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Multiple solutions for a perturbed fourth-order Kirchhoff type elliptic problem

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Abstract. The existence of three distinct weak solutions for a perturbed fourth-order Kirchhoff-type elliptic problem is investigated. Our approach is based on variational methods.

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1. Introduction

Consider the following perturbed fourth-order Kirchhoff-type elliptic problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - [M(\int_\Omega |\nabla u|^p \, dx)]^{p-1}\Delta_p u + \rho|u|^{p-2}u & \text{in } \Omega, \\ \qquad = \lambda f(x,u) + \mu g(x,u) & \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $p > \max\left\{1, \dfrac{N}{2}\right\}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\lambda > 0$ and $\mu \geq 0$ are real numbers, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded smooth domain $\rho > 0$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions and $M : [0, +\infty[\to \mathbb{R}$ is a continuous function.

The problem (1) is related to the stationary problem

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

proposed by Kirchhoff [29] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Equation (2) was developed to form

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2\, dx\Big)\Delta u = f(x,u).$$

After that, many authors studied the following nonlocal elliptic boundary value problem:

$$-M\Big(\int_\Omega |\nabla u|^2\, dx\Big)\Delta u = f(x,u). \tag{3}$$

Problems like (3) can be used for modeling several physical and biological systems where u describes a process which depends on the average of it self, such as the population density, see [1].

However, to be accurate, the problem (1) is related to the models of extensible beams and plates by Woinowsky-Krieger [44] and Berger [5]. In addition, the first stationary study of such fourth order nonlocal boundary value problem was given by Ma [33]. In [34] was named first time "fourth order problem of Kirchhoff type". The problem is also related to the so-called p-Kirchhoff problems. Note that the part $[M(\int_\Omega |\nabla u|^p\, dx)]^{p-1}\Delta_p u$ of the problem (1) has an exponent $p-1$. This was first introduced by Corrêa and Figueiredo [20] to make the problem variational.

Problems of Kirchhoff-type have been widely investigated. We refer the reader to the papers [19], [23], [26], [27], [35], [37], [40] and the references therein. For instance, B. Ricceri in an interesting paper [40] established the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problem using Theorem A of [39]. In [27], motivated by [40], based on a three critical points theorem proved in [6], the existence of two intervals of positive real parameters λ for which the boundary value problem of Kirchhoff-type

$$\begin{cases} -K\big(\int_a^b |u'(x)|^2\, dx\big)u'' = \lambda f(x,u), \\ u(a) = u(b) = 0, \end{cases}$$

where $K : [0, +\infty[\to \mathbf{R}$ is a continuous function, $f : [a,b] \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function and $\lambda > 0$ admits three weak solutions whose norms are uniformly bounded with respect to λ belonging to one of the two intervals, was established.

The fourth-order equation of nonlinearity furnishes a model to study traveling waves in suspension bridges, so it is important to physics. Due to this, many researchers have discussed the existence of at least one solution, or multiple solutions, or even infinitely many solutions for fourth-order boundary value problems by using lower and upper solution methods, Morse theory, the mountain-pass theorem, constrained minimization and concentration-compactness principle, fixed-point theorems and degree theory, and variational methods and critical point theory. We refer the reader to [4], [10], [11], [12], [13], [16], [17], [18], [24], [25], [28], [30], [31], [32] and references therein.

In [42], using the mountain pass theorem, Wang and An established the existence and multiplicity of solutions for the following fourth-order nonlocal elliptic problem

$$\begin{cases} \Delta^2 u - M(\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, in [43] the authors, by using the mountain pass techniques and the truncation method, studied the existence of nontrivial solutions for a class of fourth order elliptic equations of Kirchhoff-type.

In particular, in [36] employing a smooth version of Ricceri's variational principle [38], the authors ensured the existence of infinitely many solutions for the problem (1) when $\mu = 0$. In [22], the existence of two solutions for the problem (1) when $\mu = 0$ by combining an algebraic condition on f with the classical Ambrosetti–Rabinowitz condition was established.

In the present paper, employing two kinds of three critical points theorems obtained in [7], [14] which we recall in the next section (Theorems 2.1 and 2.2), we ensure the existence of least three weak solutions for the problem (1). These theorems have been successfully employed to establish the existence of at least three solutions for perturbed boundary value problems in the papers [8], [9], [21].

For a through on the subject, we also refer the reader to the papers [2], [3].

2. Preliminaries

Our main tools are three critical point theorems that we recall here in a convenient form. The first has been obtained in [14], and it is a more precise version of Theorem 3.2 of [7]. The second has been established in [7].

Theorem 2.1 ([14], Theorem 3.6). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$.*

Assume that there exist $r > 0$ and $\bar{v} \in X$, with $r < \Phi(\bar{v})$ such that

(a_1) $\dfrac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \dfrac{\Psi(\bar{v})}{\Phi(\bar{v})}$,

(a_2) *for each* $\lambda \in \Lambda_r := \left]\dfrac{\Phi(\bar{v})}{\Psi(\bar{v})}, \dfrac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\right[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$ *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

Theorem 2.2 ([7], Corollary 3.1). *Let* X *be a reflexive real Banach space,* $\Phi : X \to \mathbb{R}$ *be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on* X^*, $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

2. *for each* $\lambda > 0$ *and for every* $u_1, u_2 \in X$ *which are local minima for the functional* $\Phi - \lambda\Psi$ *and such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$, *one has*

$$\inf_{s \in [0,1]} \Psi\big(su_1 + (1-s)u_2\big) \geq 0.$$

Assume that there are two positive constants r_1, r_2 *and* $\bar{v} \in X$, *with* $2r_1 < \Phi(\bar{v}) < \frac{r_2}{2}$, *such that*

(b_1) $\dfrac{\sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\bar{v})}{\Phi(\bar{v})}$;

(b_2) $\dfrac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\bar{v})}{\Phi(\bar{v})}$.

