

Warped product pointwise semi-slant submanifolds of Kähler manifolds

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Abstract. It is known that there exist no warped product semi-slant submanifolds in Kähler manifolds [15]. Recently, Chen and Garay studied pointwise slant submanifolds of almost Hermitian manifolds in [9] and obtained many new results for such submanifolds. In this paper, we first introduce pointwise semi-slant submanifolds of Kähler manifolds and then we show that there exists non-trivial warped product pointwise semi-slant submanifolds of Kähler manifold by giving an example, contrary to the semi-slant case. We present a characterization theorem and establish an inequality for the squared norm of the second fundamental form in terms of the warping function for such warped product submanifolds in Kähler manifolds. The equality case is also considered.

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1. Introduction

CR-submanifolds of Kähler manifolds were introduced by Bejancu [1] as a generalization of totally real submanifolds and holomorphic submanifolds. In [5], Chen (see also, [6], [7]) studied warped product CR-submanifolds and showed that there exist no warped product CR-submanifolds of the form $M_{\perp} \times_f M_T$ such that M_{\perp} is a totally real submanifold and M_T is a holomorphic submanifold of a Kähler manifold \overline{M} . Then he introduced the CR-warped product submanifolds as follows: A submanifold M of a Kähler manifold \overline{M} is called CR-warped product if it is the warped product $M_T \times_f M_{\perp}$ of a holomorphic submanifold M_T and a totally real submanifold M_{\perp} of \overline{M} . He also established general sharp inequalities for CR-warped products in Kähler manifolds. After Chen's papers, CR-warped product submanifolds have been studied by many authors see: a survey [8] and references therein.

On the other hand, slant submanifolds of Kähler manifolds were defined by Chen in [4] as another generalization of totally real submanifolds and holomorphic

submanifolds. A slant submanifold is called proper if it is neither totally real nor holomorphic, see also [10] for slant submanifolds. We note that there exists no inclusion relation between proper CR-submanifolds and proper slant submanifolds. In [14], N. Papaghiuc introduced a class of submanifolds, called semi-slant submanifolds such that the class of CR-submanifolds and the class of slant submanifolds appear as particular classes of semi-slant submanifolds. In [15], we proved that there do not exist warped product semi-slant submanifolds of the forms $M_T \times_f M_\theta$ and $M_\theta \times_f M_T$ in Kähler manifolds, where M_T is a holomorphic submanifold and M_θ is a proper slant submanifold of a Kähler manifold \bar{M} . Pointwise slant submanifolds of almost Hermitian manifolds were introduced by Etayo in [12] and such submanifolds have been studied by Chen–Garay in [9]. They obtain simple characterizations, give a method how to construct such submanifolds in Euclidean space and investigate geometric and topological properties of pointwise slant submanifolds. In this paper we first define pointwise semi-slant submanifolds and then we show that there exists non-trivial warped product pointwise semi-slant submanifolds of the form $M_T \times_f M_\theta$ in Kähler manifolds, where M_T is a holomorphic submanifold and M_θ pointwise slant submanifolds.

The paper is organized as follows: In section 2, we present the basic information needed for this paper. In section 3, we give definition of pointwise semi-slant submanifolds. After we give two characterization theorems for pointwise semi-slant submanifolds, we investigate the geometry of leaves of distributions which are involved in the definition of pointwise semi-slant submanifolds. In section 4, we prove that there do not exist warped product submanifolds of the form $M_\theta \times_f M_T$ such that M_θ is a pointwise slant submanifold and M_T is a holomorphic submanifold of \bar{M} . In section 5, we consider warped product submanifolds of the form $M_\theta \times_f M_T$ in Kähler manifolds, give an example and present a characterization of such warped product submanifolds. We also obtain an inequality for the squared norm of the second fundamental form in terms of the warping function for warped product pointwise semi-slant submanifolds. The equality case is also considered.

In this paper, we assume that every object at hand is smooth.

2. Preliminaries

Let (\bar{M}, g) be a Kähler manifold. This means [16] that \bar{M} admits a tensor field J of type $(1, 1)$ on \bar{M} such that for all $X, Y \in \Gamma(T\bar{M})$, we have

$$J^2 = -I, \quad g(X, Y) = g(JX, JY), \quad (\bar{\nabla}_X J)Y = 0, \quad (2.1)$$

where g is the Riemannian metric and $\bar{\nabla}$ is the Levi-Civita connection on \bar{M} .

Let \bar{M} be a Kähler manifold with complex structure J and M a Riemannian manifold isometrically immersed in \bar{M} . Then M is called holomorphic (complex) if $J(T_pM) \subset T_pM$, for every $p \in M$, where T_pM denotes the tangent space of M at the point p . M is called totally real if $J(T_pM) \subset T_pM^\perp$ for every $p \in M$, where T_pM^\perp denotes the normal space of M at the point p . Besides holomorphic and totally real submanifolds, there are four other important classes of submanifolds of a Kähler manifold determined by the behavior of the tangent bundle of the submanifold under the action of the complex structure of the ambient manifold.

