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Rigidity index preservation of regular holonomic $\mathcal{D}$-modules under Fourier transform

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Abstract. This paper gives a purely algebraic proof that the Fourier transform preserves the rigidity index of irreducible regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-modules.

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1. Introduction

Riemann showed in 1857 that the local system defined by the multiform solutions of the hypergeometric equation can be reconstructed, up to isomorphism, from the knowledge of the local monodromies around its singular points 0, 1 and ∞. In modern terminology, a local system on a projective smooth connected curve X over $\mathbb{C}$ minus a nonempty finite subset Σ of X is called (physically) *rigid*, if it is determined, up to isomorphism, by the local monodromies around each point of Σ. N. Katz gave necessary and sufficient conditions for the rigidity of local systems, when X is the Riemann sphere, based on a cohomological invariant, called the *rigidity index* (see [9]). He showed in addition that in characteristic $p > 0$, this index is preserved under Fourier transform when the local system is a perverse sheaf such that neither its support nor the support of its Fourier transform is punctual (cf. [9] Theorem 3.0.2). Moreover, he conjectured that *"it should be true that Fourier transform preserves the index of rigidity in the $\mathcal{D}$-module context"* (op. cit. p. 10). This conjecture was proved some years later by S. Bloch and H. Esnault in [2]. A different and purely algebraic proof of this result is given in this paper in the case of irreducible regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-modules.

The paper is organized as follows. Section 1 reviews some results on rigidity. Section 2 recalls the notion of minimal extension for holonomic $\mathcal{D}_{\mathbb{P}^1}$-modules and introduces the analogue notion of rigidity in the context of holonomic $\mathcal{D}_{\mathbb{P}^1}$-modules. Section 3 recalls the definition of Fourier transform and then computes the rigidity index of the Fourier transform, for irreducible regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-modules. Section 4 recalls a dictionary between germs of holonomic $\mathcal{D}$-modules and pairs of vector spaces (quivers). The last section is devoted to the proof of the main result (Theorem 5.1). The appendix contains a detailed proof of Theorem A.1, which have been postponed there for ease of readability. This new result necessary for the proof of Theorem 5.1 relates the monodromy of regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\infty]$-modules at 0 and the monodromy of its Fourier transform at ∞. General references for this paper are [3], [4], [11], [13], [16], [18], [19], [23].

2. Rigidity index and minimal extensions

Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on a Riemann surface X and let $\Sigma = \{x_0, \dots, x_n\} \subset X$ be a finite set. The following notation will be used throughout the paper.

- $\mathcal{M}^* \doteq \mathcal{H}om_{\mathcal{D}_X}\big(\omega_X, \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)\big)$ denotes the *dual* of $\mathcal{M}$;
- $\mathrm{H}_{[\Sigma]}(\mathcal{M})$ denotes the *torsion* submodule of $\mathcal{M}$ supported on Σ (see [20], §1).

Definition 2.1. One says that a holonomic $\mathcal{D}$-module $\mathcal{N}$ on X is a *minimal extension* of $\mathcal{M}$ along Σ and denote it $\mathcal{M}_{\min}$, if:

i) $\mathcal{O}_{\mathrm{X}}[*\Sigma] \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{M} = \mathcal{O}_{\mathrm{X}}[*\Sigma] \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{N}$,

ii) $\mathcal{N}$ has neither nonzero submodules nor nonzero quotients with support on a subset of Σ.

Proposition 2.2 ([16], Theorem 2.7.6 and Theorem 2.7.3). *A minimal extension of $\mathcal{M}$ along Σ exists and is given by*

$$\mathcal{M}_{\min} = \big((\mathcal{M}/\mathrm{H}_{[\Sigma]}(\mathcal{M}))^*/\mathrm{H}_{[\Sigma]}\big((\mathcal{M}/\mathrm{H}_{[\Sigma]}(\mathcal{M}))^*\big)\big)^*.$$

Moreover, the minimal extension along Σ is unique up to isomorphism.

Corollary 2.3. *The $\mathcal{D}$-modules $\mathcal{M}$ and $\mathcal{M}[*\Sigma]$ have the same minimal extension along Σ, i.e., $\mathcal{M}_{\min} = \mathcal{M}[*\Sigma]_{\min}$.*

Taking into account that for each holonomic $\mathcal{D}$-module $\mathcal{M}$ on a Riemann surface X with singularities on Σ one has $\mathrm{supp}(\mathcal{M}) \subset \Sigma$, iff $\mathcal{M} = \mathrm{H}_{[\Sigma]}(\mathcal{M})$ (cf. [16], Lemma 2.7.8), the notion of minimal extension can also be defined for germs of $\mathcal{D}_x \doteq \mathbb{C}\{x\}\langle\partial_x\rangle$ (resp. $\hat{\mathcal{D}}_x \doteq \mathbb{C}[\![x]\!]\langle\partial_x\rangle$) -modules in the following way.

Definition 2.4. Let $\mathcal{M}, \mathcal{N}$ be holonomic $\mathcal{D}_x$ (resp. $\hat{\mathcal{D}}_x$) -modules. One says that $\mathcal{N}$ is a *minimal extension* of $\mathcal{M}$ and denote it $\mathcal{M}_{\min}$ if:

i) $\mathcal{M}[x^{-1}] = \mathcal{N}[x^{-1}]$,

ii) $\mathcal{N}$ has neither submodules nor quotients isomorphic to $\mathcal{D}_x/\mathcal{D}_x \cdot x^k$ (resp. $\hat{\mathcal{D}}_x/\hat{\mathcal{D}}_x \cdot x^k$) for some $k \in \mathbb{N}^+$.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_x$-module and let $\hat{\mathcal{M}} \doteq \mathbb{C}[\![x]\!] \otimes_{\mathbb{C}\{x\}} \mathcal{M}$ denote its *formalized*.

Proposition 2.5 ([16], Theorem 2.7.11). *The minimal extension commutes with the formalized, that is, if $\mathcal{M}$ is a holonomic $\mathcal{D}_x$-module, then $\widehat{\mathcal{M}_{\min}} \simeq (\hat{\mathcal{M}})_{\min}$.*

Let us now recall a result of N. Katz [9] leading to the definition of rigidity index of holonomic $\mathcal{D}$-modules. Set $U = \mathbb{P}^1 \backslash \Sigma$. Set $n = \operatorname{rank}(\mathcal{L})$ and set $k = \sharp\Sigma$. Let $j : U \hookrightarrow \mathbb{P}^1$ be the open inclusion. Given a local system $\mathcal{L}$ on U denote by $\rho : \pi_1(U, x) \to \operatorname{End}_{\mathbb{C}}(\mathcal{L}_x)$ its monodromy representation. Set $\mathrm{T}_{x_i} = \rho(\gamma_i)$, where γ_i is a small loop in U around x_i. Let $\mathrm{Z}(\mathrm{T}_{x_i}) \doteq \{A \in \operatorname{End}_{\mathbb{C}}(\mathcal{L}_{s_i}) \mid \mathrm{AT}_{x_i} = \mathrm{T}_{x_i}\mathrm{A}\}$ be the commuting algebra of T_{x_i}.

Definition 2.6. $\mathcal{L}$ is irreducible, if $\mathcal{L}$ has no non-trivial submodules.

Definition 2.7 (Rigidity index for local systems). The Euler–Poincaré characteristic $\chi\big((\mathbb{P}^1)^{\mathrm{an}}, j_* \mathcal{E}nd(\mathcal{L})\big)$ equals $(2-k)n^2 + \sum_i \dim \mathrm{Z}(\mathrm{T}_{x_i})$ and is called the *rigidity index* of $\mathcal{L}$.

Theorem 2.8 ([9], Theorem 1.1.2). *If $\mathcal{L}$ is irreducible, $\mathcal{L}$ is (physically) rigid if and only if $\chi\big((\mathbb{P}^1)^{\mathrm{an}}, j_* \mathcal{E}nd(\mathcal{L})\big) = 2$.*

It is well known that if $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module with singularities on Σ, the local system $\mathcal{L} \doteq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})_{|X\backslash\Sigma}$ satisfies the following (cf. [20], §1, pp. 1265–1266), where DR( ) denotes the de Rham complex,

$$\mathrm{DR}\big((\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M}[*\Sigma]))_{\min}\big) = j_* \mathcal{E}nd(\mathcal{L}).$$

Definition 2.9. $\mathcal{M}$ is irreducible, if $\mathcal{M}$ has no non-trivial submodules.

These facts motivate the following definition.

Definition 2.10 (Rigidity index for holonomic $\mathcal{D}_{\mathbb{P}^1}$-modules). Let $\mathcal{M}$ be an irreducible holonomic $\mathcal{D}_{\mathbb{P}^1}$-module and let $\boldsymbol{\Sigma}$ be the set of its singular points. One calls *rigidity index* of $\mathcal{M}$ to the invariant,

$$\operatorname{rig}(\mathcal{M}) \doteq \chi\big(\mathbb{P}^1, \mathrm{DR}\big((\mathcal{E}nd_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{M}[*\Sigma]))_{\min}\big)\big).$$

3. Rigidity index and Fourier transform

Let $A_1 \doteq \mathbb{C}[t]\langle\partial_t\rangle$ denote the *Weyl algebra* in the variable t and consider the isomorphism of Weyl algebras,

$$F : \mathbb{C}[t]\langle\partial_t\rangle \to \mathbb{C}[\tau']\langle\partial_{\tau'}\rangle, \qquad t \mapsto -\partial_{\tau'},\, \partial_t \mapsto \tau'.$$

This isomorphism induces on every $\mathbb{C}[t]\langle\partial_t\rangle$-module M a structure of $\mathbb{C}[\tau']\langle\partial_{\tau'}\rangle$-module denoted ${}^{\mathrm{F}}\mathrm{M}$ and is called the *Fourier transform* of M.

GAGA provides an equivalence between the categories of holonomic A_1-modules, holonomic algebraic $\mathcal{D}^{\mathrm{alg}}_{\mathbb{P}^1}[*\{\infty\}]$-modules and holonomic analytic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}] \doteq \mathcal{D}^{\mathrm{ana}}_{\mathbb{P}^1}[*\{\infty\}]$-modules (see [14], chap. I, §4). This is done the following way. Let M be a holonomic A_1-module with singularities on $\mathrm{S} = \{x_1, \ldots, x_k\} \subset \mathbb{C}$, where $k \geq 1$. Denote by $\mathcal{M}^{\mathrm{alg}}$ the $\mathcal{D}^{\mathrm{alg}}_{\mathbb{P}^1}[*\{\infty\}]$-module $\mathcal{O}^{\mathrm{alg}}_{\mathbb{P}^1}[*\{\infty\}] \otimes_{\mathbb{C}[t]} \mathrm{M}$, with $\mathbb{P}^1$ covered by charts with local coordinates t and t' and transition map $t = t'^{-1}$ on their intersection. (Likewise, let τ' be the coordinate at 0, τ the coordinate at ∞ and $\tau' = \tau^{-1}$ the transition map.) This way $\mathcal{M}^{\mathrm{alg}}(\mathbb{C}) = \mathrm{M}$. For this reason, ${}^{\mathcal{F}}\mathcal{M}$ denotes the analytic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-module associated to ${}^{\mathrm{F}}\mathrm{M}$.

Proposition 3.1 ([22], Proposition V, 2.2). *Let* M *be an holonomic A_1-module. If* M *has a regular singularity at infinity, then its Fourier transform* ${}^{\mathrm{F}}\mathrm{M}$ *has singularities only at $\tau' = 0$ and $\tau' = \infty$. The former is regular and the later is possibly irregular.*

Lemma 3.2. *If* M *is a regular holonomic A_1-module* (*including at* ∞)*, the Newton polygon of the irregular part of ${}^{\mathcal{F}}\mathcal{M}$ at ∞ is either null or has slope* 1.

