [BZ1]). So the following characterization of minimal cusco maps in the class of all set-valued maps can be of some interest:

Theorem 2.2 ([HH3]). *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let F be a map from X to Y . Then the following are equivalent:*

- (1) F is a minimal cusco map;
- (2) F has nonempty compact values and there is a quasicontinuous, subcontinuous selection f of F such that $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$;
- (3) F has nonempty compact, convex values, F has a closed graph, every extreme function of F is quasicontinuous, subcontinuous and any two extreme functions of F have the same closures of their graphs;
- (4) F has nonempty compact values, every extreme function f of F is quasicontinuous, subcontinuous and $F(x) = \overline{\text{co}} \bar{f}(x)$ for every $x \in X$.

Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Denote by $\text{MU}(X, Y)$ the set of all minimal usco maps and by $\text{MC}(X, Y)$ the set of all minimal cusco maps from X to Y . Of course $\text{MU}(X, Y) \cap \text{MC}(X, Y) \neq \emptyset$. It follows from the next example that $\text{MU}(X, Y) \setminus \text{MC}(X, Y) \neq \emptyset$ and also $\text{MC}(X, Y) \setminus \text{MU}(X, Y) \neq \emptyset$.

Example 2.1. Let $X = [-1, 1]$ with the usual Euclidean topology. Consider the maps F and G from X to \mathbb{R} defined by

$$F(x) = \begin{cases} 1, & x \in [-1, 0); \\ \{-1, 1\}, & x = 0; \\ -1, & x \in (0, 1]; \end{cases} \quad G(x) = \begin{cases} 1, & x \in [-1, 0); \\ [-1, 1], & x = 0; \\ -1, & x \in (0, 1]. \end{cases}$$

Definition 2.1. Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. We say that f is $*$ -quasicontinuous at x if for every $y \in \overline{\text{co}} \bar{f}(x)$, for every $V \in \mathcal{U}(y)$ and every $U \in \mathcal{U}(x)$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is $*$ -quasicontinuous at every point of X , we say that f is $*$ -quasicontinuous.

Example 2.2. Consider the function f from \mathbb{R} to \mathbb{R} defined by

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0; \\ 0 & x = 0. \end{cases}$$

The function f is not continuous at $x = 0$ but is $*$ -quasicontinuous at 0.

Theorem 2.3. Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let f be a function from X to Y . Then f is $*$ -quasicontinuous at x if and only if every selection of \bar{f} is quasicontinuous at x and $\bar{f}(x) = \overline{\text{co}} \bar{f}(x)$.

It follows from the previous Theorem that every $*$ -quasicontinuous function is quasicontinuous.

Theorem 2.4. Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let F be a map from X to Y . Then the following are equivalent:

- (1) $F \in \text{MU}(X, Y) \cap \text{MC}(X, Y)$;
- (2) There exist a $*$ -quasicontinuous and subcontinuous function f from X to Y such that $\bar{f} = F$;
- (3) Every selection f of F is $*$ -quasicontinuous, subcontinuous and $\bar{f} = F$.

Proof. (1) \Rightarrow (3): Let f be a selection of F . Since $F \in \text{MU}(X, Y)$, by Theorem 2.1 f is quasicontinuous, subcontinuous and $\bar{f} = F$. Since $F \in \text{MC}(X, Y)$, $\overline{\text{co}} \bar{f}(x) = F(x)$. So by Theorem 2.3 f is $*$ -quasicontinuous.

(3) \Rightarrow (2) is trivial.

(2) \Rightarrow (1): Let f be a $*$ -quasicontinuous and subcontinuous function from X to Y such that $\bar{f} = F$. Thus f is quasicontinuous and, by Theorem 2.3,

$F(x) = \bar{f}(x) = \overline{\text{co}} \bar{f}(x)$. By Theorem 2.1 F is minimal usco map and since by Lemma 2.1 F is cusco, by Proposition 2.7 in [BZ1] F is minimal cusco. \square

Denote by $F(X, Y)$ the set of all maps with nonempty closed values from a topological space X to a Hausdorff locally convex (linear topological) space Y . Define the function $\varphi : \text{MU}(X, Y) \rightarrow F(X, Y)$ as follows: $\varphi(F)(x) = \overline{\text{co}} F(x)$.