- (1) The submanifold M is called a CR-submanifold [1] if there exists a differentiable distribution $D : p \rightarrow D_p \subset T_pM$ such that D is invariant with respect to J and the complementary distribution D^\perp is anti-invariant with respect to J .
- (2) The submanifold M is called slant [4] if for each non-zero vector X tangent to M the angle $\theta(X)$ between JX and T_pM is a constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM$.
- (3) The submanifold M is called semi-slant [14] if it is endowed with two orthogonal distributions D and D' , where D is invariant with respect to J and D' is slant, i.e., $\theta(X)$ between JX and D'_p is constant for $X \in D'_p$.
- (4) The submanifold M is called pointwise slant submanifold [12], [9] if at each given point $p \in M$, the Wirtinger angle $\theta(X)$ between JX and the space T_pM is independent of the choice of the nonzero vector $X \in \Gamma(TM)$. In this case, the angle θ can be regarded as a function M , which is called the *slant function* of the pointwise slant submanifold.

A point p in a pointwise slant submanifold is called a totally real point if its slant function θ satisfies $\cos \theta = 0$ at p . Similarly, a point p is called a complex point if its slant function satisfies $\sin \theta = 0$ at p . A pointwise slant submanifold M in an almost Hermitian manifold \bar{M} is called totally real if every point of M is a totally real point. A pointwise slant submanifold of an almost Hermitian manifold is called pointwise proper slant if it contains no totally real points. A pointwise slant submanifold M is called slant when its slant function θ is globally constant, i.e., θ is also independent of the choice of the point on M . It is clear that pointwise slant submanifolds include holomorphic and totally real submanifolds and slant submanifolds. It is also clear that CR-submanifolds and slant submanifolds are particular semi-slant submanifolds with $\theta = \frac{\pi}{2}$ and $D = \{0\}$, respectively.

Let M be a Riemannian manifold isometrically immersed in \bar{M} and denote by the same symbol g for the Riemannian metric induced on M . Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(TM^\perp)$ the set of all vector fields normal to M , same notation for smooth sections of any other vector bundle E . Denote by ∇

the Levi-Civita connection of M . Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (2.2)$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \quad (2.3)$$

for any $X, Y \in \Gamma(TM)$ and any $N \in \Gamma(TM^\perp)$, where ∇^\perp is the connection in the normal bundle TM^\perp , h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N . The second fundamental form h and the shape operator A are related by

$$g(A_N X, Y) = g(h(X, Y), N). \quad (2.4)$$

For any $X \in \Gamma(TM)$ we write

$$JX = TX + FX, \quad (2.5)$$

where TX is the tangential component of JX and FX is the normal component of JX . Similarly, for any vector field N normal to M , we put

$$JN = BN + CN, \quad (2.6)$$

where BN and CN are the tangential and the normal components of JN , respectively.

3. Pointwise semi-slant submanifolds

In this section, we define and study pointwise semi-slant submanifolds in a Kähler manifold \bar{M} . We obtain characterizations, give an example and investigate the geometry of leaves of distributions.

Definition 3.1. Let \bar{M} be a Kähler manifold and M a real submanifold of \bar{M} . Then we say that M is a pointwise semi-slant submanifold if there exist two orthogonal distributions D^T and D^θ on M such that

- (a) TM admits the orthogonal direct decomposition $TM = D^T \oplus D^\theta$.
- (b) The distribution D^T is a holomorphic distribution, i.e., $JD^T = D^T$.
- (c) The distribution D^θ is pointwise slant with slant function θ .

In this case, we call the angle θ the slant function of the pointwise slant submanifold M . The holomorphic distribution D^T of a pointwise semi-slant submanifold is a pointwise slant distribution with slant function $\theta = 0$. If we denote the dimension of D^T and D^θ by m_1 and m_2 , respectively, then we have the following:

- (a) If $m_2 = 0$, then M is a holomorphic submanifold.
- (b) If $m_1 = 0$, then M is a pointwise slant submanifold.
- (c) If θ is constant then M is a proper semi-slant submanifold with slant angle θ .
- (d) If $\theta = \frac{\pi}{2}$, then M is a CR-submanifold.

We say that a pointwise semi-slant submanifold is proper if $m_1 \neq 0$ and θ is not a constant.

Example 3.2. Let M be a submanifold of \mathbf{R}^6 given by

$$\chi(t, s, u, v) = (t, s, u, \sin v, 0, \cos v).$$

It is easy to see that a local frame of TM is given by

$$Z_1 = \frac{\partial}{\partial x_1}, \quad Z_2 = \frac{\partial}{\partial x_2}, \quad Z_3 = \frac{\partial}{\partial x_3}, \quad Z_4 = \cos v \frac{\partial}{\partial x_4} - \sin v \frac{\partial}{\partial x_6}.$$

Then using the canonical complex structure of \mathbf{R}^6 , we see that $D^T = \text{span}\{Z_1, Z_2\}$. Moreover it is easy to see that $D^\theta = \text{span}\{Z_3, Z_4\}$ is a pointwise slant distribution with slant function v . Thus M is a proper pointwise semi-slant submanifold of \mathbf{R}^6 .