Proof. It is a consequence of [14], Proposition V, 1.2. □

Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\Sigma]$-module with singularities on $\Sigma \doteq \{\infty\} \cup \mathrm{S}$. Set $x_0 \doteq \infty$. Let $(\hat{\mathcal{N}}, \hat{\nabla})$ denote the formalized at ∞ of the meromorphic connexion determined by ${}^{\mathcal{F}}\mathcal{M}[*\{0, \infty\}]$ (cf. [18], Theorem 4.3.2). Let

$$(\hat{\mathcal{N}}, \hat{\nabla}) \simeq \bigoplus_{i=1}^{k} \hat{\mathcal{E}}^{\varphi_i} \otimes (\hat{\mathrm{R}}_i, \hat{\nabla}_i), \tag{1}$$

be the Turrittin decomposition of $(\hat{\mathcal{N}}, \hat{\nabla})$ with $(\hat{\mathrm{R}}_i, \hat{\nabla}_i)$ regular meromorphic connexions (cf. Turrittin [24], Levelt [10], and [16], Theorem 1.9.5 and Lemma 1.9.6). Let $\hat{\mathrm{T}}_i$ denote the monodromy of $(\hat{\mathrm{R}}_i, \hat{\nabla}_i)$ and let n_i be the dimension of $\hat{\mathrm{R}}_i$, $i = 1, \ldots, k$. Let φ_i denote $-x_i/x$. Let $\hat{\mathrm{T}}_0$ denote the monodromy at 0 of the local system $\mathcal{H}om_{\mathcal{D}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}, {}^{\mathcal{F}}\mathcal{M})|_{\mathbb{C}^*}$.[1] Let $i(\ )$ denote the Malgrange–Komatsu irregularity of a $\mathcal{D}_x$-module (see [18], chap. II, §1.3).

Corollary 3.3. *The irregularity of the term $\hat{\mathcal{E}}^{\varphi_i} \otimes (\hat{\mathrm{R}}_i, \hat{\nabla}_i)$ of the decomposition of Turrittin* (1) *equals* 0 *if $\varphi_i = 0$ and equals n_i otherwise.*

Proof. The result follows from Lemma 3.2, because the irregularity can be read off from the Newton polygon of the meromorphic connexion $(\hat{\mathcal{N}}, \hat{\nabla})$ (cf. [18], fig. 8, pp. 53–54). □

Theorem 3.4. *If $\mathcal{M}$ is a regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-module with singularities on $\Sigma \subset \mathbb{P}^1$, the rigidity index of ${}^{\mathcal{F}}\mathcal{M}$ is given by*

$$\operatorname{rig}({}^{\mathcal{F}}\mathcal{M}) = \dim \mathrm{Z}(\hat{\mathrm{T}}_0) + \sum_{i=1}^{k} \dim \mathrm{Z}(\hat{\mathrm{T}}_i) + \sum_{i=1}^{k} n_i^2 - \Big(\sum_{i=1}^{k} n_i\Big)^2. \tag{2}$$

In order to prove theorem above, let us first prove the following auxiliary result.

Lemma 3.5. *Let $(\hat{\mathcal{N}}, \hat{\nabla})$ be the formalized at ∞ of the meromorphic connexion associated to the Fourier transform of a regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-module with singularities on Σ. Assume moreover that $\varphi_1 = 0$ on the Turrittin decomposition of $(\hat{\mathcal{N}}, \hat{\nabla})$ ($(\hat{\mathrm{R}}_1, \hat{\nabla}_1)$ might be* 0*). The following holds:*

i) $\chi(\hat{\mathcal{N}}_{\min}, \mathbb{C}[\![\tau]\!]) = \dim\{\mathrm{e} \mid \hat{\mathrm{T}}_1 \mathrm{e} = \mathrm{e}\}$.

ii) $\chi\big(\mathcal{E}nd_{\hat{\mathcal{O}}_\tau}(\hat{\mathcal{N}})_{\min}, \mathbb{C}[\![\tau]\!]\big) = \sum_{i=1}^{k} \dim \mathrm{Z}(\hat{\mathrm{T}}_i)$.

iii) $i\big(\mathcal{E}nd_{\mathcal{O}_\tau}(\mathcal{N})\big) = (\sum_{i=1}^{k} n_i)^2 - \sum_{i=1}^{k} n_i^2$.

Proof. i) For each term of decomposition (1), either $i = 1$ or $i > 1$. If $i > 1$, the holonomic $\hat{\mathcal{D}}_\tau$-module $\hat{\mathcal{N}}_i \doteq \hat{\mathcal{E}}^{\varphi_i} \otimes (\hat{\mathrm{R}}_i, \hat{\nabla}_i)$ has no regular component. Therefore, $\hat{\mathcal{N}}_i \simeq \hat{\mathcal{N}}_i[\tau^{-1}]$ (cf. [18], Theorem 6.3.1) and the multiplication by τ is bijective. Hence $\mathrm{H}_{[0]}(\hat{\mathcal{N}}_i) = 0$ and $\mathrm{H}_{[0]}(\hat{\mathcal{N}}_i^*) = 0$. Thus, by Proposition 2.2, $(\hat{\mathcal{N}}_i)_{\min} = \hat{\mathcal{N}}_i[\tau^{-1}]$. Thanks to these isomorphisms, one has $\chi(\hat{\mathcal{N}}_{\min}, \mathbb{C}[\![\tau]\!]) = \chi\big((\hat{\mathcal{N}}_1)_{\min}, \mathbb{C}[\![\tau]\!]\big)$, because $\chi(\hat{\mathcal{N}}_i[\tau^{-1}], \mathbb{C}[\![\tau]\!]) = 0$, for $i > 1$. If $i = 1$, possibly after a change of basis and coordinate system, one has the isomorphism,

$$(\hat{\mathrm{R}}_1, \hat{\nabla}_1) \simeq \left(\mathbb{C}[\![\tau]\!]^n, \tau\frac{d}{d\tau} - \mathrm{A}_1\right), \qquad n = \dim \hat{\mathrm{R}}_1.$$

[1]The notation ^ is also traditionally used to denote the Fourier transform. Here it is only and exclusively used on $\hat{\mathrm{T}}_0$ to stress the fact that this is a monodromy on the Fourier transform.

By Proposition 2.3, both $\hat{\mathcal{N}}_1$ and $\hat{\mathcal{N}}_1[\tau^{-1}]$ have the same minimal extension. Hence, one may compute the minimal extension of $\hat{\mathcal{N}}_1[\tau^{-1}]$ instead of $\hat{\mathcal{N}}_1$. There is a meromorphic change of bases, which transforms A_1 into a constant matrix in the Jordan canonical form, say $\mathrm{J} = \mathrm{J}_1 \oplus \cdots \oplus \mathrm{J}_m$, with $n_1 + \cdots + n_m = n$. Thus,

$$(\hat{\mathrm{R}}_1[\tau^{-1}], \hat{\nabla}_1) \simeq \bigoplus_{j=1}^{m} \left(\mathbb{C}[\![\tau]\!]^{n_j}, \tau \frac{d}{d\tau} - \mathrm{J}_j \right).$$

If α_j is the eigenvalue of the Jordan block J_j, one has the isomorphism,

$$\hat{\mathcal{N}}_1[\tau^{-1}] \simeq \bigoplus_{j=1}^{m} \mathbb{C}[\![\tau]\!][\tau^{-1}]\langle \partial_\tau \rangle / \mathbb{C}[\![\tau]\!][\tau^{-1}]\langle \partial_\tau \rangle.(\tau\partial_\tau - \alpha_j)^{n_j}.$$

If $\alpha_j \notin \mathbb{Z}$, the Bernstein polynomial $b(\tau)$ of $\hat{\mathcal{N}}_{1j} \doteq \mathbb{C}[\![\tau]\!]\langle \partial_\tau \rangle / (\tau\partial_\tau - \alpha_j)^{n_j}$ equals $(\tau - \alpha_j)^{n_j}$. Therefore, for each $k \in \mathbb{N}$, $b(k) \neq 0$. Hence, the multiplication by τ

$$\mathbb{C}[\![\tau]\!]\langle \partial_\tau \rangle / (\tau\partial_\tau - \alpha_j)^{n_j} \xrightarrow{\tau.} \mathbb{C}[\![\tau]\!]\langle \partial_\tau \rangle / (\tau\partial_\tau - \alpha_j)^{n_j}$$

is a bijective map, i.e., $\hat{\mathcal{N}}_{1j}$ is a meromorphic connexion (cf. [18], Lemma 4.2.7). In particular, one has $\mathrm{H}_{[0]}(\hat{\mathcal{N}}_{1j}) = 0$ and $\mathrm{H}_{[0]}\big((\hat{\mathcal{N}}_{1j})^*\big) = 0$. Thus $(\hat{\mathcal{N}}_{1j})_{\min} = \hat{\mathcal{N}}_{1j}$. Since $\chi(\hat{\mathcal{N}}_{1j}[\tau^{-1}], \mathbb{C}[\![\tau]\!]) = 0$ for each $\alpha_j \notin \mathbb{Z}$,

$$\chi(\hat{\mathcal{N}}_{\min}, \mathbb{C}[\![\tau]\!]) = \sum_{\alpha_j \in \mathbb{Z}} \chi\big((\hat{\mathcal{N}}_{1j})_{\min}, \mathbb{C}[\![\tau]\!]\big).$$

If $\alpha_j \in \mathbb{Z}$, one can assume $\alpha_j = 0$. In fact after the change of basis $\mathrm{B} = \tau^{-\alpha_j}\mathbb{1}$, the matrix J_j is transformed into

$$\mathrm{B}\mathrm{J}_j\mathrm{B}^{-1} + \tau\partial\mathrm{B}/\partial\tau\mathrm{B}^{-1} = \mathrm{J}_j - \alpha_j\mathbb{1}.$$

In this case, $\hat{\mathcal{N}}_j = \mathbb{C}[\![\tau]\!]\langle \partial_\tau \rangle / (\tau\partial_\tau)^{n_j}$ and therefore $(\hat{\mathcal{N}}_j)_{\min} = \mathbb{C}[\![\tau]\!]\langle \partial_\tau \rangle / R_j$, where $R_j = \partial_\tau(\tau\partial_\tau)^{n_j - 1}$. As $\{1, \log\tau, \ldots, \log^{n_j - 1}\tau\}$ is a fundamental system of solutions the differential equation $R_j y = 0$, the kernel of R_j in $\mathbb{C}[\![\tau]\!]$ is $\langle 1 \rangle$. Hence, $\dim\ker R_j = 1$. Given that $(l+1)^{-n_j}\tau^{l+1}$ is a solution of the differential equation $R_j y = \tau^l$, $\dim\operatorname{coker} R_j = 0$. In brief $\chi(\hat{\mathcal{N}}_{\min}, \mathbb{C}[\![\tau]\!]) = \sharp\{\mathrm{J}_j \mid \alpha_j \in \mathbb{Z}\}$, so $\chi(\hat{\mathcal{N}}_{\min}, \mathbb{C}[\![\tau]\!]) = \dim\{\mathrm{e} \mid \hat{\mathrm{T}}_1\mathrm{e} = \mathrm{e}\}$.

ii) The Turrittin decomposition (1), $(\hat{\mathcal{N}}, \hat{\nabla}) \simeq \bigoplus_{i=1}^{k} \hat{\mathcal{E}}^{\varphi_i} \otimes \hat{\mathrm{R}}_i$, implies that

$$
\begin{aligned}
&\Big(\mathrm{Hom}_{\hat{\mathcal{O}}_\tau}\Big(\bigoplus_{i=1}^{k}\hat{\mathcal{E}}^{\varphi_i}\otimes\hat{\mathrm{R}}_i,\bigoplus_{i=1}^{k}\hat{\mathcal{E}}^{\varphi_i}\otimes\hat{\mathrm{R}}_i\Big),\hat{\nabla}\Big)\\
&\quad\simeq\bigoplus_{i=1}^{k}\bigoplus_{j=1}^{k}\big(\mathrm{Hom}_{\hat{\mathcal{O}}_\tau}(\hat{\mathcal{E}}^{\varphi_i}\otimes\hat{\mathrm{R}}_i,\hat{\mathcal{E}}^{\varphi_j}\otimes\hat{\mathrm{R}}_j),\hat{\nabla}\big)\\
&\quad\simeq\bigoplus_{i=1}^{k}\bigoplus_{j=1}^{k}\big(\mathrm{Hom}_{\hat{\mathcal{O}}_\tau}(\hat{\mathrm{R}}_i,\hat{\mathcal{E}}^{\varphi_j-\varphi_i}\otimes\hat{\mathrm{R}}_j),\hat{\nabla}\big)\\
&\quad\simeq\bigoplus_{i=1}^{k}\bigoplus_{j=1}^{k}\big(\mathrm{Hom}_{\hat{\mathcal{O}}_\tau}(\hat{\mathrm{R}}_i,\hat{\mathrm{R}}_j)\otimes\hat{\mathcal{E}}^{\varphi_j-\varphi_i},\hat{\nabla}\big).
\end{aligned}\tag{3}
$$