Then, for each

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \min\left\{\frac{r_1}{\sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ *has at least three distinct critical points which lie in* $\Phi^{-1}(]-\infty, r_2[)$.

Here and in the sequel, X will denote the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega \big(|\Delta u(x)|^p + |\nabla u(x)|^p + |u(x)|^p\big)\,dx\right)^{1/p}.$$

Put

$$k = \sup_{u \in X \setminus \{0\}} \frac{\max_{x \in \bar{\Omega}} |u(x)|}{\|u\|}. \tag{4}$$

For $p > \max\left\{1, \frac{N}{2}\right\}$, since the embedding $X \hookrightarrow C^0(\bar{\Omega})$ is compact, one has $k < +\infty$.

Let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be two L^1-Carathéodory functions and $M : [0, +\infty[\to \mathbb{R}$ be a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \leq M(t) \leq m_1$ for all $t \geq 0$.

Corresponding to f and g we introduce the functions $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $G : \Omega \times \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x, t) = \int_0^t f(x, \xi)\, d\xi, \qquad \text{for all } (x, t) \in \Omega \times \mathbb{R}$$

and

$$G(x, t) = \int_0^t g(x, \xi)\, d\xi, \qquad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

Moreover, set $G^c := \int_\Omega \sup_{|t| \leq c} G(x, t)\, dx$ for every $c > 0$ and $G_d := \inf_{\Omega \times [0, d]} G$ for every $d > 0$. If g is sign-changing, then $G^c \geq 0$ and $G_d \leq 0$.

Set

$$\tilde{M}(t) = \int_0^t [M(s)]^{p-1}\, ds \qquad \text{for all } t \geq 0,$$

$$M^- := \min\{1, m_0^{p-1}, \rho\}$$

and

$$M^+ := \max\{1, m_1^{p-1}, \rho\}.$$

We mean by a (weak) solution of the problem (1), any function $u \in X$ such that

$$\begin{aligned} &\int_\Omega |\Delta u(x)|^{p-2} \Delta u(x) \Delta v(x)\, dx \\ &\quad + \left[M\left(\int_\Omega |\nabla u(x)|^p\, dx \right) \right]^{p-1} \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x)\, dx \\ &\quad + \rho \int_\Omega |u(x)|^{p-2} u(x) v(x)\, dx - \lambda \int_\Omega f\big(x, u(x)\big) v(x)\, dx \\ &\quad - \mu \int_\Omega g\big(x, u(x)\big) v(x)\, dx = 0 \end{aligned}$$

for every $v \in X$.

We need the following proposition in the proofs of Theorems 3.1 and 3.2.

Proposition 2.3. *Let $T : X \to X^*$ be the operator defined by*

$$T(u)h = \int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta h(x)\,dx + \left[M\Big(\int_\Omega |\nabla u(x)|^p\,dx\Big)\right]^{p-1}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla h(x)\,dx + \rho\int_\Omega |u(x)|^{p-2}u(x)h(x)\,dx$$

for every $u, h \in X$. Then T admits a continuous inverse on X^*.

Proof. Since

$$T(u)u \geq M^-\|u\|^p,$$

T is coercive. Taking into account (2.2) of [41] for $p > 1$ there exists a positive constant C_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq \begin{cases} C_p|x-y|^p & \text{if } p \geq 2, \\ C_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2 \end{cases}$$

where $\langle .,. \rangle$ denotes the usual inner product in $\mathbb{R}^N$, for every $x, y \in \mathbb{R}^N$. Thus, for $1 < p < 2$, it is easy to see that

$$\big(T(u) - T(v)\big)(u - v) \geq C_pM^- \int_\Omega \Big(\frac{|\Delta u(x) - \Delta v(x)|^2}{\big(|\Delta u(x)| + |\Delta v(x)|\big)^{2-p}} + \frac{|\nabla u(x) - \nabla v(x)|^2}{\big(|\nabla u(x)| + |\nabla v(x)|\big)^{2-p}} + \frac{|u(x) - v(x)|^2}{\big(|u(x)| + |v(x)|\big)^{2-p}}\Big)\,dx > 0$$

for every $u, v \in X$, $u \neq v$, which means that T is strictly monotone. For, $p \geq 2$, we also observe that

$$\big(T(u) - T(v)\big)(u - v) \geq C_pM^- \int_\Omega \big(|\Delta u(x) - \Delta v(x)|^p\,dx + |\nabla u(x) - \nabla v(x)|^p + |u(x) - v(x)|^p\big)\,dx > 0,$$

which means that T, in this case, is strictly monotone too. Moreover, we observe T is the dual mapping on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ corresponding to the gauge function $\phi_p(t) = t^{p-1}$. Hence T is demicontinuous. So by [45], Theorem 26.A(d), the inverse operator T^{-1} of T exists. T^{-1} is continuous. Indeed, let (α_n) be a sequence of X^* such that $\alpha_n \to \alpha$ strongly in $\left(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right)^*$ as $n \to +\infty$. Let u_n and u in X such that $T^{-1}(\alpha_n) = u_n$ and $T^{-1}(\alpha) = u$. Taking in to account that T is coercive, one has that the sequence u_n is bounded in the reflexive space X. For a suitable subsequence, we have $u_n \to \hat{u}$ weakly in X as $n \to +\infty$, which concludes

$$\lim_{n\to+\infty} \langle T(u_n) - T(u), u_n - \hat{u} \rangle = \langle \alpha_n - \alpha, u_n - \hat{u} \rangle = 0.$$