Let M be a pointwise semi-slant submanifold of a Kähler manifold \bar{M} . We denote the projections on the distributions D^T and D^θ by P_1 and P_2 , respectively. Then we can write

$$X = P_1X + P_2X \tag{3.1}$$

for any $X \in \Gamma(TM)$. Applying J to (3.1) and using (2.5) we obtain

$$JX = JP_1X + TP_2X + FP_2X. \tag{3.2}$$

Thus we have

$$JP_1X \in \Gamma(D^T), \quad FP_1X = 0, \tag{3.3}$$

$$TP_2X \in \Gamma(D^\theta), \quad FP_2X \in \Gamma(TM^\perp). \tag{3.4}$$

Then (3.3) and (3.4) imply

$$TX = JP_1X + TP_2X \tag{3.5}$$

for $X \in \Gamma(TM)$.

It is known that M is a pointwise slant submanifold of \bar{M} if and only if

$$T^2 = -(\cos^2 \theta)I \tag{3.6}$$

for some real-valued function θ defined on M [9], where I denotes the identity transformation of the tangent bundle TM of the submanifold M . Thus we can prove the following characterization theorem.

Theorem 3.3. *Let D be a distribution on M . Then D is pointwise slant if and only if there exists a function $\lambda \in [-1, 0]$ such that $(TP_2)^2X = \lambda X$ for $X \in \Gamma(D)$, where P_2 denotes the orthogonal projection on D . Moreover in this case $\lambda = -\cos^2 \theta$.*

Actually this theorem is similar to that theorem given [3] for Sasakian case. We can use Theorem 3.3 to characterize pointwise semi-slant submanifolds. Let M be a real submanifold of an almost Hermitian manifold \bar{M} and D a distribution on M . We define $T_D : D \rightarrow TM$ by $T_D(X) = (JX)^{\mathcal{F}_D}$, where \mathcal{F}_D is the orthogonal projection of $T\bar{M}$ onto D . If M is a pointwise slant submanifold and D is its slant distribution, we have

$$T_D = P_2TI_D,$$

where I_D is the identity of D .

Theorem 3.4. *Let M be a submanifold of a Kähler manifold \bar{M} . Then M is a pointwise semi-slant submanifold if and only if there exists a function $\lambda \in [-1, 0]$ and a distribution D on M such that*

- (i) $D = \{X \in \Gamma(TM) \mid (T_D)^2X = \lambda X\}$,
- (ii) T maps D into D .

Moreover in this case $\lambda = -\cos^2 \theta$, where θ denotes the slant function of M .

Proof. Let M be a pointwise semi-slant submanifold of \bar{M} . Then $\lambda = -\cos^2 \theta$ and $D = D^\theta$. By the definition of pointwise semi-slant submanifold, (ii) is clear. Conversely (i) and (ii) imply $TM = D \oplus D^T$. Since T maps D into D , it implies that $J(D^T) = D^T$. Thus proof is complete. □

From Theorem 3.4 we have the following corollary:

Corollary. *Let M be a pointwise semi-slant submanifold of a Kähler manifold \bar{M} . Then we have*

$$g(TX, TY) = \cos^2 \theta g(X, Y) \tag{3.7}$$

$$g(FX, FY) = \sin^2 \theta g(X, Y) \tag{3.8}$$

for $X, Y \in \Gamma(D^\theta)$.

Proof. For $X, Y \in \Gamma(D^\theta)$, from (2.1) we have $g(TX, TY) = g(JX - FX, TY)$. Hence $g(TX, TY) = -g(X, JTY)$. Using Theorem 3.2 (i), we obtain (3.7). Using (3.7) we get (3.8). □

In the rest of this section, we first study the integrability of distributions and then we find the conditions under which leaves of distributions on a pointwise semi-slant submanifold M in a Kähler manifold \bar{M} are totally geodesic immersed in M . For the integrability of the distributions D^T and D^θ on a pointwise semi-slant submanifold M , we have the following.

Theorem 3.5. *Let M be a proper pointwise semi-slant submanifold of a Kähler manifold.*

(i) *The distribution D^T is integrable if and only if*

$$g(h(X, JY), FV) = g(h(JX, Y), FV) \quad \text{for all } X, Y \in \Gamma(D^T) \text{ and } V \in \Gamma(D^\theta).$$

(ii) *The distribution D^θ is integrable if and only if*

$$g(A_{FTW}V - A_{FTV}W, X) = g(A_{FW}V - A_{FV}W, JX)$$

for $W \in \Gamma(D^\theta)$.