The following theorem was proved in [HH3]. For the reader's convenience we will also provide another proof suggested by the referee.

Theorem 2.5 ([HH3], Theorem 4.1). *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let $F : X \rightarrow Y$ be a minimal cusco map. There is a unique minimal usco map contained in F .*

Proof. Let G, H be two minimal usco maps contained in F . It is sufficient to prove that $G(x) \cap H(x) \neq \emptyset$ for every $x \in X$. Then the map $L : X \rightarrow Y$ defined as $L(x) = G(x) \cap H(x)$ for every $x \in X$ is usco and $L \subset G, L \subset H$. Thus $G = L = H$.

By Lemma 2.1 and the minimality of F , $\varphi(G) = \varphi(H) = F$. Now, by the Krein–Milman and Milman theorems, we have

$$\emptyset \neq \mathcal{E}(F(x)) \subset G(x) \cap H(x) \quad \text{for all } x \in X. \quad \square$$

Theorem 2.6. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space in which the closed convex hull of a compact set is compact. The map φ is bijection from $\text{MU}(X, Y)$ to $\text{MC}(X, Y)$.*

Proof. Let $F \in \text{MU}(X, Y)$. To show that $\varphi(F) \in \text{MC}(X, Y)$ note that by Lemma 2.1 the map G defined as $G(x) = \overline{\text{co}} F(x)$ for every $x \in X$ is a cusco map and by Proposition 2.7 in [BZ1] G is minimal cusco.

Next we show that φ maps $\text{MU}(X, Y)$ onto $\text{MC}(X, Y)$. Let $G \in \text{MC}(X, Y)$ and let F be a minimal usco map contained in G . By Lemma 2.1 the map $x \rightarrow \overline{\text{co}} F(x)$ is a cusco map such that $\overline{\text{co}} F(x) \subset G(x)$ for every $x \in X$. Since G is minimal cusco, $G(x) = \overline{\text{co}} F(x)$ for every $x \in X$.

Finally, to show that φ is one-to-one, suppose that $F, G \in \text{MU}(X, Y)$ and $F \neq G$. Suppose, by way of contradiction, that $\varphi(F) = \varphi(G)$. So by Theorem 2.5 $F = G$, a contradiction. \square

3. Topological properties of φ

Let X be a Hausdorff topological space and (Y, d) be a metric space. The open d -ball with center $z_0 \in Y$ and radius $\varepsilon > 0$ will be denoted by $S_\varepsilon(z_0)$ and the ε -parallel body $\bigcup_{a \in A} S_\varepsilon(a)$ for a subset A of Y will be denoted by $S_\varepsilon(A)$.

Denote by $\text{CL}(Y)$ the space of all nonempty closed subsets of Y . By $\mathfrak{R}(X)$ and $\mathfrak{F}(X)$ we mean the family of all nonempty compact and finite subsets of X , respectively. If $A \in \text{CL}(Y)$, the distance functional $d(\cdot, A) : Y \mapsto [0, \infty)$ is described by the familiar formula

$$d(z, A) = \inf\{d(z, a) : a \in A\}.$$

Let A and B be nonempty subsets of (Y, d) . The excess of A over B with respect to d is defined by the formula

$$e_d(A, B) = \sup\{d(a, B) : a \in A\}.$$

The Hausdorff (extended-valued) metric H_d on $\text{CL}(Y)$ [Be] is defined by

$$H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}.$$

We will often use the following equality on $\text{CL}(Y)$:

$$H_d(A, B) = \inf\{\varepsilon > 0 : A \subset S_\varepsilon(B) \text{ and } B \subset S_\varepsilon(A)\}.$$

The topology generated by H_d is called the Hausdorff metric topology.