Note that if $u_n \to \hat{u}$ weakly in X as $n \to +\infty$ and $T(u_n) \to T(\hat{u})$ strongly in X^* as $n \to +\infty$, one has $u_n \to \hat{u}$ strongly in X as $m \to +\infty$, and since T is continuous we have $u_n \to \hat{u}$ weakly in X as $n \to +\infty$ and $T(u_n) \to T(\hat{u}) = T(u)$ strongly in X^* as $n \to +\infty$. Hence, taking into account that T is an injection, we have $u = \hat{u}$. □

3. Main results

Fix $x^0 \in \Omega$ and pick $s > 0$ such that $B(x^0, s) \subset \Omega$ where $B(x^0, s)$ denotes the ball with center at x^0 and radius of s. Put

$$\theta_1 := \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_{s/2}^{s} \left| \frac{12(N+1)}{s^3} r - \frac{24N}{s^2} + \frac{9(N-1)}{s} \frac{1}{r} \right|^p r^{N-1}\, dr,$$

$$\theta_2 := \int_{B(x^0,s)\setminus B(x^0,s/2)} \left[\Sigma_{i=1}^{N} \left(\frac{12\ell(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9(x_i - x_i^0)}{s\ell} \right)^2 \right]^{p/2} dx,$$

$$\theta_3 := \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \left[\frac{\left(\frac{s}{2}\right)^N}{N} + \int_{s/2}^{s} \left| \frac{4}{s^3} r^3 - \frac{12}{s^2} r^2 + \frac{9}{s} r - 1 \right|^p r^{N-1}\, dr \right]$$

where Γ denotes the Gamma function, and $\ell = \operatorname{dist}(x, x^0) = \sqrt{\Sigma_{i=1}^{N} (x_i - x_i^0)^2}$, and

$$L := \theta_1 + \theta_2 + \theta_3. \tag{5}$$

In order to introduce our first result, fixing two positive constants c and d such that

$$\frac{d^p M^+ L}{\int_{B(x^0,s/2)} F(x,d)\, dx} < \frac{M^- \left(\frac{c}{k}\right)^p}{\int_\Omega \sup_{|t| \le c} F(x,t)\, dx}$$

and taking

$$\lambda \in \Lambda := \left] \frac{\frac{d^p}{p} M^+ L}{\int_{B(x^0,s/2)} F(x,d)\, dx}, \frac{\frac{M^-}{p} \left(\frac{c}{k}\right)^p}{\int_\Omega \sup_{|t| \le c} F(x,t)\, dx} \right[,$$

put

$$\delta_{\lambda,g} := \min\left\{\frac{M^- c^p - \lambda p k^p \int_\Omega \sup_{|t|\le c} F(x,t)\,dx}{p k^p G^c}, \frac{M^+ L d^p - \lambda p \int_{B(x^0,s/2)} F(x,d)\,dx}{p|\Omega| G_d}\right\} \tag{6}$$

and

$$\bar{\delta}_{\lambda,g} := \min\left\{\delta_{\lambda,g}, \frac{1}{\max\left\{0, \frac{pk^p|\Omega|}{M^-} \limsup_{|t|\to\infty} \frac{\sup_{x\in\Omega} G(x,t)}{t^p}\right\}}\right\} \tag{7}$$

where that for instance $\bar{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in\Omega} G(x,t)}{t^p} \le 0,$$

and $G_d = G^c = 0$.

Now, we formulate our main result.

Theorem 3.1. *Assume that there exist two positive constants c and d with* $\frac{c}{\sqrt[p]{L}} < d$ *such that*

(A1) $\int_{\Omega\setminus B(x^0,s/2)} F(x,\xi)\,dx \ge 0$ *for each* $\xi \in [0,d]$*;*

(A2) $\frac{\int_\Omega \sup_{|t|\le c} F(x,t)\,dx}{c^p} < \frac{M^-}{M^+ L k^p} \frac{\int_{B(x^0,s/2)} F(x,d)\,dx}{d^p}$ *where L is given by* (5)*;*

(A3) $\limsup_{|t|\to+\infty} \frac{\sup_{x\in\Omega} F(x,t)}{t^p} \le 0.$

Then, for each $\lambda \in \Lambda$ *and for every* L^1*-Carathéodory function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfying the condition*

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in\Omega} G(x,t)}{t^p} < +\infty,$$

there exists $\bar{\delta}_{\lambda,g} > 0$ *given by* (7) *such that, for each* $\mu \in [0, \bar{\delta}_{\lambda,g}[$*, the problem* (1) *admits at least three distinct weak solutions in X.*

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u(x)|^p\,dx + \frac{1}{p}\tilde{M}\left[\int_\Omega |\nabla u(x)|^p\,dx\right] + \frac{\rho}{p}\int_\Omega |u(x)|^p\,dx$$

and

$$\Psi(u) = \int_\Omega \left[F\big(x, u(x)\big) + \frac{\mu}{\lambda} G\big(x, u(x)\big) \right] dx.$$

Let us prove that the functionals Φ and Ψ satisfy the required conditions. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_\Omega \left[f\big(x, u(x)\big) + \frac{\mu}{\lambda} g\big(x, u(x)\big) \right] v(x)\, dx$$

for every $v \in X$ as well as is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$, then u_n converges uniformly to u on Ω as $n \to \infty$; see [45]. Since f, g are continuous functions, f, g are continuous in $\mathbb{R}$ for every $x \in \Omega$, so

$$f(x, u_n) + \frac{\mu}{\lambda} g(x, u_n) \to f(x, u) + \frac{\mu}{\lambda} g(x, u)$$

as $n \to \infty$. Hence $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [45].

Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned}\Phi'(u)(v) = {} & \int_\Omega |\Delta u(x)|^{p-2} \Delta u(x) \Delta v(x)\, dx \\ & + \left[M\left(\int_\Omega |\nabla u(x)|^p\, dx \right) \right]^{p-1} \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x)\, dx \\ & + \rho \int_\Omega |u(x)|^{p-2} u(x) v(x)\, dx\end{aligned}$$

for every $v \in X$, while Proposition 2.3 gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous.