Proof. We prove (i), (ii) can be obtained in a similar way. From (2.1), (2.3) and (2.5) we have

$$\begin{aligned} g([X, Y], V) &= -g(\bar{\nabla}_X Y, T^2V + FTV) + g(h(X, JY), FV) \\ &\quad + g(\bar{\nabla}_Y X, T^2V + FTV) - g(h(JX, Y), FV). \end{aligned}$$

Then the symmetric h and (3.6) imply that

$$\sin^2 \theta g([X, Y], V) = g(h(X, JY), FV) - g(h(JX, Y), FV)$$

which gives the assertion. □

Next we give necessary and sufficient conditions for the distributions D^T and D^θ whose leaves are totally geodesic.

Theorem 3.6. *Let M be a proper pointwise semi-slant submanifold of a Kähler manifold.*

(a) *The holomorphic distribution D^T defines a totally geodesic foliation if and only if*

$$g(h(X, Y), FTV) = g(h(X, JY), FV) \tag{3.9}$$

for $X, Y \in \Gamma(D^T)$ and $V \in \Gamma(D^\theta)$.

(b) *The slant distribution D^θ defines a totally geodesic foliation on M if and only if*

$$g(h(U, X), FTV) = g(h(U, JX), FV)$$

for $X \in \Gamma(D^T)$ and $U, V \in \Gamma(D^\theta)$.

Proof. Let M be a proper pointwise semi-slant submanifold of a Kähler manifold \bar{M} . Then we have $g(\nabla_X Y, V) = g(\bar{\nabla}_X JY, JV)$ for $X, Y \in \Gamma(D^T)$ and $V \in \Gamma(D^\theta)$. Thus using (2.5) and (2.6) we get

$$g(\nabla_X Y, V) = -g(\bar{\nabla}_X Y, JTV) + g(\bar{\nabla}_X JY, FV).$$

Then (3.6) implies that

$$\sin^2 \theta g(\nabla_X Y, V) = -g(h(X, Y), FTV) + g(h(X, JY), FV)$$

which gives (a). In a similar way, we obtain (b). □

Thus from Theorem 3.6, we have the following result:

Corollary. *Let M be a pointwise semi-slant submanifold of a Kähler manifold \bar{M} . Then M is a locally Riemannian product manifold $M = M_T \times M_\theta$ if and only if*

$$A_{FTV}X = A_{FV}JX$$

for $V \in \Gamma(D^\theta)$ and $X \in \Gamma(D^\perp)$, where M_T is a holomorphic submanifold and M_θ is a pointwise slant submanifold of \bar{M} .

4. Warped products $M_\theta \times_f M_T$ in Kähler manifolds

Let (B, g_1) and (F, g_2) be two Riemannian manifolds, $f : B \rightarrow (0, \infty)$ and $\pi : B \times F \rightarrow B, \eta : B \times F \rightarrow F$ the projection maps given by $\pi(p, q) = p$ and

$\eta(p, q) = q$ for every $(p, q) \in B \times F$. The warped product ([2]) $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_*X, \pi_*Y) + (f\circ\pi)^2 g_2(\eta_*X, \eta_*Y)$$

for every X and Y of M , where $*$ denotes the tangent map. The function f is called the warping function of the warped product manifold. In particular, if the warping function is constant, then the warped product manifold M is said to be trivial.

Let X, Y be vector fields on B and V, W vector fields on F , then from Lemma 7.3 of [2], we have

$$\nabla_X V = \nabla_V X + \left(\frac{Xf}{f}\right) V \tag{4.1}$$

where ∇ is the Levi-Civita connection on M .

In this section we investigate the existence of non-trivial warped product submanifolds $M_\theta \times_f M_T$ of Kähler manifolds such that M_T is a holomorphic submanifold and M_θ is a proper pointwise slant submanifold of \overline{M} .

Theorem 4.1. *Let \overline{M} be a Kähler manifold. Then there exist no non-trivial warped product submanifolds $M = M_\theta \times_f M_T$ of a Kähler manifold \overline{M} such that M_T is a holomorphic submanifold and M_θ is a proper pointwise slant submanifold of \overline{M} .*

Proof. From (4.1), (2.2), (2.3), (2.4) and (2.5) we have

$$V(\ln f)g(X, Y) = -g(\overline{\nabla}_X T^2 V + FTV, Y) - g(A_{FV}X, JY).$$

Using (3.6) we get

$$V(\ln f)g(X, Y) = g(\overline{\nabla}_X \cos^2 \theta V, Y) - g(\overline{\nabla}_X FTV, Y) - g(A_{FV}X, JY).$$

Thus from (2.3) and (2.4) we obtain

$$\begin{aligned} V(\ln f)g(X, Y) &= -\sin 2\theta X(\theta)g(V, Y) + \cos^2 \theta g(\nabla_X V, Y) \\ &\quad + g(h(X, Y), FTV) - g(h(X, JY), FV). \end{aligned}$$