Following [HM] we will define the topology τ_p of pointwise convergence on $F(X, Y)$. The topology τ_p of pointwise convergence on $F(X, Y)$ is induced by the uniformity \mathfrak{U}_p of pointwise convergence which has a base consisting of sets of the form

$$W(A, \varepsilon) = \{(\Phi, \Psi) : H_d(\Phi(x), \Psi(x)) < \varepsilon \text{ for all } x \in A\},$$

where $A \in \mathfrak{F}(X)$ and $\varepsilon > 0$. The general τ_p -basic neighborhood of $\Phi \in F(X, Y)$ will be denoted by $W(\Phi, A, \varepsilon)$, i.e., $W(\Phi, A, \varepsilon) = W(A, \varepsilon)[\Phi] = \{\Psi : H_d(\Phi(x), \Psi(x)) < \varepsilon \text{ for every } x \in A\}$. If $A = \{a\}$, we will write $W(\Phi, a, \varepsilon)$ instead of $W(\Phi, \{a\}, \varepsilon)$.

We will define the topology τ_{UC} of uniform convergence on compact sets on $F(X, Y)$ [HM]. This topology is induced by the uniformity \mathfrak{U}_{UC} which has a base consisting of sets of the form

$$W(K, \varepsilon) = \{(\Phi, \Psi) : H_d(\Phi(x), \Psi(x)) < \varepsilon \text{ for all } x \in K\},$$

where $K \in \mathfrak{R}(X)$ and $\varepsilon > 0$. The general τ_{UC} -basic neighborhood of $\Phi \in F(X, Y)$ will be denoted by $W(\Phi, K, \varepsilon)$, i.e., $W(\Phi, K, \varepsilon) = W(K, \varepsilon)[\Phi]$.

Finally we will define the topology τ_U of uniform convergence on $F(X, Y)$ [HM]. Let ϱ be the (extended-valued) metric on $F(X, Y)$ defined by

$$\varrho(\Phi, \Psi) = \sup\{H_d(\Phi(x), \Psi(x)) : x \in X\}$$

for each $\Phi, \Psi \in F(X, Y)$. Then the topology of uniform convergence for the space $F(X, Y)$ is the topology generated by the metric ρ .

Lemma 3.1. *Let Y be a normed linear space. Let A, B be nonempty closed subsets of Y . Then $H_d(\overline{\text{co}} A, \overline{\text{co}} B) \leq H_d(A, B)$.*

Proof. At first we show that $e_d(\overline{\text{co}} A, \overline{\text{co}} B) \leq e_d(A, B)$. It is known (see [Be] exercise 1.5.3 b), that if C is convex then $e_d(\overline{\text{co}} A, C) = e_d(A, C)$. So $e_d(\overline{\text{co}} A, \overline{\text{co}} B) = e_d(A, \overline{\text{co}} B)$. Since $B \subset \overline{\text{co}} B$ we have that

$$e_d(\overline{\text{co}} A, \overline{\text{co}} B) = e_d(A, \overline{\text{co}} B) \leq e_d(A, B).$$

Similarly we can show that

$$e_d(\overline{\text{co}} B, \overline{\text{co}} A) = e_d(B, \overline{\text{co}} A) \leq e_d(B, A).$$

Since for every $C, D \in \text{CL}(Y)$

$$H_d(C, D) = \max\{e_d(C, D), e_d(D, C)\},$$

we are done. □

Theorem 3.1. *Let X be a topological space and Y be a Banach space. The map φ from $(\text{MU}(X, Y), \tau)$ onto $(\text{MC}(X, Y), \tau)$ is continuous if τ is one of the following topologies $\tau_p, \tau_{\text{UC}}, \tau_U$.*

Proof. The proof follows from the above lemma. □

The following example shows that the map φ^{-1} from $(\text{MC}([-1, 1]), \tau_p)$ onto $(\text{MU}([-1, 1]), \tau_p)$ is not continuous.