Clearly, the weak solutions of the problem (1) are exactly the solutions of the equation $\Phi'(u) - \lambda \Psi'(u) = 0$. Put $r := \dfrac{M^-}{p} \left(\dfrac{c}{k} \right)^p$ and

$$w(x) := \begin{cases} 0 & \text{if } x \in \overline{\Omega} \backslash B(x^0, s) \\ d\big(\frac{4}{s^3} \ell^3 - \frac{12}{s^2} \ell^2 + \frac{9}{s} \ell - 1 \big) & \text{if } x \in B(x^0, s) \backslash B\big(x^0, \frac{s}{2}\big) \\ d & \text{if } x \in B\big(x^0, \frac{s}{2}\big). \end{cases} \tag{8}$$

It is easy to see that $w \in X$ and,

$$\frac{\partial w(x)}{\partial x_i} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \backslash B(x^0, s) \cup B\left(x^0, \frac{s}{2}\right) \\ d\left(\frac{12\ell(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{\ell}\right) & \text{if } x \in B(x^0, s) \backslash B\left(x^0, \frac{s}{2}\right) \end{cases}$$

and

$$\frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \backslash B(x^0, s) \cup B\left(x^0, \frac{s}{2}\right) \\ d\left(\frac{12}{s^3}\frac{(x_i - x_i^0)^2 + \ell^2}{\ell} - \frac{24}{s^2} + \frac{9}{s}\frac{\ell^2 - (x_i - x_i^0)^2}{\ell^3}\right) & \text{if } x \in B(x^0, s) \backslash B\left(x^0, \frac{s}{2}\right), \end{cases}$$

and so that

$$\sum_{i=1}^{N} \frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \backslash B(x^0, s) \cup B\left(x^0, \frac{s}{2}\right) \\ d\left(\frac{12l(N+1)}{s^3} - \frac{24N}{s^2} + \frac{9}{s}\frac{N-1}{\ell}\right) & \text{if } x \in B(x^0, s) \backslash B\left(x^0, \frac{s}{2}\right). \end{cases}$$

It is easy to see that $w \in X$ and, in particular, since

$$\int_\Omega |\Delta w(x)|^p \, dx = d^p \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_{s/2}^{s} \left| \frac{12(N+1)}{s^3} r - \frac{24N}{s^2} + \frac{9(N-1)}{s} \frac{1}{r} \right|^p r^{N-1} \, dr,$$

$$\begin{aligned} &\int_\Omega |\nabla w(x)|^p \, dx \\ &= \int_{B(x^0,s)\backslash B(x^0,s/2)} \left[\sum_{i=1}^{N} d^2 \left(\frac{12l(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{l} \right)^2 \right]^{p/2} dx \\ &= d^p \int_{B(x^0,s)\backslash B(x^0,s/2)} \left[\sum_{i=1}^{N} \left(\frac{12l(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{l} \right)^2 \right]^{p/2} dx \end{aligned}$$

and

$$\int_\Omega |w(x)|^p \, dx = d^p \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \left(\frac{\left(\frac{s}{2}\right)^N}{N} + \int_{s/2}^{s} \left| \frac{4}{s^3} r^3 - \frac{12}{s^2} r^2 + \frac{9}{s} r - 1 \right|^p r^{N-1} \, dr \right),$$

In particular, one has

$$\begin{aligned} \frac{d^p}{p} M^- L &\le \frac{1}{p}\left(\theta_1 d^p + m_0^{p-1}\theta_2 d^p + \rho\theta_3 d^p\right) \le \Phi(w) = \frac{1}{p}\left(\theta_1 d^p + \tilde{M}(\theta_2 d^p) + \rho\theta_3 d^p\right) \\ &\le \frac{1}{p}\left(\theta_1 d^p + m_1^{p-1}\theta_2 d^p + \rho\theta_3 d^p\right) \le \frac{d^p}{p} M^+ L. \end{aligned}$$

Taking the condition $\frac{c}{\sqrt[p]{L}} < d$ into account, we observe

$$0 < r < \Phi(w).$$

Since $\frac{M^-}{p}\|u\|^p \le \Phi(u)$ for each $u \in X$ and bearing (4) in mind, we see that

$$\begin{aligned}\Phi^{-1}(]-\infty, r]) &= \{u \in X; \Phi(u) \le r\}\\ &\subseteq \left\{u \in X; \frac{M^-}{p}\|u\|^p \le r\right\}\\ &\subseteq \{u \in X; |u(x)| \le c \text{ for each } x \in \Omega\},\end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) &= \sup_{u \in \Phi^{-1}(]-\infty, r])} \int_\Omega \left[F\big(x, u(x)\big) + \frac{\mu}{\lambda} G\big(x, u(x)\big)\right] dx\\ &\le \int_\Omega \sup_{|t| \le c} F(x, t)\, dx + \frac{\mu}{\lambda} G^c.\end{aligned}$$