Hence by i),

$$
\begin{aligned}
\chi\big(\mathrm{End}(\hat{\mathcal{N}})_{\min},\mathbb{C}[\![\tau]\!]\big)&=\dim\{\mathrm{M}=\mathrm{M}_1\oplus\cdots\oplus\mathrm{M}_k\mid\mathrm{ad}_{\hat{\mathrm{T}}_1}\oplus\cdots\oplus\mathrm{ad}_{\hat{\mathrm{T}}_k}\mathrm{M}=\mathrm{M}\}\\
&=\sum_{i=1}^{k}\dim\mathrm{Z}(\hat{\mathrm{T}}_i).
\end{aligned}
$$

iii) Each term $(\hat{\mathrm{R}}_i,\hat{\nabla}_i)$ of (1) is a regular meromorphic connexion. By Corollary 3.3 the irregularity of $\hat{\mathcal{E}}^{\varphi_i}\otimes(\hat{\mathrm{R}}_i,\hat{\nabla}_i)$ equals 0, if $i=1$ and equals $n_i=\mathrm{rank}(\hat{\mathrm{R}}_i)=\mathrm{rank}(\hat{\mathrm{T}}_i)$ otherwise. Applying the same reasoning to the Turrittin decomposition of the endomorphisms of $\hat{\mathcal{N}}$, i.e., to the identity (3), one has,

$$
i\big(\mathcal{E}nd_{\mathcal{O}_\tau}(\mathcal{N})_{\min}\big)=\sum_{\substack{i,j=1\\ i\neq j}}^{k}\mathrm{rank}\big(\mathrm{Hom}_{\hat{\mathcal{O}}_\tau}(\hat{\mathrm{R}}_i,\hat{\mathrm{R}}_j)\big)=\Big(\sum_{i=1}^{k}n_i\Big)^2-\sum_{i=1}^{k}n_i^2. \qquad\square
$$

Proof of Theorem 3.4. This proof relies on triangulated categories. For an overview on this subject, see [8], chap. 1, §5. Let $\mathcal{E}$ be the $\mathcal{D}_{\mathbb{P}^1}$-module $\mathcal{E}nd_{\mathcal{O}_{\mathbb{P}^1}}({}^{\mathcal{F}}\mathcal{M}[*\{0,\infty\}])$. Let $\mathcal{F}^\bullet$ be the de Rham complex $\mathrm{DR}(\mathcal{E}_{\min})$ on $\mathbb{P}^1$ and let $j:\mathbb{C}^*\hookrightarrow\mathbb{P}^1$ be the open inclusion. Let $\eta: j_!j^{-1}\mathcal{F}^\bullet\to\mathcal{F}^\bullet$ the open inclusion. By taking the mapping cone of η, denoted by $C(\eta)$, the morphism η fits in the distinguished triangle below

$$
j_!j^{-1}\mathcal{F}^\bullet\xrightarrow{\eta}\mathcal{F}^\bullet\to C(\eta)\to,\tag{4}
$$

which yields the identity, where $\chi(\ )$ denotes the Euler–Poincaré index (see [8], Exercise I.32),

$$
\chi(\mathcal{F}^\bullet)=\chi(j_!j^{-1}\mathcal{F}^\bullet)+\chi\big(C(\eta)\big).\tag{5}
$$

By Proposition 3.1 $\{0, \infty\}$ are the only singularities of $\mathcal{F}^\bullet$ and therefore $\mathbb{P}^1\backslash\{0, \infty\}$ is homotopic to $\mathbb{S}^1$. Setting $\mathcal{L} \doteq h^0(j^{-1}\mathcal{F}^\bullet)$, one gets

$$\chi(j_! j^{-1}\mathcal{F}^\bullet) = (1-1)\operatorname{rank}\mathcal{L} = 0 \quad \text{and} \quad \chi\big(C(\eta)\big) = \chi\big(C(\eta)_0\big) + \chi\big(C(\eta)_\infty\big),$$

because $(1, 1)$ are the Betti numbers of $\mathbb{S}^1$. Thus,

$$\chi(\mathcal{F}^\bullet) = \chi\big(C(\eta)_0\big) + \chi\big(C(\eta)_\infty\big). \tag{6}$$

To compute $\chi\big(C(\eta)_0\big)$, take a disk $\mathrm{D} \subset \mathbb{C}$ centered at 0, the inclusion $i : \mathrm{D}^* \hookrightarrow \mathrm{D}$ and set $\mathcal{G}^\bullet \doteq \mathrm{DR}(\mathcal{E}_{\min}|_\mathrm{D})$. By Proposition 3.1, $\mathcal{E}_{\min}|_\mathrm{D}$ is a regular holonomic $\mathcal{D}_\mathrm{D}$-module. Therefore, $\mathcal{G}^\bullet = i_* \,\mathcal{E}nd(\mathcal{L}')$ with $\mathcal{L}' = \mathcal{H}om_{\mathcal{D}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}, {}^{\mathcal{F}}\!\mathcal{M})|_{\mathrm{D}^*}$. Since $\mathcal{G}^\bullet$ is a perverse sheaf on D, it gives rise to the short exact sequence

$$0 \longrightarrow i_! i^{-1}\mathcal{G}^\bullet \xrightarrow{\eta|_\mathrm{D}} \mathcal{G}^\bullet \longrightarrow (\operatorname{coker}\eta|_\mathrm{D}) \longrightarrow 0. \tag{7}$$

Thus,

$$\chi(\mathcal{G}^\bullet) = \chi(i_! i^{-1}\mathcal{G}^\bullet) + \chi(\operatorname{coker}\eta|_\mathrm{D}).$$

Since $h^0(i^{-1}\mathcal{G}^\bullet) = \mathcal{E}nd(\mathcal{L}')$, $\chi(i_! i^{-1}\mathcal{G}^\bullet) = (1-1)\operatorname{rank}\mathcal{E}nd(\mathcal{L}') = 0$, because D^* is homotopic to $\mathbb{S}^1$. Therefore $\chi(\mathcal{G}^\bullet) = \chi\big((\operatorname{coker}\eta|_\mathrm{D})_0\big)$. The exact sequence (7) implies that $\mathcal{G}^\bullet_0 = (\operatorname{coker}\eta|_\mathrm{D})_0$. Moreover, $\mathcal{G}^\bullet_0 = \{\mathrm{M} \in \mathrm{End}(\mathrm{E}) \mid \mathrm{T}_{\mathcal{E}nd(\mathcal{L}')}(\mathrm{M}) = \mathrm{M}\}$, where $\mathrm{E} = h^0(\mathrm{D}\backslash\mathbb{R}^+, \mathcal{L}')$, $\mathrm{T}_{\mathcal{E}nd(\mathcal{L}')} = \mathrm{ad}_{\hat{\mathrm{T}}_0}$ and $\hat{\mathrm{T}}_0$ is the monodromy of $\mathcal{L}'$ at 0. Thus $\mathcal{G}^\bullet_0 = \{\mathrm{M} \in \mathrm{End}(\mathrm{E}) \mid \hat{\mathrm{T}}_0\mathrm{M} = \mathrm{M}\hat{\mathrm{T}}_0\} \doteq \mathrm{Z}(\hat{\mathrm{T}}_0)$. By Mayer-Vietoris, one shows that $\chi(\operatorname{coker}\eta_0) = \chi(\operatorname{coker}\eta|_\mathrm{D})$. On the other hand, the restriction of the distinguished triangle (4) to D yields the identity $\chi(\mathcal{G}^\bullet) = \chi\big(C(\eta)_0\big)$. Hence,

$$\chi\big(C(\eta)_0\big) = \dim \mathrm{Z}(\hat{\mathrm{T}}_0). \tag{8}$$

By the identity (5), one has $\chi(\mathcal{F}^\bullet_\infty) = \chi(j_! j^{-1}\mathcal{F}^\bullet_\infty) + \chi\big(C(\eta)_\infty\big)$. Without loss of generality ∞ is purely irregular, as the regular part reduces to the previous case. Since $(j_! j^{-1}\mathcal{F}^\bullet)_\infty = 0$, $\chi(\mathcal{F}^\bullet_\infty) = \chi\big(C(\eta)_\infty\big)$. On the other hand, $\chi(\mathcal{F}^\bullet_\infty) = \chi\big((\mathcal{E}_{\min})_\infty, \mathbb{C}\{\tau\}\big)$. Therefore by definition of irregularity

$$\chi(\mathcal{F}^\bullet_\infty) = \chi\big((\widehat{\mathcal{E}_{\min}})_\infty, \mathbb{C}[\![\tau]\!]\big) - i\big((\mathcal{E}_{\min})_\infty\big). \tag{9}$$

Owing to Lemma 3.5 and Proposition 2.5

$$\chi\big((\widehat{\mathcal{E}_{\min}})_\infty, \mathbb{C}[\![\tau]\!]\big) = \sum_{i=1}^{k} \dim \mathrm{Z}(\hat{\mathrm{T}}_i). \tag{10}$$

Furthermore, by Lemma 3.5

$$i\big((\mathcal{E}_{\min})_\infty\big) = \Big(\sum_{i=1}^{k} n_i\Big)^2 - \sum_{i=1}^{k} n_i^2. \tag{11}$$

The identity (2) is now an immediate consequence of (6), (8), (9), (10) and (11).

□

4. An equivalence of categories

The previous two sections showed that both the rigidity index of a regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-module as well as the rigidity index of its Fourier transform can be expressed in terms of the monodromy at its singular points (cf. Theorems 2.8 and 3.4). With this in mind, the category $\mathcal{C}$ of quivers consisting of pairs of vector spaces linked by morphisms seems the natural choice for a dictionary, because the monodromy figures there prominently. For details on the category $\mathcal{C}$, refer to [18], chap. I, ¶6.2.3, or [3], [23]. Furthermore, not only the category $\mathcal{C}$ is equivalent to the category of regular holonomic $\mathcal{D}_t$-modules, but also dually equivalent to the category of germs of complexes of perverse sheaves. For the equivalence, refer to [18], chap. I, §6, and Theorem I 6.2.5. For the duality, refer to [5], §II and Theorem II.2.3.

Theorem 4.1 ([16], Proposition 2.4.5). *For every complex of perverse sheaves $\mathcal{F}^\bullet$ on a disk D centered at* 0 *one has the following quivers in the category $\mathcal{C}$, where*

i)	$j_!(\mathcal{L})$	ii)	$j_*(\mathcal{L})$	iii)	$Rj_*(\mathcal{L})$
	$\mathrm{E} \underset{\mathrm{T}-\mathbb{1}_\mathrm{E}}{\overset{\mathbb{1}_\mathrm{E}}{\rightleftarrows}} \mathrm{E}$		$\mathrm{E} \underset{\mathrm{inc}}{\overset{\mathrm{T}-\mathbb{1}_\mathrm{E}}{\rightleftarrows}} \mathrm{F}$, *where* $\mathrm{F} \subset \mathrm{E}$		$\mathrm{E} \underset{\mathbb{1}_\mathrm{E}}{\overset{\mathrm{T}-\mathbb{1}_\mathrm{E}}{\rightleftarrows}} \mathrm{E}$

$j : D^* \hookrightarrow D$ is the inclusion map, $\mathrm{E} \doteq h^0\big(\mathcal{F}^\bullet(D\backslash\mathbb{R}^+)\big)$ and T is the monodromy of the local system $\mathcal{L} \doteq h^0(\mathcal{F}^\bullet|_{D^*})$.

A distinct proof of Theorem 4.1 with a slightly change of notation can also be found in [17], Proposition 4.7.