Example 3.1. Let $X = [-1, 1]$ with the usual Euclidean topology. Let F and G are maps from Example 2.1. Then $F = \varphi^{-1}(G)$. We claim that φ^{-1} is not continuous at G . For every $n \in \mathbb{Z}^+$ let P_n be the map from $[-1, 1]$ to \mathbb{R} defined by

$$P_n(x) = \begin{cases} 1, & x \in [-1, 0); \\ [-1, 1], & x = 0; \\ \sin \frac{1}{x}, & x \in (0, \frac{2}{(4n-1)\pi}); \\ -1, & x \in [\frac{2}{(4n-1)\pi}, 1]. \end{cases}$$

It is easy to see that for every $A \in \mathfrak{F}(X)$ and every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{Z}^+$ such that $P_n \in W(G, A, \varepsilon)$ for every $n \geq n_0$. For every $n \in \mathbb{Z}^+$ we have that $H_d(F(0), \varphi^{-1}(P_n)(0)) = 1$. Then for every $n \in \mathbb{Z}^+$ $P_n \notin W(F, 0, \frac{1}{2})$ and so the map φ^{-1} is not continuous at G .

Remark 3.1. If X is a Baire space, Y is a normed linear space and $F \in \text{MU}(X, Y)$, then there is a dense G_δ -subset E of X such that $F(x)$ is single-valued for each $x \in E$. In fact, let $f : X \rightarrow Y$ be a quasicontinuous subcontinuous selection of F such that $\tilde{f} = F$ (see Theorem 2.1). By Theorem 4.1 in [HP], the set $C(f)$ of the points of continuity of f is a dense G_δ -set in X . It is easy to verify that F must be single-valued at every point of the set $C(f)$. The same holds also for $F \in \text{MC}(X, Y)$.

Theorem 3.2. *Let X be a locally compact space and Y be a Banach space. The map φ from $(\text{MU}(X, Y), \tau_{\text{UC}})$ onto $(\text{MC}(X, Y), \tau_{\text{UC}})$ is homeomorphism.*

Proof. By Theorem 2.6, φ is a bijection. By Theorem 3.1 it is sufficient to prove that φ^{-1} is continuous. Let $G \in \text{MC}(X, Y)$ and $F = \varphi^{-1}(G)$. Let $K \in \mathfrak{K}(X)$ and $\varepsilon > 0$. We show that there exist $K_1 \in \mathfrak{K}(X)$ and $\varepsilon_1 > 0$ such that $\varphi^{-1}(W(G, K_1, \varepsilon_1)) \subset W(F, K, \varepsilon)$. Let $K_1 \in \mathfrak{K}(X)$ be such that $K \subset \text{Int } K_1$. Put $\varepsilon_1 = \frac{\varepsilon}{3}$. Let $H \in W(G, K_1, \varepsilon_1)$ and $x \in K$. We show that $F(x) \subset S_\varepsilon(\varphi^{-1}(H)(x))$. Let $y \in F(x)$. By Remark 3.1 and Theorem 2.1 for $\frac{\varepsilon}{3}$ and for every $U \in \mathcal{U}(x)$ there exist $x_U \in U \cap \text{Int } K_1$ such that $F(x_U)$ is single-valued and $F(x_U) \in S_{\varepsilon/3}(y)$. From the fact that $F(x_U)$ is single-valued it follows that $G(x_U)$ is single-valued too and consequently $F(x_U) = G(x_U)$. Since $H(x_U) \subset S_{\varepsilon/3}(G(x_U))$ we have that $\varphi^{-1}(H)(x_U) \subset S_{\varepsilon/3}(F(x_U))$ and hence there exist $y_U \in \varphi^{-1}(H)(x_U)$ such that $d(y_U, F(x_U)) < \frac{\varepsilon}{3}$. Hence $d(y, y_U) < \frac{2\varepsilon}{3}$. So there exists a subnet of the net $\{(x_U, y_U) : U \in \mathcal{U}(x)\}$ which converges to a point (x, z) , where $z \in \varphi^{-1}(H)(x)$. So $F(x) \subset S_\varepsilon(\varphi^{-1}(H)(x))$. The inclusion $\varphi^{-1}(H)(x) \subset S_\varepsilon(F(x))$ can be proved similarly. □

The following example shows that the condition of local compactness in Theorem 3.2 is essential.