Since D^T and D^θ are orthogonal, using (4.1) we arrive at

$$\sin^2 \theta V(\ln f)g(X, Y) = g(h(X, Y), FTV) - g(h(X, JY), FV).$$

Interchanging the role of X and Y in above equation and then subtracting each other, we derive

$$g(h(JX, Y), FV) = g(h(X, JY), FV). \quad (4.2)$$

On the other hand, from (2.3), (2.1), (2.5) and (4.1) we have

$$g(h(X, JY), FV) = -V(\ln f)g(X, Y) + TV(\ln f)g(X, JY). \quad (4.3)$$

Then from (4.2) and (4.3) we conclude

$$TV(\ln f)g(X, JY) = 0.$$

Replacing V by TV and X by JX we find

$$\cos^2 \theta V(\ln f)g(X, Y) = 0$$

which implies

$$V(\ln f) = 0$$

due to M_θ is proper pointwise slant and M_T is a Riemannian manifold. Thus it follows that f is a constant. \square

Remark 4.2. We note that Theorem 4.1 is a generalization of Theorem 3.1 in [5] and Theorem 3.1 in [15].

5. Non-trivial warped products $M_T \times_f M_\theta$ in Kähler manifolds

Theorem 4.1 shows that there do not exist non-trivial warped product pointwise semi-slant submanifolds of the form $M_\theta \times_f M_T$ in Kähler manifolds. In this section, we consider non-trivial warped product pointwise semi-slant submanifolds of the form $M_T \times_f M_\theta$, where M_T is a holomorphic submanifold and M_θ is a proper pointwise slant submanifold of \overline{M} . First, we are going to give an example of non-trivial warped product pointwise semi-slant submanifold of the form $M_T \times_f M_\theta$.

Example 5.1. For $t, s \neq 0, 1, u, v \in (0, \frac{\pi}{2})$, consider a submanifold M in R^{10} given by the equations

$$\begin{aligned} x_1 &= t \cos u, & x_2 &= s \cos u, & x_3 &= t \cos v, & x_4 &= s \cos v, & x_5 &= t \sin u \\ x_6 &= s \sin u, & x_7 &= t \sin v, & x_8 &= s \sin v, & x_9 &= u, & x_{10} &= v. \end{aligned}$$

Then the tangent bundle TM is spanned by Z_1, Z_2, Z_3 and Z_4 where

$$\begin{aligned} Z_1 &= \cos u \frac{\partial}{\partial x_1} + \cos v \frac{\partial}{\partial x_3} + \sin u \frac{\partial}{\partial x_5} + \sin v \frac{\partial}{\partial x_7} \\ Z_2 &= \cos u \frac{\partial}{\partial x_2} + \cos v \frac{\partial}{\partial x_4} + \sin u \frac{\partial}{\partial x_6} + \sin v \frac{\partial}{\partial x_8} \\ Z_3 &= -t \sin u \frac{\partial}{\partial x_1} - s \sin u \frac{\partial}{\partial x_2} + t \cos u \frac{\partial}{\partial x_5} + s \cos u \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_9} \\ Z_4 &= -t \sin v \frac{\partial}{\partial x_3} - s \sin v \frac{\partial}{\partial x_4} + t \cos v \frac{\partial}{\partial x_7} + s \cos v \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}}. \end{aligned}$$

Then $D^T = \text{span}\{Z_1, Z_2\}$ is a holomorphic distribution and $D^\theta = \text{span}\{Z_3, Z_4\}$ is a pointwise slant distribution with the slant function $\cos^{-1}\left(\frac{1}{t^2+s^2+1}\right)$. Thus M is a pointwise semi-slant submanifold of R^{10} . It is easy to see that D^θ and D^T are integrable. We denote the integral manifolds of D^T and D^θ by M_T and M_θ , respectively. Then the metric tensor g of M is

$$g = 2dx_1^2 + 2dx_2^2 + (t^2 + s^2 + 1)(dx_3^2 + dx_4^2) = g_{M_T} + (t^2 + s^2 + 1)g_{M_\theta}.$$

Thus M is a non-trivial warped product submanifold of R^{10} of the form $M_T \times_f M_\theta$ with warping function $\sqrt{(t^2 + s^2 + 1)}$.

Remark 5.2. Non-trivial warped product pointwise semi-slant submanifolds of the form $M_T \times_f M_\theta$ are natural extension of warped product CR-submanifolds. Indeed, every CR-warped product submanifold is a non-trivial warped product pointwise semi-slant submanifold of the form $M_T \times_f M_\theta$ with the slant function $\theta = 0$.

From now on, we will consider non-trivial warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ such that M_θ is a proper pointwise slant submanifold and M_T is a holomorphic submanifold. First we give some preparatory lemmas.

Lemma 5.3. *Let $M = M_T \times_f M_\theta$ be a non-trivial warped product pointwise proper semi-slant submanifold of a Kähler manifold \bar{M} . Then we have*

$$g(A_{FV}W, X) = g(A_{FW}V, X) \tag{5.1}$$

for $V, W \in \Gamma(D^\theta)$ and $X \in \Gamma(D^T)$.