On the other hand, by using condition (A1), we deduce

$$\begin{aligned}\Psi(w) &\ge \int_{B(x^0, s/2)} F(x, d)\, dx + \frac{\mu}{\lambda}\int_\Omega G(x, d)\, dx\\ &\ge \int_{B(x^0, s/2)} F(x, d)\, dx + |\Omega|\frac{\mu}{\lambda} \inf_{\Omega \times [0, d]} G\\ &= \int_{B(x^0, s/2)} F(x, d)\, dx + |\Omega|\frac{\mu}{\lambda} G_d.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} &= \frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \int_\Omega \left[F\big(x, u(x)\big) + \frac{\mu}{\lambda} G\big(x, u(x)\big)\right] dx}{r}\\ &\le \frac{\int_\Omega \sup_{|t| \le c} F(x, t)\, dx + \frac{\mu}{\lambda} G^c}{\frac{M^-}{p}\left(\frac{c}{k}\right)^p},\end{aligned} \tag{9}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\ge \frac{\int_{B(x^0, s/2)} F(x, d)\, dx + \frac{\mu}{\lambda}\int_\Omega G\big(x, w(x)\big)\, dx}{\frac{d^p}{p} M^+ L}\\ &\ge \frac{\int_{B(x^0, s/2)} F(x, d)\, dx + |\Omega|\frac{\mu}{\lambda} G_d}{\frac{d^p}{p} M^+ L}.\end{aligned} \tag{10}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{M^- c^p - \lambda p k^p \int_\Omega \sup_{|t| \le c} F(x,t)\,dx}{p k^p G^c},$$

that is,

$$\frac{\int_\Omega \sup_{|t| \le c} F(x,t)\,dx + \frac{\mu}{\lambda} G^c}{\frac{M^-}{p}\left(\frac{c}{k}\right)^p} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{M^+ L d^p - \lambda p \int_{B(x^0, s/2)} F(x,d)\,dx}{p|\Omega| G_d},$$

that is,

$$\frac{\int_{B(x^0, s/2)} F(x,d)\,dx + |\Omega| \frac{\mu}{\lambda} G_d}{\frac{d^p}{p} M^+ L} > \frac{1}{\lambda}.$$

Then,

$$\frac{\int_\Omega \sup_{|t| \le c} F(x,t)\,dx + \frac{\mu}{\lambda} G^c}{\frac{M^-}{p}\left(\frac{c}{k}\right)^p} < \frac{1}{\lambda} < \frac{\int_{B(x^0, s/2)} F(x,d)\,dx + |\Omega| \frac{\mu}{\lambda} G_d}{\frac{d^p}{p} M^+ L}. \tag{11}$$

Hence from (9)–(11), we observe that the condition (a_1) of Theorem 2.1 is fulfilled.

Finally, since $\mu < \bar{\delta}_{\lambda,g}$, we can fix $l > 0$ such that $\limsup_{|t|\to\infty} \frac{\sup_{x\in\Omega} G(x,t)}{t^p} < l$ and $\mu l < \frac{M^-}{p k^p |\Omega|}$. Therefore, there exists a function $h \in L^1(\Omega)$ such that

$$G(x,t) \le l t^p + h(x), \tag{12}$$

for every $x \in \Omega$ and $t \in \mathbb{R}$.

Now, fix $0 < \varepsilon < \frac{M^-}{p k^p |\Omega| \lambda} - \frac{\mu l}{\lambda}$. From (A3) there is a function $h_\varepsilon \in L^1(\Omega)$ such that

$$F(x,t) \le \varepsilon t^p + h_\varepsilon(x), \tag{13}$$

for every $x \in \Omega$ and $t \in \mathbb{R}$.

Taking (4) into account, it follows that, for each $u \in X$,

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &\geq \frac{M^-}{p}\|u\|^p - \lambda\int_\Omega \left[F\big(x, u(x)\big) + \frac{\mu}{\lambda}G\big(x, u(x)\big)\right] dx \\ &\geq \frac{M^-}{p}\|u\|^p - \lambda\varepsilon\int_\Omega u^p(x)\,dx - \lambda\|h_\varepsilon\|_{L^1(\Omega)} \\ &\quad - \mu l\int_\Omega u^p(x)\,dx - \mu\|h\|_{L^1(\Omega)} \\ &\geq \left(\frac{M^-}{p} - \lambda k^p|\Omega|\varepsilon - \mu k^p|\Omega| l\right)\|u\|^p - \lambda\|h_\varepsilon\|_{L^1(\Omega)} - \mu\|h\|_{L^1(\Omega)},\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}\big(\Phi(u) - \lambda\Psi(u)\big) = +\infty,$$

which means the functional $\Phi - \lambda\Psi$ is coercive, and the condition (a_2) of Theorem 2.1 is verified.

Since from relations (9)–(11),

$$\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right[,$$

Theorem 2.1 (with $\bar{v} = w$) assures the desired conclusion. □

Now, a variant of Theorem 2.2 in which no asymptotic condition on g is requested, is pointed out. In such a case f and g are supposed to be nonnegative.

Theorem 3.2. *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function satisfies the condition $f(x,t) \geq 0$ for all $x \in \Omega$ and $t \geq 0$. Assume that there exist three positive constants c_1, c_2 and d with $c_1 < k\left(\frac{L}{2}\right)^{1/p} d$ and $k\left(\frac{2M^+L}{M^-}\right)^{1/p} d < c_2$ such that*

$$\max\left\{\frac{\int_\Omega \sup_{|t|\leq c_1} F(x,t)\,dx}{c_1^p}, \frac{2\int_\Omega \sup_{|t|\leq c_2} F(x,t)\,dx}{c_2^p}\right\} < \frac{2}{3}\frac{M^-}{M^+Lk^p}\frac{\int_{B(x^0,s/2)} F(x,d)\,dx}{d^p}.$$