Proposition 4.2. *The quiver in statement* iii) *of theorem above is the representative in the category $\mathcal{C}$ of the germs of localized regular holonomic $\mathcal{D}_t$-modules.*

Proof. By [18], Theorem I 6.2.5, each regular holonomic $\mathcal{D}_t$-module is represented by a quiver $u : \mathrm{E} \rightleftarrows \mathrm{F} : v$ in the category $\mathcal{C}$. Since left multiplication by t is bijective for localized holonomic $\mathcal{D}_t$-modules, v is an isomorphism (see [22], proof of Proposition V, 2.2 (2)). The result then follows by diagram chasing. □

The following theorem shows that the notion of minimal extension in the category $\mathcal{C}$ is meaningful.

Theorem 4.3 ([16], Proposition 2.7.13). *Let D be a disk centered at the origin, $j : D^* \hookrightarrow D$ the inclusion map and let $\mathcal{L}$ be a locally constant sheaf on D^*. If $\mathcal{F}^\bullet$ is a complex of perverse sheaves on D such that $\mathcal{F}^\bullet|_{D^*} = \mathcal{L}$, then:*

i) *There exists $\eta : j_*\mathcal{L} \to \mathcal{F}^\bullet$ morphism of complexes of perverse sheaves such that $\eta|_{D^*} = \mathrm{id}_{\mathcal{L}}$,*
ii) *$j_*\mathcal{L}$ has neither non trivial subobjects nor quotients with support at the origin.*

An alternative proof can be found in [7]. For i), cf. [7] Proposition 8.2.5(i). For ii), cf. [7] Proposition 8.2.7.

Definition 4.4 (Minimal extension for perverse sheaves). Let D be a disk centered at the origin and let $j : D^* \hookrightarrow D$ be the inclusion. Take $\mathcal{F}^\bullet$ a complex of perverse sheaves on D and set $\mathcal{L} = h^0(\mathcal{F}^\bullet|_{D^*})$. One calls *minimal extension* of $\mathcal{F}^\bullet$ the complex $j_*\mathcal{L}$.

Definition 4.5 (Minimal extension for quivers). Let $u : \mathrm{E} \rightleftarrows \mathrm{F} : v$ be a quiver. One defines the minimal extension of this quiver via the equivalence of categories between germs of holonomic $\mathcal{D}_x$-modules and quivers (see [18], Theorem I, 6.2.5).

Proposition 4.6. *If the quiver $u : \mathrm{E} \rightleftarrows \mathrm{F} : v$ is isomorphic to its minimal extension, then* $\dim \mathrm{Z}(v \circ u) - \dim \mathrm{Z}(u \circ v) = \big(\dim \ker(v \circ u)\big)^2$.

Proof. By diagram chasing $u : \mathrm{E} \rightleftarrows \mathrm{F} : v$ is equivalent to $\mathrm{E} \rightleftarrows \mathrm{im}(v \circ u)$ with the natural arrows. Let $\alpha_1, \dots, \alpha_k$ be the distinct eigenvalues of $v \circ u$ and let $\mathrm{E} = \mathrm{E}_1 \oplus \cdots \oplus \mathrm{E}_k$ be the decomposition of E in terms of the eigenspaces of $v \circ u$. By choosing a convenient Jordan basis $v \circ u$ is represented by $\mathrm{J} = \mathrm{J}_1 \oplus \cdots \oplus \mathrm{J}_k$, where $\mathrm{J}_1, \dots, \mathrm{J}_k$ are the Jordan matrices associated to the eigenvalues $\alpha_1, \dots, \alpha_k$ respectively. Furthermore, in such basis each J_i is block diagonal, i.e., $\mathrm{J}_i = \mathrm{J}_{i1} \oplus \cdots \oplus \mathrm{J}_{il_i}$. By [15], 6.4.7, not only $\dim \mathrm{Z}(\mathrm{J}) = \dim \mathrm{Z}(\mathrm{J}_1) + \cdots + \dim \mathrm{Z}(\mathrm{J}_k)$, but also the corresponding blocks C_{ij} of each matrix C commuting with J_i has one of the following shapes:

$$\mathrm{C}_{ij} = \begin{bmatrix} 0 & \cdots & 0 & c_1^{ij} & c_2^{ij} & c_3^{ij} & \cdots & c_{n_j}^{ij} \\ 0 & \cdots & 0 & 0 & c_1^{ij} & c_2^{ij} & \cdots & c_{n_j-1}^{ij} \\ 0 & \cdots & 0 & 0 & 0 & c_1^{ij} & \cdots & c_{n_j-2}^{ij} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & c_1^{ij} \end{bmatrix}, \quad \mathrm{C}_{ij} = \begin{bmatrix} c_1^{ij} & c_2^{ij} & c_3^{ij} & \cdots & c_{n_j}^{ij} \\ 0 & c_1^{ij} & c_2^{ij} & \cdots & c_{n_j-1}^{ij} \\ 0 & 0 & c_1^{ij} & \cdots & c_{n_j-2}^{ij} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1^{ij} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Therefore,

$$\dim \mathrm{Z}(\mathrm{J}_i) = \sum_{jk} \min\{\dim \mathrm{E}_{ij}, \dim \mathrm{E}_{ik}\}. \tag{12}$$

Let $\mathrm{F} = \mathrm{F}_1 \oplus \cdots \oplus \mathrm{F}_k$ be the corresponding Jordan decomposition associated to $u \circ v$, (notice that F_{ij} might be 0). As $v \circ u$ is onto, $\dim \mathrm{E}_{ij} = \dim \mathrm{F}_{ij}$, if $\alpha_i \neq 0$ and $\dim \mathrm{E}_{ij} = \dim \mathrm{F}_{ij} + 1$ otherwise. Let i_0 be the index for which $\alpha_{i_0} = 0$. Thus,

$$\begin{aligned} \dim \mathrm{Z}(v \circ u) - \dim \mathrm{Z}(u \circ v) &= \sum_{ijk} \min\{\dim \mathrm{E}_{ij}, \dim \mathrm{E}_{ik}\} - \min\{\dim \mathrm{F}_{ij}, \dim \mathrm{F}_{ik}\} \\ &= \sum_{i=i_0, jk} 1 \\ &= \big(\dim \ker(v \circ u)\big)^2. \end{aligned}$$

□

Let us now turn our attention to the final preparatory material for the proof of the main theorem. So far, one has already seen two distinct notations for the monodromy. On the one hand, the monodromy of a local system associated to $\mathcal{M}$ at the (regular) singular point $x_i \in \Sigma$ is denoted by T_{x_i} (cf. Theorem 2.8). On the other hand, the germ $\mathcal{M}_{x_i}$ is represented by a quiver in the category $\mathcal{C}$,

$$\mathrm{T}_{\mathrm{E}_i} - \mathbb{1} \circlearrowright \mathrm{E}_i \underset{v_i}{\overset{u_i}{\rightleftarrows}} \mathrm{F}_i \circlearrowleft \mathrm{T}_{\mathrm{F}_i} - \mathbb{1}, \tag{13}$$

but $\mathrm{T}_{x_i} = \mathrm{T}_{\mathrm{E}_i}$. Likewise for the germ ${}^{\mathcal{F}}\mathcal{M}_0$ (cf. footnote on p. 4) where $\hat{\mathrm{T}}_0 = \mathrm{T}_{\hat{\mathrm{E}}}$:

$$\mathrm{T}_{\hat{\mathrm{E}}} - \mathbb{1} \circlearrowright \hat{\mathrm{E}} \underset{v}{\overset{u}{\rightleftarrows}} \hat{\mathrm{F}} \circlearrowleft \mathrm{T}_{\hat{\mathrm{F}}} - \mathbb{1}. \tag{14}$$

The proof of the main theorem hinges on the irreducibility of the starting $\mathcal{D}$-module $\mathcal{M}$. The irreducibility of $\mathcal{M}$ implies that $\mathcal{M}_{\min} = \mathcal{M}$ (if $\mathcal{M}_{\min} \neq 0$). By Theorem 4.1 ii) and Definition 4.4, for each $x_i \in \Sigma$ the quiver (13) is equivalent to

$$\mathrm{T}_{x_i} - \mathbb{1} \circlearrowright \mathrm{E}_i \underset{\mathrm{inc}}{\overset{\mathrm{T}_{x_i}-\mathbb{1}}{\rightleftarrows}} \mathrm{F}_i \circlearrowleft \mathrm{T}_{\mathrm{F}_i} - \mathbb{1}. \tag{15}$$

Recall that $x_0 \doteq \infty$. Since $\mathcal{M}$ is localized at ∞, one has by Theorem 4.1 iii), $\mathrm{E}_\infty = \mathrm{F}_\infty$. The same holds for the germ ${}^{\mathcal{F}}\mathcal{M}_0$, because the Fourier transform preserves the irreducibility (cf. [4] Proposition 5 2.1),

$$\hat{\mathrm{T}}_0 - \mathbb{1} \circlearrowright \hat{\mathrm{E}} \underset{\mathrm{inc}}{\overset{\hat{\mathrm{T}}_0-\mathbb{1}}{\rightleftarrows}} \hat{\mathrm{F}} \circlearrowleft \mathrm{T}_{\hat{\mathrm{F}}} - \mathbb{1}. \tag{16}$$

As M is holonomic, $\mathrm{M} = A_1/I$. In particular, if one takes a division basis $(P_p, \ldots, P_q)$ of I, then $\dim \mathrm{E}_i = \deg_{\partial_x} P_p$. For a proof, see [23] Theorem I.1.1. Without loss of generality one can assume $\mathrm{E}_i = \mathrm{E}$. Thus the quivers in (15) can be rewritten as follows

$$\mathrm{T}_{x_i} - \mathbb{1} \circlearrowright \mathrm{E} \underset{\mathrm{inc}}{\overset{\mathrm{T}_{x_i}-\mathbb{1}}{\rightleftarrows}} \mathrm{F}_i \circlearrowleft \mathrm{T}_{\mathrm{F}_i} - \mathbb{1}. \tag{17}$$

There is also the information provided by the Turrittin decomposition of the Fourier transform at infinity. Let $(\hat{\mathcal{N}}, \hat{\nabla})$ be the formalized of the meromorphic connexion $(\mathcal{N}, \nabla) \doteq {}^{\mathcal{F}}\mathcal{M}[*\{\infty\}]$. Its Turrittin decomposition is

$$(\hat{\mathcal{N}}, \hat{\nabla}) \simeq \bigoplus_{i=1}^{k} \hat{\mathcal{E}}^{\varphi_i} \otimes (\hat{\mathrm{R}}_i, \hat{\nabla}_i).$$

By Proposition 4.2 iii), each meromorphic connexion $(\hat{\mathrm{R}}_i, \hat{\nabla}_i)$ is equivalent to

$$\mathrm{T}_{\hat{\mathrm{F}}_i} - \mathbb{1} \circlearrowright \hat{\mathrm{F}}_i \underset{=}{\overset{\mathrm{T}_{\hat{\mathrm{F}}_i}-\mathbb{1}}{\rightleftarrows}} \hat{\mathrm{F}}_i \circlearrowleft \mathrm{T}_{\hat{\mathrm{F}}_i} - \mathbb{1}, \tag{18}$$

with $\hat{\mathrm{T}}_i \doteq \mathrm{T}_{\hat{\mathrm{F}}_i}$ (see Theorem 3.4).

When $\mathcal{M}$ is an irreducible regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-module, the F_i's and $\mathrm{T}_{\mathrm{F}_i}$'s in (17) are "preserved" by Fourier transform in the sense of the following two lemmas.