Example 3.2. Let $X = [-1, 1]$ with the topology, where the open sets in X are all subsets of X not containing 0 and all subsets of X containing 0 that have countable complement. Every compact set in X is finite. Thus the topology τ_{UC} on $\text{MU}(X, \mathbb{R})$ and $\text{MC}(X, \mathbb{R})$ reduces to the topology τ_p . So we can use Example 3.1.

Theorem 3.3. *Let X be a Baire space and Y be a Banach space. The map φ from $(\text{MU}(X, Y), \tau_U)$ onto $(\text{MC}(X, Y), \tau_U)$ is homeomorphism.*

Proof. The proof is similar to the proof of Theorem 3.2. □

In the last part of our paper we will consider the Vietoris topology V on $\text{MU}(X, \mathbb{R})$ and on $\text{MC}(X, \mathbb{R})$. First we will consider the Vietoris topology V on

the space $\text{CL}(X \times \mathbb{R})$ of nonempty closed subsets of $X \times \mathbb{R}$. The basic open subsets of $\text{CL}(X \times \mathbb{R})$ in V are the subsets of the form

$$W^+ \cap W_1^- \cap \cdots \cap W_n^-,$$

where W, W_1, \dots, W_n are open subsets of $X \times \mathbb{R}$, $W^+ = \{F \in \text{CL}(X \times \mathbb{R}) : F \subset W\}$, and each $W_i^- = \{F \in \text{CL}(X \times \mathbb{R}) : F \cap W_i \neq \emptyset\}$.

Under the identification of every element of $\text{MU}(X, \mathbb{R})$ and $\text{MC}(X, \mathbb{R})$ with its graph, we can consider $\text{MU}(X, \mathbb{R})$ and $\text{MC}(X, \mathbb{R})$ as subsets of $\text{CL}(X, \mathbb{R})$. We will consider the induced Vietoris topology V on $\text{MU}(X, \mathbb{R})$ and on $\text{MC}(X, \mathbb{R})$.

Theorem 3.4. *Let X be a locally connected space. The map φ from $(\text{MU}(X, \mathbb{R}), V)$ onto $(\text{MC}(X, \mathbb{R}), V)$ is continuous.*

Proof. Let $F \in \text{MU}(X, \mathbb{R})$ and $W^+ \cap W_1^- \cap \cdots \cap W_n^-$ be a basic open set in $(\text{MC}(X, \mathbb{R}), V)$ such that $\varphi(F) \in W^+ \cap W_1^- \cap \cdots \cap W_n^-$.

Let $G = \varphi(F)$. By Lemma 4.1 in [HJM] there is an open set $H \subset X \times \mathbb{R}$ such that $G \subset H \subset W$ and $H(x)$ is connected for every $x \in X$. Without loss of generality we can also suppose that for every $i = 1, 2, \dots, n$, $W_i \subset H$ and $W_i = U_i \times V_i$, U_i open in X , V_i an open interval in \mathbb{R} .

For every $i \in \{1, 2, \dots, n\}$ we will define an open set \mathcal{H}_i as follows. Let $i \in \{1, 2, \dots, n\}$. Let $(x_i, y_i) \in G \cap W_i$. If $y_i = \inf F(x_i)$ or $y_i = \sup F(x_i)$, we will put $\mathcal{H}_i = W_i^-$. Otherwise, let C_i be a connected set in X such that $x_i \in \text{Int } C_i \subset C_i \subset U_i$ and $\epsilon > 0$ be such that $\inf F(x_i) + \epsilon < y_i < \sup F(x_i) - \epsilon$. Put $\mathcal{H}_i = (\text{Int } C_i \times (\inf F(x_i) - \epsilon, \inf F(x_i) + \epsilon))^- \cap (\text{Int } C_i \times (\sup F(x_i) - \epsilon, \sup F(x_i) + \epsilon))^-$.