Proof. Using (2.1), (2.2) and (2.5) we have

$$g(A_{FV}X, W) = g(\nabla_X V, TW) + g(\bar{\nabla}_X V, FW) + g(\nabla_X TV, W)$$

for $X \in \Gamma(D^T)$ and $V, W \in \Gamma(D^0)$. Then from (4.1) and (2.2) we obtain

$$g(A_{FV}X, W) = g(h(X, V), FW)$$

which gives the assertion. □

Lemma 5.4. *Let $M = M_T \times_f M_0$ be a non-trivial warped product pointwise semi-slant submanifold of a Kähler manifold \bar{M} . Then we have*

$$g(A_{FTW}V, X) = -JX(\text{Inf})g(TW, V) - X(\text{Inf})\cos^2\theta g(V, W) \tag{5.2}$$

and

$$g(A_{FW}V, JX) = X(\text{Inf})g(W, V) + JX(\text{Inf})g(V, TW) \tag{5.3}$$

for $V, W \in \Gamma(D^0)$ and $X \in \Gamma(D^T)$.

Proof. From (5.1) we write $g(A_{FTW}V, X) = g(A_{FV}TW, X)$. Then using (2.1), (2.2), (2.3) and (2.5) we have

$$g(A_{FTW}X, W) = g(\nabla_{TW}V, JX) + g(\nabla_{TW}TV, X).$$

Thus from (4.1) and (3.7) we obtain (5.2). (5.2) gives (5.3). □

In the sequel we give a characterization for non-trivial warped product pointwise semi-slant submanifolds of the form $M_T \times_f M_0$. Recall that we have the following result of Hiepko [13], see also [11]: *Let D_1 be a vector subbundle in the tangent bundle of a Riemannian manifold M and D_2 be its normal bundle. Suppose that the two distributions are involutive. We denote the integral manifolds of D_1 and D_2 by M_1 and M_2 , respectively. Then M is locally isometric to non-trivial warped product $M_1 \times_f M_2$ if the integral manifold M_1 is totally geodesic and the integral manifold M_2 is an extrinsic sphere, i.e., M_2 is a totally umbilical submanifold with parallel mean curvature vector.*

Theorem 5.5. *Let M be a pointwise semi-slant submanifold of a Kähler manifold \bar{M} . Then M is locally a non-trivial warped product manifold of the form $M = M_T \times_f M_0$ such that M_0 is a proper pointwise slant submanifold and M_T is a holomorphic submanifold in \bar{M} if the following condition is satisfied*

$$A_{FTW}X - A_{FW}JX = -(1 + \cos^2\theta)X(\mu)W, \tag{5.4}$$

where μ is a function on M such that $W(\mu) = 0$ for every $W \in \Gamma(D^0)$ and $X \in \Gamma(D^T)$.

Proof. Let $M = M_T \times_f M_\theta$ be a non-trivial warped product pointwise semi-slant submanifold of a Kähler manifold \bar{M} . Then from (2.1), (2.3) and (2.5) we obtain

$$g(A_{FV}X, Y) = g(\nabla_X V, JY) + g(\nabla_X TV, Y)$$

for $X, Y \in \Gamma(D^T)$ and $V \in \Gamma(D^\theta)$. Then using (4.1) we derive

$$g(A_{FV}X, Y) = 0$$

which shows that $A_{FV}X$ belongs to D^θ . Conversely, suppose that M is a pointwise semi-slant submanifold of a Kähler manifold \bar{M} such that

$$A_{FTW}X - A_{FW}JX = -(1 + \cos^2 \theta)X(\mu)W \tag{5.5}$$

for $W \in \Gamma(D^\theta)$ and $X \in \Gamma(D^T)$. Then from Theorem 3.5 (ii), D^θ is integrable. Also from Theorem 3.6 (b), we find that the integral manifold M_T of D^T is totally geodesic. Let M_θ be the integral manifold of D^θ and denote the second fundamental form of M_θ in M by h_θ . Since Weingarten operator A_N is self-adjoint, using (2.3) we get

$$g(A_{FTV}X - A_{FV}JX, W) = -g(X, \bar{\nabla}_W FTV) + g(JX, \bar{\nabla}_W FV)$$

for $V, W \in \Gamma(D^\theta)$ and $X \in \Gamma(TM)$. Then from (2.1), (2.2) and (2.5) we have

$$g(A_{FTV}X - A_{FV}JX, W) = g(X, \nabla_W T^2V) + g(X, \nabla_W V).$$

Thus from (3.6) we obtain

$$\begin{aligned} g(A_{FTV}X - A_{FV}JX, W) &= \sin 2\theta W(\theta)g(X, V) - \cos^2 \theta g(X, \nabla_W V) + g(X, \nabla_W V) \\ &= \sin^2 \theta g(X, \nabla_W V). \end{aligned}$$