Lemma 4.7. *If* ${}^{\mathcal{F}}\mathcal{M}_{\min} = {}^{\mathcal{F}}\mathcal{M}$, *then* $(\hat{\mathrm{F}}, \hat{\mathrm{T}}_{\hat{\mathrm{F}}})$ *is isomorphic to* $(\mathrm{E}, \mathrm{T}_\infty)$, *i.e.,* ${}^{\mathcal{F}}\mathcal{M}_0$ *is equivalent to*

$$\hat{\mathrm{T}}_0 - \mathbb{1} \circlearrowright \hat{\mathrm{E}} \underset{\mathrm{inc}}{\overset{\hat{\mathrm{T}}_0 - \mathbb{1}}{\rightleftarrows}} \hat{\mathrm{F}} \circlearrowleft \mathrm{T}_\infty - \mathbb{1}.$$

Proof. It is an easy corollary of Theorem A.1, because the monodromy is defined up to conjugation. □

Lemma 4.8. *If* $\mathcal{M}_{\min} = \mathcal{M}$, *then for each* $x_i \in \Sigma \cap \mathbb{C}$ $(\mathrm{F}_i, \mathrm{T}_{\mathrm{F}_i})$ *is isomorphic to* $(\hat{\mathrm{E}}_i, \hat{\mathrm{T}}_i)$, *i.e.,* $\mathcal{M}_{x_i}$ *is equivalent to*

$$\mathrm{T}_{x_i} - \mathbb{1} \circlearrowright \mathrm{E} \underset{\mathrm{inc}}{\overset{\mathrm{T}_{x_i} - \mathbb{1}}{\rightleftarrows}} \mathrm{F}_i \circlearrowleft \hat{\mathrm{T}}_i - \mathbb{1}.$$

Before proving Lemma 4.8, let us present some notations and then state the necessary results for its proof. Let $\mathcal{E}_{\mathbb{P}^1} \doteq \mathbb{C}[\partial_t, \partial_t^{-1}] \otimes_{\mathbb{C}[\partial_t]} \mathcal{D}_{\mathbb{P}^1}$ denote the sheaf of micro-differential operators on $\mathbb{P}^1$. Given a holonomic A_1-module M one shall denote by $\mathscr{M} \doteq \mathcal{O}_{\mathbb{P}^1}^{\mathrm{ana}} \otimes_{\mathbb{C}[t]} \mathrm{M}$ the corresponding $\mathcal{D}_{\mathbb{P}^1}^{\mathrm{ana}}$-module. Notice that $\mathcal{M}|_{\mathbb{C}} = \mathscr{M}|_{\mathbb{C}}$. Therefore $\mathcal{M}_x = \mathscr{M}_x$, for each $x \in \mathbb{C}$.

Lemma 4.9 ([22], Lemma V, 3.3). *The microlocalized module* $\mathscr{M}^\mu \doteq \mathcal{E}_{\mathbb{P}^1} \otimes_{\mathcal{D}_{\mathbb{P}^1}} \mathscr{M}$ *has support in the set* Σ *of singular points of* $\mathscr{M}$.

As a corollary of this lemma, $\Gamma(\mathbb{C}, \mathscr{M}^\mu) = \bigoplus_{c \in \Sigma \cap \mathbb{C}} \mathscr{M}_c^\mu$.

Proposition 4.10 ([22], Proposition V, 3.4). *At any singular point c of* $\mathscr{M}$, *the germ* $\mathcal{E}^{c/\tau} \otimes \mathscr{M}_c^\mu$ *is a* $(\hat{k} \doteq \mathbb{C}[\![\tau]\!][\tau^{-1}], \hat{\nabla})$*-vector bundle with regular singularities.*

Proposition 4.11 ([22], Proposition V, 3.6). *The composed* $\mathbb{C}[\![\tau]\!]$*-linear mapping*

$$\hat{\mathbb{G}} \doteq \hat{k} \otimes_{\mathbb{C}[\tau^{-1}]} {}^{\mathrm{F}}\mathrm{M} \to \Gamma(\mathbb{C}, \hat{k} \otimes_{\mathbb{C}[\partial_t]} \mathscr{M}) \to \Gamma(\mathbb{C}, \mathscr{M}^\mu)$$

is an isomorphism.

Proof of Lemma 4.8. Thanks to Lemma 4.9, for each $x_i \in \Sigma \cap \mathbb{C}$ the microlocalization morphism μ gives rise to the exact sequence of holonomic $\mathcal{D}_{x_i}$-modules

$$0 \to \ker \mu \hookrightarrow \mathcal{M}_{x_i} \xrightarrow{\mu} (\mathcal{M}_{x_i})^\mu \to \operatorname{coker} \mu \to 0. \tag{19}$$

As $\ker \mu \simeq \mathcal{D}_{x_i} / \mathcal{D}_{x_i}.(\partial_{x_i})^k$ (resp. $\operatorname{coker} \mu \simeq \mathcal{D}_{x_i} / \mathcal{D}_{x_i}.(\partial_{x_i})^{k'}$) for a given $k \in \mathbb{N}$ (resp. $k' \in \mathbb{N}$), $\ker \mu$ (resp. $\operatorname{coker} \mu$) is equivalent to $\mathbb{C}^k \rightleftarrows 0$ (resp. $\mathbb{C}^{k'} \rightleftarrows 0$). By

Proposition 4.10, $(\mathcal{M}_{x_i})^{\mu}$ is a regular meromorphic connexion therefore propositions 4.2 and 4.11 imply that it is equivalent to the quiver (18). Thus the exact sequence (19) is equivalent to the exact sequence of quivers

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}^k & \hookrightarrow & \mathrm{E} & \xrightarrow{\alpha} & \hat{\mathrm{F}}_i & \longrightarrow & \mathbb{C}^{k'} & \longrightarrow & 0 \\
 & & \uparrow\downarrow & & {\scriptstyle j}\uparrow\downarrow{\scriptstyle \mathrm{T}_{x_i}-\mathbb{1}} & & {\scriptstyle \mathbb{1}}\uparrow\downarrow{\scriptstyle \hat{\mathrm{T}}_i-\mathbb{1}} & & \uparrow\downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{F}_i & \underset{\beta}{\xrightarrow{\sim}} & \hat{\mathrm{F}}_i & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

Since the rows are exact, β is an isomorphism. Furthermore the commutativity of this diagram implies that $\alpha \circ j = \beta$ and $\beta \circ (\mathrm{T}_{x_i} - \mathbb{1}) = (\hat{\mathrm{T}}_i - \mathbb{1}) \circ \alpha$. Thus $\beta \circ (\mathrm{T}_{x_i} - \mathbb{1}) \circ j = (\hat{\mathrm{T}}_i - \mathbb{1}) \circ \alpha \circ j$. Hence $\beta \circ \mathrm{T}_{x_i} = \hat{\mathrm{T}}_i \circ \beta$. □

Corollary 4.12. $\dim \mathrm{Z}(\mathrm{T}_{x_i}) - \dim \mathrm{Z}(\hat{\mathrm{T}}_i) = \big(\dim \ker(\mathrm{T}_{x_i} - \mathbb{1})\big)^2 = (\dim \mathrm{E} - \dim \hat{\mathrm{F}}_i)^2$.

Proof. As the quiver $\mathrm{T}_{x_i} - \mathbb{1} : \mathrm{E} \rightleftarrows \mathrm{F}_i : j$ is minimal, it follows from Proposition 4.6 that $\dim \mathrm{Z}(\mathrm{T}_{x_i}) - \dim \mathrm{Z}(\hat{\mathrm{T}}_i) = \big(\dim \ker(\mathrm{T}_{x_i} - \mathbb{1})\big)^2$. On the other hand, $\dim \ker(\mathrm{T}_{x_i} - \mathbb{1}) + \dim \mathrm{im}(\mathrm{T}_{x_i} - \mathbb{1}) = \dim \mathrm{E}$. Since $\mathrm{T}_{x_i} - \mathbb{1} : \mathrm{E} \to \mathrm{F}_i$ is onto, $\dim \mathrm{F}_i = \dim \mathrm{im}(\mathrm{T}_{x_i} - \mathbb{1})$. Therefore, $\big(\dim \ker(\mathrm{T}_{x_i} - \mathbb{1})\big)^2 = (\dim \mathrm{E} - \dim \hat{\mathrm{F}}_i)^2$. □

Corollary 4.13. $\dim \mathrm{Z}(\mathrm{T}_{\infty}) - \dim \mathrm{Z}(\hat{\mathrm{T}}_0) = -\big(\dim \ker(\hat{\mathrm{T}}_0 - \mathbb{1})\big)^2 = -(\dim \hat{\mathrm{F}} - \dim \mathrm{F}_{\infty})^2$.

Proof. Similar to Corollary 4.12. □

Corollary 4.14. $\dim \ker(\hat{\mathrm{T}}_0 - \mathbb{1}) = \dim \hat{\mathrm{E}} - \dim \mathrm{E} = \sum_{i=1}^{k} \dim \hat{\mathrm{F}}_i - \dim \mathrm{E}$.

Proof. Thanks to Lemma 4.7, $\hat{\mathrm{T}}_0 - \mathbb{1} : \hat{\mathrm{E}} \to \hat{\mathrm{F}}$ is onto. Therefore $\dim \ker(\hat{\mathrm{T}}_0 - \mathbb{1}) = \dim \hat{\mathrm{E}} - \dim \hat{\mathrm{F}}$. This Lemma also implies that $\dim \hat{\mathrm{F}} = \dim \mathrm{F}_{\infty} = \dim \mathrm{E}$, because $\mathcal{M}$ is regular and localized at infinity. It follows from Proposition 4.11 that $\dim \ker(\hat{\mathrm{T}}_0 - \mathbb{1}) = \sum_{i=1}^{k} \dim \hat{\mathrm{F}}_i - \dim \hat{\mathrm{F}}$. □

5. Main result

The main result of this paper can now be proved.

Theorem 5.1. *The Fourier transform preserves the rigidity index of irreducible regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-modules with nonzero minimal extension.*

Proof. Let $\mathcal{M}$ be an irreducible regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-module and let Σ be the set of its singular points. The irreducibility condition ensures that

$\mathcal{M}_{\min} = \mathcal{M}$, unless $\mathrm{H}_{[\Sigma]}(\mathcal{M}) = \mathcal{M}$ (cf. Proposition 2.2). It follows from Theorem 3.4 that,

$$\mathrm{rig}({}^{\mathcal{F}}\mathcal{M}) = \dim \mathrm{Z}(\hat{\mathrm{T}}_0) + \sum_{i=1}^{k} \dim \mathrm{Z}(\hat{\mathrm{T}}_i) + \sum_{i=1}^{k}(\dim \hat{\mathrm{F}}_i)^2 - (\dim \hat{\mathrm{E}})^2.$$

Thanks to corollaries 4.12 and 4.13 the equality above can be rewritten as follows,

$$\begin{aligned}\mathrm{rig}({}^{\mathcal{F}}\mathcal{M}) = {} & \dim \mathrm{Z}(\mathrm{T}_\infty) + \big(\dim \ker(\hat{\mathrm{T}}_0 - \mathbb{1})\big)^2 \\ & + \sum_{i=1}^{k}\Big[\dim \mathrm{Z}(\mathrm{T}_{x_i}) - \big(\dim \ker(\mathrm{T}_{x_i} - \mathbb{1})\big)^2\Big] + \sum_{i=1}^{k}(\dim \hat{\mathrm{F}}_i)^2 - (\dim \hat{\mathrm{E}})^2.\end{aligned}$$

By corollaries 4.14 and 4.12, one has

$$\begin{aligned}\mathrm{rig}({}^{\mathcal{F}}\mathcal{M}) &= \dim \mathrm{Z}(\mathrm{T}_\infty) + (\dim \hat{\mathrm{E}} - \dim \mathrm{E})^2 \\ &\quad + \sum_{i=1}^{k}[\dim \mathrm{Z}(\mathrm{T}_{x_i}) - (\dim \mathrm{E} - \dim \hat{\mathrm{F}}_i)^2] + \sum_{i=1}^{k}(\dim \hat{\mathrm{F}}_i)^2 - (\dim \hat{\mathrm{E}})^2 \\ &= \dim \mathrm{Z}(\mathrm{T}_\infty) + \sum_{i=1}^{k} \dim \mathrm{Z}(\mathrm{T}_{x_i}) + (\dim \hat{\mathrm{E}} - \dim \mathrm{E})^2 \\ &\quad - \sum_{i=1}^{k}[(\dim \mathrm{E})^2 - 2\dim \mathrm{E} \dim \hat{\mathrm{F}}_i + (\dim \hat{\mathrm{F}}_i)^2] \\ &\quad + \sum_{i=1}^{k}(\dim \hat{\mathrm{F}}_i)^2 - (\dim \hat{\mathrm{E}})^2 \\ &= \dim \mathrm{Z}(\mathrm{T}_\infty) + \sum_{i=1}^{k} \dim \mathrm{Z}(\mathrm{T}_{x_i}) + (\dim \hat{\mathrm{E}} - \dim \mathrm{E})^2 \\ &\quad - k(\dim \mathrm{E})^2 + 2\dim \mathrm{E} \sum_{i=1}^{k} \dim \hat{\mathrm{F}}_i - (\dim \hat{\mathrm{E}})^2 \\ &= \dim \mathrm{Z}(\mathrm{T}_\infty) + \sum_{i=1}^{k} \dim \mathrm{Z}(\mathrm{T}_{x_i}) + (\dim \hat{\mathrm{E}})^2 - 2\dim \hat{\mathrm{E}} \dim \mathrm{E} \\ &\quad + (\dim \mathrm{E})^2 - k(\dim \mathrm{E})^2 + 2\dim \mathrm{E} \dim \hat{\mathrm{E}} - (\dim \hat{\mathrm{E}})^2 \\ &= \big(2 - (k+1)\big)(\dim \mathrm{E})^2 + \dim \mathrm{Z}(\mathrm{T}_\infty) + \sum_{i=1}^{k} \dim \mathrm{Z}(\mathrm{T}_{x_i}) \\ &= \mathrm{rig}(\mathcal{M}).\end{aligned}$$

□

A. Appendix

The purpose of this appendix is to give a proof of the following theorem:

Theorem A.1. *Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$-module and ${}^{\mathcal{F}}\mathcal{M}$ its Fourier transform. If $\mathrm{E}_\infty \rightleftarrows \mathrm{F}_\infty$ (resp. $\hat{\mathrm{E}} \rightleftarrows \hat{\mathrm{F}}$) is the quiver equivalent to $\mathcal{M}_\infty$ (resp. ${}^{\mathcal{F}}\mathcal{M}_0$), then $\hat{\mathrm{F}} = \mathrm{F}_\infty$ and $\mathrm{T}_{\hat{\mathrm{F}}} = \mathrm{T}_\infty$.*

The proof relies on V-filtrations and on [18], Theorem I, 6.2.5.