It is easy to verify that $L \in \text{MU}(X, \mathbb{R}) \cap \mathcal{H}_i$ implies that $\varphi(L) \in \mathcal{H}_i$. Since $\varphi(L)$ is upper semicontinuous set-valued map with connected values, $\varphi(L)(\text{Int } C_i)$ must be a connected set ([Be], Proposition 6.2.12); i.e., $y_i \in \varphi(L)(\text{Int } C_i)$. Thus $\varphi(L) \in \bigcap W_i^-$.

Put $\mathcal{G} = W^+ \cap \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_n$. Then $F \in \mathcal{G}$ and $\varphi(S) \in W^+ \cap W_1^- \cap \cdots \cap W_n^-$ for every $S \in \mathcal{G}$. \square

The following Example shows that the condition of local connectedness in the above Theorem is essential.

Example 3.3. Let $X = [-1, 1] \setminus \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ with the usual Euclidean topology. Consider the maps F and G from X to \mathbb{R} defined by

$$F(x) = \begin{cases} 1, & x \in X \cap [-1, 0); \\ \{-1, 1\}, & x = 0; \\ -1, & x \in X \cap (0, 1]; \end{cases} \quad G(x) = \begin{cases} 1, & x \in X \cap [-1, 0); \\ [-1, 1], & x = 0; \\ -1, & x \in X \cap (0, 1]. \end{cases}$$

Then $G = \varphi(F)$ and we claim that φ is not continuous at F . For every $n \in \mathbb{Z}^+$ let f_n be the function from X to \mathbb{R} defined by

$$f_n(x) = \begin{cases} 1, & x \in X \cap [-1, \frac{1}{n}); \\ -1, & x \in X \cap (\frac{1}{n}, 1]. \end{cases}$$

We have $f_n = \varphi(f_n)$ for every $n \in \mathbb{Z}^+$. The sequence $\{f_n : n \in \mathbb{Z}^+\}$ converges in $(\text{MU}(X, \mathbb{R}), V)$ to F , but $\{f_n : n \in \mathbb{Z}^+\}$ does not converge to G in $(\text{MC}(X, \mathbb{R}), V)$.

The following example shows that the map φ^{-1} from $(\text{MC}([-1, 1], \mathbb{R}), V)$ onto $(\text{MU}([-1, 1], \mathbb{R}), V)$ is not continuous.

Example 3.4. Let $X = [-1, 1]$ with the usual Euclidean topology. Let F, G be maps from Example 2.2. Then $F = \varphi^{-1}(G)$ and we claim that φ^{-1} is not continuous at G . For every $n \in \mathbb{Z}^+$ let g_n be the function from $[-1, 1]$ to \mathbb{R} defined by

$$g_n(x) = \begin{cases} 1, & x \in [-1, 0]; \\ 1 - 2nx, & x \in (0, \frac{1}{n}); \\ -1, & x \in [\frac{1}{n}, 1]. \end{cases}$$

Evidently $g_n = \varphi^{-1}(g_n)$ for every $n \in \mathbb{Z}^+$. It is easy to see that the sequence $\{g_n : n \in \mathbb{Z}^+\}$ converges in $(\text{MC}([-1, 1], \mathbb{R}), V)$ to G , but $\{g_n : n \in \mathbb{Z}^+\}$ does not converge to F in $(\text{MU}([-1, 1], \mathbb{R}), V)$.

Acknowledgement. The authors would like to thank the referee.

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Received February 28, 2013; revised July 10, 2013

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