Hence we derive

$$g(A_{FTV}X - A_{FV}JX, W) = \sin^2 \theta g(X, h_\theta(V, W)). \tag{5.6}$$

Then (5.6) and (5.5) imply that

$$h_\theta(V, W) = -(\csc^2 \theta + \cot^2 \theta)\nabla\mu g(V, W)$$

which shows that M_θ is a totally umbilical submanifold in M with the mean curvature vector field $-(\csc^2 \theta + \cot^2 \theta)\nabla\mu$, where $\nabla\mu$ is the gradient of μ . On the other hand, by direct computations, we get

$$\begin{aligned} g(\nabla_V \nabla \mu, X) &= [Vg(\nabla\mu, X) - g(\nabla\mu, \nabla_V X)] \\ &= [V(X(\mu)) - [V, X]\mu - g(\nabla\mu, \nabla_X V)] \\ &= [[V, X]\mu + X(V(\mu)) - [V, X]\mu - g(\nabla\mu, \nabla_X V)] \\ &= [X(V(\mu)) - g(\nabla\mu, \nabla_X V)]. \end{aligned}$$

Since $V(\mu) = 0$, we obtain

$$g(\nabla_V \nabla \mu, X) = g(\nabla\mu, \nabla_X V).$$

On the other hand, since $\nabla\mu \in \Gamma(TM_T)$ and M_T is totally geodesic in M , it follows that $\nabla_X V \in \Gamma(TM_\theta)$ for $X \in \Gamma(D^T)$ and $V \in \Gamma(D^\theta)$. Hence $g(\nabla_V \nabla \mu, X) = 0$. Then the spherical condition is also fulfilled, that is M_θ is an extrinsic sphere in M . Thus we conclude that M is a non-trivial warped product and proof is complete. □

We now give an inequality in terms of the length of the second fundamental form. First we give a lemma which will be useful for the theorem.

Lemma 5.6. *Let $M = M_T \times_f M_\theta$ be a non-trivial warped product pointwise semi-slant submanifold of a Kähler manifold \bar{M} . Then we have*

$$g(h(X, Y), FV) = 0 \tag{5.7}$$

and

$$g(h(X, V), FW) = -JX(\ln f)g(V, W) - X(\ln f)g(V, TW) \tag{5.8}$$

for $V, W \in \Gamma(D^\theta)$ and $X, Y \in \Gamma(D^T)$.

Proof. From (2.5), (2.1) and (2.2) we get

$$g(h(X, Y), FV) = -g(\nabla_X JY, V) - g(\nabla_X Y, TV).$$

Since D^T and D^θ are orthogonal, using (4.1) we derive

$$g(h(X, Y), FV) = X(\ln f)g(V, JY) + X(\ln f)g(TV, Y) = 0$$

which gives (5.7). (5.8) comes from (5.2) and (5.3). □

Let M be an $(m + n)$ dimensional proper pointwise semi-slant submanifold of a Kähler manifold \overline{M}^{m+2n} , where \overline{M} is of real dimension $m + 2n$ and it is obvious that m is also even. Then we choose a canonical orthonormal frame $\{e_1, \dots, e_m, \bar{e}_1, \dots, \bar{e}_n, e_1^*, \dots, e_n^*\}$ of \overline{M} such that, restricted to M , $e_1, \dots, e_m, \bar{e}_1, \dots, \bar{e}_n$ are tangent to M . Then $\{e_1, \dots, e_m, \bar{e}_1, \dots, \bar{e}_n\}$ form an orthonormal frame of M . We can take $\{e_1, \dots, e_m, \bar{e}_1, \dots, \bar{e}_n\}$ in such a way that $\{e_1, \dots, e_m\}$ form an orthonormal frame of D^T and $\{\bar{e}_1, \dots, \bar{e}_n\}$ form an orthonormal frame of D^θ , where $\dim(D^T) = m$ and $\dim(D^\theta) = n$. We can take $\{e_1^*, \dots, e_n^*\}$ as an orthonormal frame of $F(D^\theta)$. It is known that a proper pointwise slant submanifold is always even dimensional. Hence, $n = 2p$. Then we can choose orthonormal frames $\{\bar{e}_1, \dots, \bar{e}_{2p}\}$ of D^θ and $\{e_1^*, \dots, e_{2p}^*\}$ of $F(D^\theta)$ in such a way that

$$\begin{aligned} \bar{e}_2 &= \sec \theta T \bar{e}_1, \dots, \bar{e}_{2p} = \sec \theta T \bar{e}_{2p-1}, \\ e_1^* &= \csc \theta F \bar{e}_1, \dots, e_{2p}^* = \csc \theta F \bar{e}_{2p}, \end{aligned}$$

where θ is the slant function. We will call this orthonormal frame an adapted frame as for slant submanifold case [4].