Then, for each

$$\lambda \in \Lambda' := \left]\frac{3}{2}\frac{M^+Ld^p}{p\int_{B(x^0,s/2)} F(x,d)\,dx}, \frac{M^-}{pk^p}\min\left\{\frac{c_1^p}{\int_\Omega \sup_{|t|\leq c_1} F(x,t)\,dx}, \frac{c_2^p}{2\int_\Omega \sup_{|t|\leq c_2} F(x,t)\,dx}\right\}\right[$$

and for every nonnegative L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta^*_{\lambda,g} > 0$ given by

$$\min\left\{\frac{M^- c_1^p - \lambda p k^p \int_\Omega \sup_{|t|\le c_1} F(x,t)\,dx}{pk^p G^{c_1}}, \frac{M^- c_2^p - 2\lambda p k^p \int_\Omega \sup_{|t|\le c_2} F(x,t)\,dx}{2pk^p G^{c_2}}\right\}.$$

such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$, the problem (1) *admits at least three distinct weak solutions u_i for $i = 1,2,3$, such that*

$$0 \le u_i(x) < c_2, \qquad \forall x \in \Omega,\ (i = 1,2,3).$$

Proof. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on Φ and Ψ are satisfied. Then, our aim is to verify (b_1) and (b_2). To this end, put w as given in (8), as well as $r_1 := \frac{M^-}{p}\left(\frac{c_1}{k}\right)^p$ and $r_2 := \frac{M^-}{p}\left(\frac{c_2}{k}\right)^p$. Using the conditions $c_1 < k\left(\frac{L}{2}\right)^{1/p} d$ and $k\left(\frac{2M^+L}{M^-}\right)^{1/p} d < c_2$ and bearing in mind that

$$\frac{d^p}{p} M^- L \le \Phi(w) \le \frac{d^p}{p} M^+ L,$$

we get

$$2r_1 < \Phi(w) < \frac{r_2}{2}.$$

Since $\mu < \delta^*_{\lambda,g}$ and $G_d = 0$, one has

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{r_1} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\int_\Omega \left[F\big(x,u(x)\big) + \frac{\mu}{\lambda} G\big(x,u(x)\big)\right] dx}{r_1} \\
&\le \frac{\int_\Omega \sup_{|t|\le c_1} F(x,t)\,dx + \frac{\mu}{\lambda} G^{c_1}}{\frac{M^-}{p}\left(\frac{c_1}{k}\right)^p} \\
&< \frac{1}{\lambda} < \frac{2}{3}\, \frac{\int_{B(x^0,s/2)} F(x,\eta)\,dx + |\Omega| \frac{\mu}{\lambda} G^d}{\frac{d^p}{p} M^+ L} \\
&\le \frac{2}{3}\, \frac{\Psi(w)}{\Phi(w)},
\end{aligned}$$

and

$$\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2])}\Psi(u)}{r_2}=\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2])}\int_\Omega\big[F\big(x,u(x)\big)+\frac{\mu}{\lambda}G\big(x,u(x)\big)\big]\,dx}{r_2}$$
$$\le\frac{2\int_\Omega\sup_{|t|\le c_2}F(x,t)\,dx+2\frac{\mu}{\lambda}G^{c_2}}{\frac{M^-}{p}\left(\frac{c_2}{k}\right)^p}$$
$$<\frac{1}{\lambda}<\frac{2}{3}\frac{\int_{B(x^0(s/2))}F(x,d)\,dx+|\Omega|\frac{\mu}{\lambda}G_d}{\frac{d^p}{p}M^+L}$$
$$\le\frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.$$

Therefore, (b_1) and (b_2) of Theorem 2.2 are verified.

Finally, we verify that $\Phi-\lambda\Psi$ satisfies the assumption 2. of Theorem 2.2. Let u_1 and u_2 be two local minima for $\Phi-\lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi-\lambda\Psi$, and so, they are weak solutions for the problem (1). We claim that they are nonnegative. As the same way given in [21], Theorem 3.2 let u be a weak solution of problem (1). Arguing by a contradiction, assume that the set $A=\{x\in\Omega:u(x)<0\}$ is non-empty and of positive measure. Put $\bar{u}(x)=\min\{0,u(x)\}$ for all $x\in\Omega$. Clearly, $\bar{u}\in X$ and one has

$$\begin{aligned}&\int_\Omega|\Delta u(x)|^{p-2}\Delta u(x)\Delta\bar{u}(x)\,dx\\&\quad+\left[M\left(\int_\Omega|\nabla u(x)|^p\,dx\right)\right]^{p-1}\int_\Omega|\nabla u(x)|^{p-2}\nabla u(x)\nabla\bar{u}(x)\,dx\\&\quad+\rho\int_\Omega|u(x)|^p u(x)\bar{u}(x)\,dx-\lambda\int_\Omega f\big(x,u(x)\big)\bar{u}(x)\,dx\\&\quad-\mu\int_\Omega g\big(x,u(x)\big)\bar{u}(x)\,dx=0.\end{aligned}$$

Thus, from our sign assumptions on the data, we have

$$0\le\int_A|\Delta u(x)|^p\,dx+\left[M\left(\int_A|\nabla u(x)|^p\,dx\right)\right]^{p-1}\int_A|\nabla u(x)|^p\,dx+\rho\int_\Omega|u(x)|^p\,dx\le 0.$$

Hence, $u=0$ in A and this is absurd. Then, we deduce $u_1(x)\ge 0$ and $u_2(x)\ge 0$ for every $x\in\Omega$, and our claim holds true. Thus, it follows that $su_1+(1-s)u_2\ge 0$ for all $s\in[0,1]$, and that

$$(\lambda f+\mu g)\big(x,su_1+(1-s)u_2\big)\ge 0,$$

and consequently, $\Psi\big(su_1+(1-s)u_2\big)\ge 0$, for every $s\in[0,1]$.

From Theorem 2.2, for every

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)}, \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}, \frac{r_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are the weak solutions of the problem (1) and the conclusion is achieved. □

A special case of Theorem 3.1 is the following theorem.

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and denote $F(t) := \int_0^t f(\xi)\, d\xi$ for each $t \in \mathbb{R}$. Assume that $F(d) > 0$ for some $d > 0$ and $F(\xi) \geq 0$ in $[0, d]$ and*

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^p} = \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} = 0.$$

Then there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the asymptotical condition

$$\limsup_{|t|\to\infty}\frac{\sup_{x\in\Omega}\int_0^t g(x,s)\, ds}{t^p} < +\infty,$$

there exists $\bar{\delta}_{\lambda,g} > 0$ such that for each $\mu \in [0, \bar{\delta}_{\lambda,g}[$, the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - [M(\int_\Omega |\nabla u|^p\, dx)]^{p-1}\Delta_p u + \rho|u|^{p-2}u = \lambda f(u) + \mu g(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least three weak solutions.