B. Outline of the proof

The proof is inspired by the methods used by C. Sabbah in [19] §2 to characterize the Fourier transform in a cohomological way. Let $\mathbb{A}^1$, $\mathbb{A}'^1$ and $\hat{\mathbb{A}}^1$ be respectively the affine line with affine coordinate t, t' and τ respectively and let A_1, A'_1, $\hat{\mathrm{A}}_1$ be the respective Weyl algebras. Throughout the appendix one uses the following notations:

- $\mathcal{M}$ denotes the holonomic $\mathcal{D}^{\mathrm{alg}}_{\mathbb{A}^1}$-module $\mathcal{M}^{\mathrm{alg}}|_{\mathbb{A}^1}$.
- $\mathcal{M}'$ denotes the holonomic $\mathcal{D}^{\mathrm{alg}}_{\mathbb{A}'^1}$-module $\mathcal{M}^{\mathrm{alg}}|_{\mathbb{A}'^1}$.
- p, $\hat{p}$ denote the canonical projections $\mathbb{A}^1 \xleftarrow{p} \mathbb{A}^1 \times \hat{\mathbb{A}}^1 \xrightarrow{\hat{p}} \hat{\mathbb{A}}^1$.
- q, $\hat{q}$ denote the canonical projections $\mathbb{A}'^1 \xleftarrow{q} \mathbb{A}'^1 \times \hat{\mathbb{A}}^1 \xrightarrow{\hat{q}} \hat{\mathbb{A}}^1$.

Let $\mathrm{M}[\tau] = \mathrm{M} \otimes_{\mathbb{C}} \mathbb{C}[\tau] = p^+\mathrm{M}$ be the inverse image of $\mathrm{M} = \Gamma(\mathbb{A}^1, \mathcal{M})$ on $\mathbb{A}^1 \times \hat{\mathbb{A}}^1$ with its natural structure of $\mathbb{C}[t, \tau]\langle \partial_t, \partial_\tau \rangle$-module. For a detailed account on inverse and direct images, refer to [1], [4], [6]. Put $\mathrm{M}' \doteq \Gamma(\mathbb{A}'^1, \mathcal{M}')$. Analogously let $\mathrm{M}'[\tau] = \mathrm{M}' \otimes_{\mathbb{C}} \mathbb{C}[\tau] = q^+\mathrm{M}'$ be the inverse image of M' on $\mathbb{A}'^1 \times \hat{\mathbb{A}}^1$ endowed with its natural structure of $\mathbb{C}[t', t'^{-1}, \tau]\langle \partial_{t'}, \partial_\tau \rangle$-module. With the these notations one has

$$\Gamma(\mathbb{A}^1 \times \hat{\mathbb{A}}^1, p^+\mathcal{M}) = \mathrm{M}[\tau],$$
$$\Gamma(\mathbb{A}'^1 \times \hat{\mathbb{A}}^1, q^+\mathcal{M}') = \mathrm{M}'[\tau].$$

$\mathrm{M}[\tau]$ can be identified to the $\mathbb{C}[t, \tau]$-module $\mathrm{M}[\tau] \otimes_{\mathbb{C}[t,\tau]} e^{-t\tau}$ with the following twisted actions of $\partial_t \cdot \doteq \partial_t - \tau$ and $\partial_\tau \cdot \doteq \partial_\tau - t$, for $m \in \mathrm{M}[\tau]$:

$$\partial_t(m \otimes e^{-t\tau}) = [(\partial_t - \tau)m] \otimes e^{-t\tau}, \qquad \partial_\tau(m \otimes e^{-t\tau}) = [(\partial_\tau - t)m] \otimes e^{-t\tau}.$$

Similarly, $\mathrm{M}'[\tau]$ can be identified to the $\mathbb{C}[t', t'^{-1}, \tau]$-module $\mathrm{M}'[\tau] \otimes_{\mathbb{C}[t', t'^{-1}, \tau]} e^{-\tau/t'}$ with the twisted actions of $\partial_{t'} \cdot \doteq \partial_{t'} + \tau t'^{-2}$ and $\partial_\tau \cdot \doteq \partial_\tau - t'^{-1}$, for $m \in \mathrm{M}'[\tau]$:

$$\partial_{t'}(m \otimes e^{-\tau/t'}) = [(\partial_{t'} + \tau t'^{-2})m] \otimes e^{-\tau/t'},$$
$$\partial_{\tau}(m \otimes e^{-\tau/t'}) = [(\partial_{\tau} - t'^{-1})m] \otimes e^{-\tau/t'}.$$

Thanks to these identifications one has

$$\Gamma(\mathbb{A}^1 \times \hat{\mathbb{A}}^1, p^+\mathcal{M} \otimes_{\mathcal{O}^{\mathrm{alg}}_{\mathbb{A}^1\times\hat{\mathbb{A}}^1}} \mathcal{O}^{\mathrm{alg}}_{\mathbb{A}^1\times\hat{\mathbb{A}}^1}\, e^{-t\tau}) = \mathrm{M}[\tau],$$
$$\Gamma(\mathbb{A}'^1 \times \hat{\mathbb{A}}^1, q^+\mathcal{M}' \otimes_{\mathcal{O}^{\mathrm{alg}}_{\mathbb{A}'^1\times\hat{\mathbb{A}}^1}} \mathcal{O}^{\mathrm{alg}}_{\mathbb{A}'^1\times\hat{\mathbb{A}}^1}\, e^{-\tau/t'}) = \mathrm{M}'[\tau].$$

Thus the direct images are respectively the complexes,

$$\Gamma\big(\hat{\mathbb{A}}^1, \hat{p}_+(p^+\mathcal{M} \otimes_{\mathcal{O}^{\mathrm{alg}}_{\mathbb{A}^1\times\hat{\mathbb{A}}^1}} \mathcal{O}^{\mathrm{alg}}_{\mathbb{A}^1\times\hat{\mathbb{A}}^1}\, e^{-t\tau})\big) = \mathrm{M}[\tau] \xrightarrow{\partial_t\cdot} \mathrm{M}[\tau], \tag{20}$$
$$\Gamma\big(\hat{\mathbb{A}}^1, \hat{q}_+(q^+\mathcal{M}' \otimes_{\mathcal{O}^{\mathrm{alg}}_{\mathbb{A}'^1\times\hat{\mathbb{A}}^1}} \mathcal{O}^{\mathrm{alg}}_{\mathbb{A}'^1\times\hat{\mathbb{A}}^1}\, e^{-\tau/t'})\big) = \mathrm{M}'[\tau] \xrightarrow{\partial_{t'}\cdot} \mathrm{M}'[\tau], \tag{21}$$

where the right hand term of each complex has degree 0 and have cohomology in degree 0 only.

Owing to the following two lemmas, the complexes (20), (21) give rise respectively to the following short exact sequences:

$$0 \to \mathrm{M}[\tau] \xrightarrow{\partial_t\cdot} \mathrm{M}[\tau] \xrightarrow{\pi} {}^{\mathrm{F}}\mathrm{M} \to 0, \tag{22}$$
$$0 \to \mathrm{M}'[\tau] \xrightarrow{\partial_{t'}\cdot} \mathrm{M}'[\tau] \xrightarrow{\pi'} {}^{\mathrm{F}}\mathrm{M}[t^{-1}] \to 0. \tag{23}$$

Therefore,

$${}^{\mathrm{F}}\mathrm{M} = \mathrm{H}^0\big(\hat{\mathbb{A}}^1, \hat{p}_+(p^+\mathcal{M} \otimes_{\mathcal{O}^{\mathrm{alg}}_{\mathbb{A}^1\times\hat{\mathbb{A}}^1}} \mathcal{O}^{\mathrm{alg}}_{\mathbb{A}^1\times\hat{\mathbb{A}}^1}\, e^{-t\tau})\big),$$
$${}^{\mathrm{F}}\mathrm{M}[t^{-1}] = \mathrm{H}^0\big(\hat{\mathbb{A}}^1, \hat{q}_+(q^+\mathcal{M}' \otimes_{\mathcal{O}^{\mathrm{alg}}_{\mathbb{A}'^1\times\hat{\mathbb{A}}^1}} \mathcal{O}^{\mathrm{alg}}_{\mathbb{A}'^1\times\hat{\mathbb{A}}^1}\, e^{-\tau/t'})\big).$$

Lemma ([16], Lemma 2.6.18). *Let* M *be an* A_1*-module with the twisted action*

$$\partial_t\cdot = \partial_t - \tau : \mathrm{M}[\tau] \to \mathrm{M}[\tau], \qquad \sum_{i=0}^{n} \tau^i \otimes m_i \mapsto \sum_{i=0}^{n} \tau^i \otimes \partial_t m_i - \sum_{i=0}^{n} \tau^{i+1} \otimes m_i.$$

This map satisfies the following conditions:

i) *is a map of* $\mathbb{C}[\tau]\langle\partial_\tau\cdot\rangle$*-modules,*

ii) *is injective,*

iii) $\operatorname{coker}(\partial_t - \tau) = {}^{\mathrm{F}}\mathrm{M}$ (*and therefore* $\partial_t = \tau$ *in* ${}^{\mathrm{F}}\mathrm{M}$).

Proof. This lemma is essentially a restatement of [19], §§2.1. □

Lemma ([16], Lemma 2.6.20). *Let* M′ *be an* $A'_1[t'^{-1}]$*-module with the twisted action*

$$\partial_{t'}\cdot = \partial_{t'} + \tau/t'^2 : \mathrm{M}'[\tau] \to \mathrm{M}'[\tau],$$
$$\sum_{i=0}^{n} \tau^i \otimes m_i \mapsto \sum_{i=0}^{n} \tau^i \otimes \partial_{t'} m_i + \sum_{i=0}^{n} \tau^{i+1} \otimes t'^{-2} m_i.$$

This map satisfies the following conditions:

i) *it is a map of* $\mathbb{C}[\tau]\langle\partial_\tau\cdot\rangle$*-modules,*

ii) *it is injective,*

iii) $\mathrm{coker}(\partial_{t'} + \tau/t'^2) = {}^{\mathrm{F}}\mathrm{M}[t^{-1}]$.

Proof. The proof is similar to the preceding lemma. □

The key idea behind the proof of Lemma A.1 is to first take the compactification $\mathbb{P}^1 \xleftarrow{\pi_{\mathbb{P}^1}} \mathbb{P}^1 \times \hat{\mathbb{A}}^1 \xrightarrow{\pi_{\hat{\mathbb{A}}^1}} \hat{\mathbb{A}}^1$. The reason for this lies in the fact that the map $\pi_{\hat{\mathbb{A}}^1}$ is proper and therefore its direct image preserves the V-filtration for each degree of cohomology (cf. Theorem C.5). Then one uses the V-filtration to prove Lemma A.1. For a nice overview on V-filtrations in dimension refer to [18] §I.6. In higher dimensions, refer to [11] or [13] lecture 7.