Theorem 5.7. *Let M be an $(m + n)$ -dimensional non-trivial warped product pointwise semi-slant submanifold of the form $M_T \times_f M_\theta$ in a Kähler manifold \overline{M}^{m+2n} , where M_T is a holomorphic submanifold and M_θ is a proper pointwise slant submanifold of \overline{M}^{m+2n} . Then we have*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq 2n(\csc^2 \theta + \cot^2 \theta) \|\nabla(\ln f)\|^2, \quad \dim(M_\theta) = n. \tag{5.9}$$

(ii) *If the equality of (5.9) holds identically, then M_T is a totally geodesic submanifold and M_θ is a totally umbilical submanifold of \overline{M} . Moreover, M is a minimal submanifold of \overline{M} .*

Proof. Since

$$\|h\|^2 = \|h(D^T, D^T)\|^2 + \|h(D^\theta, D^\theta)\|^2 + 2\|h(D^T, D^\theta)\|^2,$$

we have

$$\begin{aligned} \|h\|^2 &= \sum_{k=1}^{m+2p} \sum_{i,j=1}^m g(h(e_i, e_j), \tilde{e}_k)^2 + \sum_{k=1}^{m+2p} \sum_{r,s=1}^{2p} g(h(\bar{e}_r, \bar{e}_s), \tilde{e}_k)^2 \\ &\quad + 2 \sum_{k=1}^{m+2p} \sum_{r=1}^{2p} \sum_{i=1}^m g(h(e_i, \bar{e}_r), \tilde{e}_k)^2 \end{aligned}$$

where $\{\tilde{e}_k\}$ is an orthonormal basis of TM^\perp . Now, considering the adapted frame, we can write the above equation as

$$\begin{aligned} \|h\|^2 &= \sum_{a=1}^{2p} \sum_{i,j=1}^m g(h(e_i, e_j), \csc \theta F \bar{e}_a)^2 + \sum_{a,r,s=1}^{2p} g(h(\bar{e}_r, \bar{e}_s), \csc \theta F \bar{e}_a)^2 \\ &\quad + 2 \sum_{i=1}^m \sum_{a,r=1}^{2p} g(h(\bar{e}_r, e_i), \csc \theta F \bar{e}_a)^2. \end{aligned}$$

Then, from (5.7) and (5.8), we obtain

$$\begin{aligned} \|h\|^2 &= \sum_{a,r,s=1}^{2p} g(h(\bar{e}_r, \bar{e}_s), \csc \theta F \bar{e}_a)^2 + 2 \sum_{i=1}^m \sum_{a,r=1}^{2p} (\csc \theta)^2 [(Je_i(\ln f)g(\bar{e}_r, \bar{e}_a))^2 \\ &\quad + 2Je_i(\ln f)g(\bar{e}_r, \bar{e}_a)e_i(\ln f)g(\bar{e}_r, T\bar{e}_a) + (e_i(\ln f)g(\bar{e}_r, T\bar{e}_a))^2]. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{i=1}^m \sum_{a,r=1}^{2p} Je_i(\ln f)g(\bar{e}_r, \bar{e}_a)e_i(\ln f)g(\bar{e}_r, T\bar{e}_a) \\ &= \sum_{i=1}^m \sum_{a,r=1}^{2p} g(\nabla(\ln f), Je_i)g(\nabla(\ln f), e_i)g(\bar{e}_r, \bar{e}_a)g(\bar{e}_r, T\bar{e}_a) \\ &= - \sum_{a,r=1}^{2p} \left[\sum_{i=1}^m g(g(\nabla(\ln f), e_i)e_i, J\nabla(\ln f)) \right] g(\bar{e}_r, \bar{e}_a)g(\bar{e}_r, T\bar{e}_a) = 0, \end{aligned}$$

by using (3.7) we obtain

$$\|h\|^2 = \sum_{a,r,s=1}^{2p} g(h(\bar{e}_r, \bar{e}_s), \csc \theta F \bar{e}_a)^2 + 2n\|\nabla \ln f\|^2[\csc^2 \theta + \cot^2 \theta].$$

Thus we obtain the inequality (5.9). If the equality sign of (5.9) holds, we have

$$\sum_{a=1}^{2p} \sum_{r,s=1}^{2p} g(h(\bar{e}_r, \bar{e}_s), \csc \theta F \bar{e}_a)^2 = 0. \tag{5.10}$$

Since M_T is totally geodesic in M , (5.7) implies that M_T is totally geodesic in \bar{M} . On the other hand, (5.10) implies that h vanishes on D^θ . Since D^θ is a spherical distribution in M , it follows that M_θ is a totally umbilical submanifold of \bar{M} . Moreover, from (5.7) and (5.10) it follows that M is minimal in \bar{M} . \square

Remark 5.8. It is well known that the semi-slant submanifolds were introduced as a generalization of proper slant and proper CR-submanifolds. From Theorem 3.1 and Theorem 3.2 of [15], it follows that the semi-slant submanifolds in the sense of N. Papaghiuc are not useful to generalize the CR-warped products. But, from Example 5.1, one can conclude that non-trivial warped product pointwise semi-slant submanifolds of the form $M_T \times_f M_\theta$ are a generalization of CR-warped products in Kähler manifolds.

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