Proof. Fix $\lambda > \lambda^* := \dfrac{d^p M^+ L}{p\left(\frac{s}{2}\right)^N \frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})} F(d)}$ for some $d > 0$. Since

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^2} = 0,$$

there is a sequence $\{c_n\} \subset]0, +\infty[$ such that $\lim_{n\to\infty} c_n = 0$ and

$$\lim_{n\to\infty}\frac{\sup_{|\xi|\leq c_n} F(\xi)}{c_n^p} = 0.$$

Indeed, one has

$$\lim_{n\to\infty}\frac{\sup_{|\xi|\leq c_n} F(\xi)}{c_n^p} = \lim_{n\to\infty}\frac{F(\xi_{c_n})}{\xi_{c_n}^p}\frac{\xi_{c_n}^p}{c_n^p} = 0,$$

where $F(\xi_{c_n}) = \sup_{|\xi| \le c_n} F(\xi)$.

Hence, there exists $\bar{c} > 0$ such that

$$\frac{\sup_{|\xi| \le \bar{c}} F(\xi)}{\bar{c}^p} < \min\left\{\frac{M^-}{|\Omega| M^+ L k^p}\left(\frac{s}{2}\right)^N \frac{\pi^{N/2}}{\Gamma\left(1+\frac{N}{2}\right)} \frac{F(d)}{d^p}; \frac{M^-}{pk^p|\Omega|\lambda}\right\}$$

and $\dfrac{\bar{c}}{\sqrt[p]{L}} < d$. From Theorem 3.1 the conclusion follows. □

Here, as an example, we point out a consequence of Theorem 3.2 as follows.

Theorem 3.4. *Suppose that* $\dfrac{M^+}{M^-} < \dfrac{1}{58.18309}$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a nonnegative continuous function such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t^2} = 0$$

and

$$\int_0^2 f(\xi)\, d\xi < \frac{2}{27 \times 58.18309} \frac{M^-}{M^+} \int_0^1 f(\xi)\, d\xi.$$

Then, for every $\lambda \in \left]\dfrac{58.18309 M^+}{2\int_0^1 f(\xi)\, d\xi}, \dfrac{M^-}{27\int_0^2 f(\xi)\, d\xi}\right[$ *and for every* L^1*-Carathéodory nonnegative function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *there exists* $\bar{\delta}^*_{\lambda,g} > 0$ *such that for each* $\mu \in [0, \bar{\delta}^*_{\lambda,g}[$, *the problem*

$$\begin{cases} \Delta(|\Delta u|\Delta u) - [M(\int_\Omega |\nabla u|^3\, dx)]^2 \Delta_3 u + |u|u = \lambda f(u) + \mu g(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega = \{(x, y); x^2 + y^2 < 9\}$, *admits at least three weak solutions.*

Proof. Our aim is to employ Theorem 3.2 by choosing $\rho = 1$, $p = 3$, $x^0 = 0$, $s = 2$, $c_2 = 2$ and $d = 1$. Therefore, since $k = \sqrt[3]{\dfrac{4}{\pi}}$, $L = 58.18309\pi$, we see that

$$\frac{3}{2} \frac{M^+ L d^p}{p\int_{B(x^0, s/2)} F(x, d)\, dx} = \frac{58.18309 M^+}{2\int_0^1 f(\xi)\, d\xi}$$

and

$$\frac{M^-}{pk^p} \frac{c_2^p}{2\int_\Omega \sup_{|t| \le c_2} F(x, t)\, dx} = \frac{M^-}{27\int_0^2 f(\xi)\, d\xi}.$$

Moreover, since $\lim_{t\to 0^+} \dfrac{f(t)}{t^2} = 0$, one has

$$\lim_{t\to 0^+} \frac{\int_0^t f(\xi)\,d\xi}{t^3} = 0.$$

Then, there exists a positive constant $c_1 < k\Big(\dfrac{58.18309}{2}\Big)^{1/p} d$ such that

$$\frac{\int_0^{c_1} f(\xi)\,d\xi}{c_1^3} < \frac{M^-}{58.18309M^+}\frac{\int_0^1 f(\xi)\,d\xi}{54}$$

and

$$\frac{c_1^3}{\int_0^{c_1} f(\xi)\,d\xi} > \frac{4}{\int_0^2 f(\xi)\,d\xi}.$$

Finally, from our hypotheses, a simple computation show that all assumptions of Theorem 3.2 are fulfilled. The desired conclusion follows. □

Now, let $\alpha \geq 0$ and $\beta > 0$ be two real numbers, let $M : [\alpha, \beta] \subseteq [0, +\infty[\to \mathbb{R}$ be a function defined by $M(t) = a + bt$ for each $t \in [\alpha, \beta]$ where $a, b > 0$. Put $M_0 = \min\{1, (a + b\alpha)^{p-1}, \rho\}$. We consider the following problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - (a + b\int_\Omega |\nabla u|^p\,dx)^{p-1}\Delta_p u + \rho|u|^{p-2}u & \text{in } \Omega \\ \qquad = \lambda f(x,u) + \mu g(x,u) & \\ u = \Delta u = 0 & \text{on } \Omega. \end{cases} \tag{14}$$

The following existence results are immediate consequences of Theorems 3.1 and 3.2, respectively.