Let us therefore take a close look at $\pi_{\hat{\mathbb{A}}^1+}\big(\pi^+_{\mathbb{P}^1}(j_+\mathcal{M}[t^{-1}])\big)$, where $j : U_0 \hookrightarrow \mathbb{P}^1$ is the inclusion, $U_0 \doteq \mathbb{C}(=\mathbb{A}^1)$ and $U_\infty \doteq \mathbb{P}^1\backslash\{0\}$. $j_+\mathcal{M}[t^{-1}]$ is the sheaf having the following sections:

$$\Gamma(U_0, j_+\mathcal{M}[t^{-1}]) = \mathrm{M}[t^{-1}],$$
$$\Gamma(U_\infty, j_+\mathcal{M}[t^{-1}]) = \mathrm{M}[t^{-1}] \quad \text{(with the actions of } t' \text{ and } \partial_{t'}),$$
$$\Gamma(U_0 \cap U_\infty, j_+\mathcal{M}[t^{-1}]) = \mathrm{M}[t^{-1}].$$

$\pi^+_{\mathbb{P}^1}(j_+\mathcal{M})$ is the sheaf having the following sections:

$$\Gamma\big(U_0, \pi^+_{\mathbb{P}^1}(j_+\mathcal{M}[t^{-1}])\big) = \mathrm{M}[t^{-1}][\tau],$$
$$\Gamma\big(U_\infty, \pi^+_{\mathbb{P}^1}(j_+\mathcal{M}[t^{-1}])\big) = \mathrm{M}[t^{-1}][\tau] \quad \text{(with the actions of } t' \text{ and } \partial_{t'}\cdot),$$
$$\Gamma\big(U_0 \cap U_\infty, \pi^+_{\mathbb{P}^1}(j_+\mathcal{M}[t^{-1}])\big) = \mathrm{M}[t^{-1}][\tau].$$

$\pi_{\hat{\mathbb{A}}^1+}\big(\pi^+_{\mathbb{P}^1}(j_+\mathcal{M})\big)$ gives rise to the double complex

$$\begin{array}{ccc} \mathrm{M}[t^{-1}][\tau] \oplus \mathrm{M}[t^{-1}][\tau] & \xrightarrow{\delta} & \mathrm{M}[t^{-1}][\tau] \\ \partial_t\cdot \downarrow \qquad \partial_{t'}\cdot\downarrow & & \partial_t\cdot\downarrow \\ \mathrm{M}[t^{-1}][\tau] \oplus \mathrm{M}[t^{-1}][\tau] & \xrightarrow{\delta} & \mathrm{M}[t^{-1}][\tau], \end{array}$$

where the rows are Čech complexes, the columns are relative de Rham complexes. δ is the onto map $(m_1, m_2) \mapsto m_1 - m_2$ and $\ker\delta \simeq \mathrm{M}[t^{-1}][\tau]$ (with the actions of t' and $\partial_{t'}\cdot$). This complex is quasi-isomorphic to

$$\begin{array}{ccc} \ker\delta & & \mathrm{M}[t^{-1}][\tau] \\ \downarrow \partial_{t'}\cdot & \simeq & \downarrow \partial_{t'}\cdot \\ \ker\delta & & \mathrm{M}[t^{-1}][\tau] \end{array}$$

The complex above gives rise to the short exact sequence (23). The preservation of the V-filtration for each degree of cohomology (cf. Theorem C.5) implies, for each $k \in \mathbb{Z}$, that

$$\pi'[{}^{\tau}V_k(\mathrm{M}'[\tau])] = V_k({}^{\mathrm{F}}\mathrm{M}[t^{-1}]).$$

C. Monodromy at infinity and Fourier transform

This section relates the quiver $\mathrm{E}_\infty \rightleftarrows \mathrm{F}_\infty$ associated to $\mathcal{M}_\infty$ and the quiver $\hat{\mathrm{E}} \rightleftarrows \hat{\mathrm{F}}$ associated to ${}^{\mathcal{F}}\mathcal{M}_0$. This is done by computing the canonical V-filtration of $\mathrm{M}'[\tau]$ and then using the preservation of the V-filtration by proper direct images [Theorem C.5] to compute the V-filtration of ${}^{\mathcal{F}}\mathcal{M}_0$. Before computing the canonical V-filtration of $\mathrm{M}'[\tau]$, let us state and prove the following two auxiliary lemmas.

Lemma C.1 ([16], Lemma 2.6.19). *Let* M *be an holonomic* A_1*-module and* $p = \sum_{i=0}^{n} \tau^i \otimes m_i \in \mathrm{M}[\tau]$, *then* $p = 0$ *iff* $m_i = 0$ *for* $i = 0, \ldots, n$.

Proof. Both $\{1, \ldots, \tau^n\}$ and $\{m_0, \ldots, m_n\}$ span finite dimensional $\mathbb{C}$-vector spaces whose tensor product is contained in $\mathrm{M}[\tau]$ and so its dimension is also finite. □

Lemma C.2. *If* M' *is a holonomic* A_1'*-module localized at* $t' = 0$, *then* $\mathrm{M}'[\tau]$ *is microlocalized at* $\tau = 0$, *i.e., left multiplication by* $\partial_\tau\cdot$ *is bijective.*

Proof. By Lemma C.1, left multiplication by $\partial_\tau\cdot$ is injective. As for each $i \in \mathbb{N}$ and $m \in \mathrm{M}'$, $\partial_\tau \cdot \left(\sum_{j=0}^{i} \frac{i!}{(i-j)!} \tau^{i-j} \otimes (-t'^{j+1} m)\right) = \tau^i \otimes m$, multiplication by $\partial_\tau\cdot$ is onto, so it is bijective. □

Here $\boxtimes$ denotes the external product. For more details on external products, refer to [4], chap. 13.

Lemma C.3. *Let* $V_k\mathrm{M}'$, $k \in \mathbb{Z}$, *be the canonical* V*-filtration of* M' *along* $t' = 0$ *and let* $b'(s) \in \mathbb{C}[s]$ *be its Bernstein polynomial. Then the family of vector spaces* ${}^{\tau}U_k(\mathrm{M}'[\tau])$,

$$ {}^{\tau}U_k(\mathrm{M}'[\tau]) = \begin{cases} \sum_{i \geq 0} (\partial_{t'} \cdot)^i (1 \otimes V_{k+1}\mathrm{M}') & \text{if } k \geq 0, \\ \tau^{k\,\tau}U_0(\mathrm{M}'[\tau]) & \text{if } k < 0, \end{cases} $$

is the canonical V*-filtration of* $\mathrm{M}'[\tau]$ *along* $\tau = 0$. *Furthermore, for each* $k \in \mathbb{Z}$, ${}^{\tau}U_k(\mathrm{M}'[\tau]) = (\partial_\tau \cdot)^{k\,\tau} U_0(\mathrm{M}'[\tau])$.

Proof. Firstly, one shows that ${}^{\tau}U_0(\mathrm{M}'[\tau])$ is a left $\mathrm{A}'_1 \boxtimes V_0\hat{\mathrm{A}}_1$-module. By construction, ${}^{\tau}U_0(\mathrm{M}'[\tau]) \doteq \sum_{i \geq 0} (\partial_{t'} \cdot)^i (1 \otimes V_1\mathrm{M}')$ is closed under the action of $\partial_{t'} \cdot$. As $t' V_k \mathrm{M}' \subset V_{k-1}\mathrm{M}'$, is also closed under the action of t'. Moreover, $\tau = \partial_{t'} \cdot - \partial_{t'}$ and $\tau \partial_\tau \cdot (1 \otimes V_1\mathrm{M}') = (\partial_{t'} \cdot - \partial_{t'}) t' (1 \otimes V_1 \mathrm{M}')$. Recall that $\partial_{t'} \cdot \doteq \partial_{t'} + \tau t'^{-2}$ and $\partial_\tau \cdot \doteq \partial_\tau - t'^{-1}$. Therefore ${}^{\tau}U_0(\mathrm{M}'[\tau])$ is also closed under the actions of τ and $\tau\partial_\tau \cdot$.

Secondly, one shows that ${}^{\tau}U_0(\mathrm{M}'[\tau])$ is a $\mathrm{A}'_1 \boxtimes V_0\hat{\mathrm{A}}_1$-module of finite type. Since $V_k\mathrm{M}'$ is a good filtration, there exists an A'_1-module map $\varphi : \bigoplus_{i=1}^n A'_1 \to \mathrm{M}'$ and $k_1, \ldots, k_n \in \mathbb{Z}$ such that $\varphi\big(V_k(\bigoplus_{i=1}^n \mathrm{A}'_1) \doteq \bigoplus_{i=1}^n V_{k+k_i}\mathrm{A}'_1\big) = V_k\mathrm{M}'$. In particular $V_1\mathrm{M}'$ is a $V_0\,\mathrm{A}'_1$-module of finite type. Let $e_1, \ldots, e_s$ be a finite set of generators of $V_1\mathrm{M}'$ over $V_0\,\mathrm{A}'_1$. Let us now show that $1 \otimes e_1, \ldots, 1 \otimes e_s$ generate ${}^{\tau}U_0(\mathrm{M}'[\tau])$ under the action of $\mathrm{A}'_1 \boxtimes V_0\hat{\mathrm{A}}_1$. Note that for each $m \in \mathrm{M}'$ and $i \in \mathbb{N}$

$$ [\tau\partial_\tau \cdot + \partial_{t'} \cdot t' + i](\partial_{t'} \cdot)^i (1 \otimes m) = (\partial_{t'} \cdot)^i (1 \otimes \partial_{t'} t' m). $$

Hence

$$ \mathbb{C}[\tau\partial_\tau \cdot + \partial_{t'} \cdot t' + i](\partial_{t'} \cdot)^i (1 \otimes m) = (\partial_{t'} \cdot)^i (1 \otimes \mathbb{C}[\partial_{t'} t'] m). $$

Therefore

$$ \mathbb{C}[\tau\partial_\tau \cdot + \partial_{t'} \cdot t'](\partial_{t'} \cdot)^i (1 \otimes m) = (\partial_{t'} \cdot)^i (1 \otimes \mathbb{C}[t' \partial_{t'}] m). $$

Consequently

$$ \begin{aligned} \sum_{j=1}^{s} \mathbb{C}[\tau\partial_\tau \cdot + \partial_{t'} \cdot t'] \mathbb{C}[\partial_{t'} \cdot] \mathbb{C}[t'](1 \otimes e_j) &= \sum_{i \geq 0} (\partial_{t'} \cdot)^i \Big(1 \otimes \sum_{j=1}^{s} \mathbb{C}[t' \partial_{t'}] \mathbb{C}[t'] e_j\Big) \\ &= \sum_{i \geq 0} (\partial_{t'} \cdot)^i (1 \otimes V_1 \mathrm{M}'). \end{aligned} $$

As the actions of $A_1' \boxtimes V_0\hat{A}_1$ and $\mathbb{C}[\tau\partial_\tau \cdot + \partial_{t'} \cdot t']\mathbb{C}[\partial_{t'}\cdot]\mathbb{C}[t']$ on $1 \otimes m$ generate the same set, $1 \otimes e_1, \dots, 1 \otimes e_s$ generate ${}^\tau U_0(M'[\tau])$ under the action of $A_1' \boxtimes V_0\hat{A}_1$.

Next, one shows that ${}^\tau U_k(M'[\tau]) = (\partial_\tau\cdot)^{k\,\tau} U_0(M'[\tau])$ for each $k \in \mathbb{Z}$. This is a consequence of $t'^{-k}V_0M' = V_kM'$, for each $k \in \mathbb{Z}$. The later can be proved using a partition analogous to the following one (cf. [16], proof of Lemma 2.6.13). As $\partial_\tau \cdot (1 \otimes V_kM') = (1 \otimes t'^{-1}V_kM')$,

$$\partial_\tau \cdot \sum_{i\geq 0} (\partial_{t'}\cdot)^i (1 \otimes V_kM') = \sum_{i \geq 0} (\partial_{t'}\cdot)^i (1 \otimes V_{k+1}M').$$

Therefore ${}^\tau U_{k+1}(M'[\tau]) = \partial_\tau \cdot {}^\tau U_k(M'[\tau]) + {}^\tau U_k(M'[\tau]) = (\partial_\tau\cdot)^{k+1} U_0(M'[\tau])$, for every $k \geq 0$. It remains to be shown that ${}^\tau U_k(M'[\tau]) = (\partial_\tau\cdot)^k U_0(M'[\tau])$, for every $k \leq 0$. In order to prove it, one takes the partition $\left(W_i(M'[\tau]\backslash\{0\})\right)_{i\in\mathbb{Z}} \doteq {}^\tau U_i(M'[\tau])\backslash{}^\tau U_{i-1}(M'[\tau])$, which satisfies the following conditions:

i) $\bigcup_{i\in\mathbb{Z}} W_i(M'[\tau]\backslash\{0\}) = M'[\tau]\backslash\{0\}$.

ii) For each $i, j \in \mathbb{Z}$, if $i \neq j$, then $W_i(M'[\tau]\backslash\{0\}) \cap W_j(M'[\tau]\backslash\{0\}) = \emptyset$.

Let $m \in W_i(M'[\tau]\backslash\{0\})$. By construction, $m \neq 0$. Thus $(\partial_\tau\cdot)^{-1}m \in M'[\tau]\backslash\{0\}$, because $M'[\tau]$ is a holonomic $\hat{A}_1$-module microlocalized in $\tau = 0$ (cf. Lemma C.2). By i), there exists only one $i \in \mathbb{Z}$ such that $(\partial_\tau\cdot)^{-1}m \in W_i(M'[\tau]\backslash\{0\})$. Therefore, $m \in W_{i+1}(M'[\tau]\backslash\{0\})$. Moreover, by ii), $i+1 = k$. Hence, for each $k \in \mathbb{Z}$, $(\partial_\tau\cdot)^{-1\,\tau}U_kM'[\tau] \subset {}^\tau U_{k-1}M'[\tau]$. Consequently $(\partial_\tau\cdot)^{-1\,\tau}U_kM'[\tau] = {}^\tau U_{k-1}M'[\tau]$ so, for each $k \in \mathbb{N}$, ${}^\tau U_{-k}M'[\tau] = (\partial_\tau\cdot)^{-k\,\tau}U_0M'[\tau]$.

By construction, ${}^\tau U_k(M'[\tau]) = (A_1' \boxtimes V_k\hat{A}_1)^\tau U_0(M'[\tau])$. Thus, it is an increasing filtration compatible with the action of $A_1' \boxtimes V_k\hat{A}_1$. Besides, it is exhaustive because

$$\begin{aligned}\bigcup_{k\in\mathbb{Z}} {}^\tau U_k(M'[\tau]) &= A_1' \boxtimes \hat{A}_1 \sum_{i\geq 0}(\partial_{t'}\cdot)^i(1 \otimes V_1M') \\ &= A_1'\mathbb{C}[\tau]\sum_{i\geq 0}(\partial_{t'}\cdot)^i(1\otimes M') \supset M'[\tau].\end{aligned}$$

Finally, let us now compute the Bernstein polynomial in order to show that ${}^\tau U_k(M'[\tau])$ is the canonical V-filtration. Notice that for each $k \geq 1$, one has the congruence modulo ${}^\tau U_{k-1}(M'[\tau])$:

$$\begin{aligned}\tau\partial_\tau \cdot {}^\tau U_k(M'[\tau]) &= \sum_{i\geq 0}(\partial_{t'}\cdot)^i(\partial_{t'}\cdot - \partial_{t'})t'^2\left(1 \otimes (-t'^{-1})V_{k+1}M'\right) \\ &\subset \sum_{i\geq 0}(\partial_{t'}\cdot)^i\left(1\otimes(\partial_{t'}t')V_{k+1}M'\right) + \sum_{i\geq 0}(\partial_{t'}\cdot)^{i+1}(1\otimes t'V_{k+1}M') \\ &\equiv \sum_{i\geq 0}(\partial_{t'}\cdot)^i\left(1\otimes(t'\partial_{t'}+1)V_{k+1}M'\right).\end{aligned}$$

Therefore, for each $a(s) \in \mathbb{C}[s]$, one has the congruence modulo ${}^{\tau}U_{k-1}(\mathrm{M}'[\tau])$

$$a(\tau\partial_\tau \cdot)^{\tau}U_k(\mathrm{M}'[\tau]) \equiv \sum_{i\geq 0}(\partial_{t'}\cdot)^i\big(1 \otimes a(t'\partial_{t'} + 1)V_{k+1}\mathrm{M}'\big).$$

In particular, the congruence holds for $a(s) = b'(s)$. Hence,

$$b'(\tau\partial_\tau \cdot + k)^{\tau}U_k(\mathrm{M}'[\tau]) \equiv \sum_{i\geq 0}(\partial_{t'}\cdot)^i\big(1 \otimes b'(t'\partial_{t'} + k + 1)V_{k+1}\mathrm{M}'\big) \equiv 0.$$

Consequently,

$$b'(\tau\partial_\tau \cdot + k)^{\tau}U_k(\mathrm{M}'[\tau]) \subset {}^{\tau}U_{k-1}(\mathrm{M}'[\tau]).$$

Thus,

$$b'(\tau\partial_\tau \cdot + 1)^{\tau}U_1(\mathrm{M}'[\tau]) \subset {}^{\tau}U_0(\mathrm{M}'[\tau]).$$

As a result of this,

$$(\partial_\tau \cdot)^{-1}b'(\tau\partial_\tau \cdot + 1)^{\tau}U_1(\mathrm{M}'[\tau]) \subset (\partial_\tau \cdot)^{-1\,\tau}U_0(\mathrm{M}'[\tau]).$$

In other words,

$$b'(\tau\partial_\tau \cdot)(\partial_\tau \cdot)^{-1\,\tau}U_1(\mathrm{M}'[\tau]) \subset (\partial_\tau \cdot)^{-1\,\tau}U_0(\mathrm{M}'[\tau]).$$

Hence, $b'(\tau\partial_\tau \cdot)^{\tau}U_0(\mathrm{M}'[\tau]) \subset {}^{\tau}U_{-1}(\mathrm{M}'[\tau])$. By recurrence, for each $k \in \mathbb{Z}^-$,

$$b'(\tau\partial_\tau \cdot + k)^{\tau}U_k(\mathrm{M}'[\tau]) \subset {}^{\tau}U_{k-1}(\mathrm{M}'[\tau]).$$

Let $\beta(s) \in \mathbb{C}[s]$ the Bernstein polynomial of ${}^{\tau}U_k(\mathrm{M}'[\tau])$. The inclusion above implies that $\beta(s)$ divides $b'(s)$. Put $[1 \otimes V_k\mathrm{M}'] \doteq {}^{\tau}U_k(\mathrm{M}'[\tau])/\mathrm{im}\,\partial_{t'}\cdot$. As for each $k \geq 1$, $\beta(\tau\partial_\tau \cdot + k)[1 \otimes V_{k+1}\mathrm{M}'] \subset [1 \otimes V_k\mathrm{M}']$,

$$\beta(t'\partial_{t'} + k + 1)V_{k+1}\mathrm{M}' \subset V_k\mathrm{M}'.$$

Therefore, for each $l \geq 0$, $t'^l\beta(t'\partial_{t'} + k + 1)V_{k+1}\mathrm{M}' \subset t'^l V_k\mathrm{M}'$. In other words, for each $k \in \mathbb{Z}$, $\beta(t'\partial_{t'} + k - l + 1)V_{k-l+1}\mathrm{M}' \subset V_{k-l}\mathrm{M}'$. Hence, for each $k \in \mathbb{Z}$,

$$\beta(t'\partial_{t'} + k + 1)V_{k+1}\mathrm{M}' \subset V_k\mathrm{M}'.$$

Given that $b'(s) \in \mathbb{C}[s]$ is the monic polynomial of smallest degree amongst those satisfying the inclusion above, $\beta(s) = b'(s)$. As both V-filtrations $V_k\mathrm{M}'$, $U_k(\mathrm{M}'[\tau])$ have the same Bernstein polynomial and $V_k\mathrm{M}'$ is the canonical V-filtration of M', then $U_k(\mathrm{M}'[\tau])$ is the canonical V-filtration of $\mathrm{M}'[\tau]$. □

Let ${}^{\tau}V_k(\mathrm{M}'[\tau])$ be the canonical V-filtration of $\mathrm{M}'[\tau]$ along $\tau = 0$. Accordingly, the quotient ${}^{\tau}V_1(\mathrm{M}'[\tau])/{}^{\tau}V_0(\mathrm{M}'[\tau])$ is an A'_1-module with the endomorphism induced by $\tau\partial_\tau\cdot$. Conversely, $\mathrm{F}_\infty \doteq V_{-1}\mathrm{M}'/V_{-2}\mathrm{M}'$ is a $\mathbb{C}$-vector space with an endomorphism induced by the action of $t'\partial_{t'}$. As a result, $\mathrm{F}_\infty \boxtimes \mathrm{A}'_1/\mathrm{A}'_1.t'$ is an A'_1-module with an endomorphism induced by the action of $t'\partial_{t'}$ on F_∞.

Lemma C.4. *The two A'_1-modules above, equipped with the respective endomorphisms, are isomorphic.*

Proof. See point (ii) (6) in the proof of [21], Proposition 4.1. □

Theorem C.5 ([13], Theorem 7.5.5 (1)). *Let $f : X \to X'$ be a holomorphic map between complex analytic manifolds and let $t \in \mathbb{C}$ be a new variable. Put $F = f \times \mathbb{1}_{\mathbb{C}} : X \times \mathbb{C} \to X' \times \mathbb{C}$. Let $\mathcal{M}$ be a right $\mathcal{D}_{X\times\mathbb{C}}$-module equipped with the canonical V-filtration $V_\bullet\mathcal{M}$ (relative to the hypersurface $Y = X \times \{0\}$). Then $V_\bullet\mathcal{M}$ defines canonically and functorially a V-filtration $\mathrm{U}_\bullet\mathcal{H}^i(F_+\mathcal{M})$. Moreover, if F is proper on the support of $\mathcal{M}$, $\mathrm{U}_\bullet\mathcal{H}^i(F_+\mathcal{M})$ is a good V-filtration.*

Now one has all the required conditions which enable us to prove Theorem A.1.

Proof of Theorem A.1. By taking $X' = \{0\}$ in theorem above, $\{0\} \times \hat{\mathbb{A}}^1$ can be identified to $\hat{\mathbb{A}}^1$ and F to $\pi_{\hat{\mathbb{A}}^1} : \mathbb{P}^1 \times \hat{\mathbb{A}}^1 \to \hat{\mathbb{A}}^1$. Since $\pi_{\hat{\mathbb{A}}^1}$ is proper, the theorem above implies that $\pi'\big({}^{\tau}V_k(\mathrm{M}'[\tau])\big) = V_k({}^{\mathrm{F}}\mathrm{M}[t^{-1}])$, for each $k \in \mathbb{Z}$. Hence,

$$\pi'\big({}^{\tau}V_k(\mathrm{M}'[\tau])\big) = [1 \otimes V_{K+1}\mathrm{M}'] = V_k({}^{\mathrm{F}}\mathrm{M}[t^{-1}]),$$

for each $k \in \mathbb{N}$. In particular, one has

$$V_1({}^{\mathrm{F}}\mathrm{M}[t^{-1}]) = [1 \otimes V_2\mathrm{M}'] \quad \text{and} \quad V_0({}^{\mathrm{F}}\mathrm{M}[t^{-1}]) = [1 \otimes V_1\mathrm{M}'].$$

Since $\hat{\mathrm{F}} \doteq V_1({}^{\mathrm{F}}\mathrm{M}[t^{-1}])/V_0({}^{\mathrm{F}}\mathrm{M}[t^{-1}])$, by Lemma C.4, $\hat{\mathrm{F}} = \mathrm{F}_\infty$ and $\mathrm{T}_{\hat{\mathrm{F}}} = \mathrm{T}_\infty$. □

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