Corollary 3.5. *Assume that there exist two positive constants ac and d such that Assumptions* (A1) *and* (A3) *in Theorem* 2.2 *hold, and*

(A4) $\theta_1 d^p + \dfrac{1}{ap}(a + b\theta_2 d^p)^p + \rho\theta_3 d^p - \dfrac{1}{ap}a^p > M_0\Big(\dfrac{c}{k}\Big)^p$;

(A5) $\dfrac{\int_\Omega \sup_{|t|\leq c} F(x,t)\,dx}{c^p} < \dfrac{M_0}{k^p}\dfrac{\int_{B(x^0, s_1/2)} F(x,d)\,dx}{\theta_1 d^p + \frac{1}{bp}(a + b\theta_2 d^p)^p + \rho\theta_3 d^p - \frac{1}{bp}a^p}$.

Then, for each

$$\lambda \in \bar{\Lambda} := \left]\frac{1}{p}\frac{\theta_1 d^p + \frac{1}{bp}(a + \theta_2 d^p)^p + \rho\theta_3 d^p - \frac{1}{bp}a^p}{\int_{B(x^0, s_1/2)} F(x,d)\,dx}, \frac{1}{p}\frac{M_0\big(\frac{c}{k}\big)^p}{\int_\Omega \sup_{|t|\leq c} F(x,t)\,dx}\right[$$

and for every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|t|\to+\infty} \frac{\sup_{x\in\Omega} G(x,t)}{t^p} < +\infty,$$

there exists $\bar{\delta}'_{\lambda,g} > 0$ where

$$\bar{\delta}'_{\lambda,g} := \min\left\{\delta'_{\lambda,g}, \frac{1}{\max\left\{0, \frac{pk^p|\Omega|}{M^-} \limsup_{|t|\to\infty} \frac{\sup_{x\in\Omega} G(x,t)}{t^p}\right\}}\right\}$$

with

$$\delta'_{\lambda,g} := \min\left\{\frac{M_0 c^p - \lambda p k^p \int_\Omega \sup_{|t|\le c} F(x,t)\,dx}{pk^p G^c}, \right.$$
$$\left. \frac{\frac{1}{p}\left(\theta_1 d^p + \frac{1}{bp}(a+b\theta_2 d^p)^p + \rho\theta_3 d^p - \frac{1}{bp}a^p\right) - p\lambda \int_{B(x^0,s_1/2)} F(x,d)\,dx}{|\Omega| p G^d}\right\}$$

such that for each $\mu \in [0, \bar{\delta}'_{\lambda,g}[$, the problem (14) *admits at least there distinct weak solutions in X.*

Proof. Bearing in mind that $m_0 = a + b\alpha$, like for Theorem 3.2, since in this case $\Phi(w) = \theta_1 d^p + \frac{1}{bp}(a+\theta_2 d^p)^p d^p + \rho\theta_3 d^p - \frac{1}{bp}a^p$ where w is given as in (8), owing to our assumptions, the conclusion follows from Theorem 2.1. □

Corollary 3.6. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the condition $f(x,t) \ge 0$ for all $(x,t) \in \Omega \times (\mathbb{R}^+ \cup \{0\})$. Assume that there exist three positive constants c_1, c_2 and d such that*

(B1) $2M_0\left(\frac{c_1}{k}\right)^p < \theta_1 d^p + \frac{1}{bp}(a+b\theta_2 d^p)^p + \rho\theta_3 d^p - \frac{1}{bp}a^p < \frac{M_0}{2}\left(\frac{c_2}{k}\right)^p$

(B2)

$$\max\left\{\frac{\int_\Omega \sup_{|t|\le c_1} F(x,t)\,dx}{c_1^p}, \frac{2\int_\Omega \sup_{|t|\le c_2} F(x,t)\,dx}{c_2^p}\right\}$$
$$< \frac{2}{3}\frac{M_0}{k^p} \frac{\int_{B(x^0,s/2)} F(x,d)\,dx}{\theta_1 d^p + \frac{1}{bp}(a+b\theta_2 d^p)^p + \rho\theta_3 d^p - \frac{1}{bp}a^p}.$$

Then, for each

$$\lambda \in \bar{\Lambda}' := \left]\frac{3}{2p}\frac{\theta_1 d^p + \frac{1}{bp}(a + b\theta_2 d^p)^p d^p + \rho\theta_3 d^p - \frac{1}{bp}a^p}{\int_{B(x^0, s_1/2)} F(x,d)\,dx},\right.$$
$$\left.\frac{M_0}{pk^p}\min\left\{\frac{c_1^p}{\int_\Omega \sup_{|t|\le c_1} F(x,t)\,dx}, \frac{c_2^p}{2\int_\Omega \sup_{|t|\le c_2} F(x,t)\,dx}\right\}\right[,$$

and for every nonnegative L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta'^*_{\lambda,g} > 0$ where

$$\delta'^*_{\lambda,g} := \min\left\{\frac{M_0 c_1^p - pk^p\lambda\int_\Omega \sup_{|t|\le c_1} F(x,t)\,dx}{pk^p G^{c_1}},\right.$$
$$\left.\frac{M_0 c_2^p - 2p\lambda k^p\int_\Omega \sup_{|t|\le c_2} F(x,t)\,dx}{2pk^p G^{c_2}}\right\}$$

such that for each $\mu \in [0, \delta'^*_{\lambda,g}[$, the problem (14) *admits at least three distinct weak solutions u_i $(i = 1,2,3)$, such that*

$$0 \le u_i(x) < c_2, \qquad \forall x \in \Omega.$$

Proof. Bearing in mind that $m_0 = a + b\alpha$, like for Theorem 3.2, since in this case $\Phi(w) = \theta_1 d^p + \dfrac{1}{bp}(a + \theta_2 d^p)^p + \rho\theta_3 d^p - \dfrac{1}{bp}a^p$ where w is given as in (8), owing to our assumptions, the conclusion follows from Theorem 2.2. □

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