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Discrete convexity built on differences

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Abstract. We shall study a new class of functions in discrete convexity which is defined using difference operators. The relation between this new class and the class of integrally convex functions is studied.

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1. Introduction

Three important problems related to the properties of convex functions of real variables are the following:

(I) Local minima: A local minimum of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is global.

(II) Marginal functions: The marginal function $H(x) = \inf_{y \in \mathbb{R}^m} F(x, y)$ of a convex function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is convex.

(III) Separating hyperplanes: there exists a separating hyperplane between two disjoint convex set.

A natural way to define discrete convexity of a function defined on the integer points is to call it convex if it admits a convex extension. This definition leads to difficulties when we try to prove the three properties mentioned. Some examples are given in [3], [4], [5] and [9]. Therefore it would be of interest to define a discrete analogue of convex functions which can serve in this context.

To improve the notion of convexity in $\mathbb{Z}^n$ and to get discrete analogues of these theorems, especially that on local minima, several types of discrete convexity have been studied. Miller [6] investigated discrete-convex functions. These functions guarantee that a local minimum is global; however they are not in general convex extensible (Murota and Shioura [11]).

Favati and Tardella [1] introduced the concept of integrally convex functions. This class was defined by using concepts from real convex analysis. The definition of integral convexity depends on the convex extension of a function on each unit hypercube and also the convexity of these extensions on the whole domain (definitions will be given in section 3). As an obvious conclusion from its definition, an integrally convex function is convex extensible. These functions are of interest because they preserve the property of local minimum. The two other properties mentioned were not studied in this paper. A study on the three properties for the two-dimensional case was done by Kiselman [3], [4], and Kiselman and Samieinia [5]. Adding submodularity to this class, Favati and Tardella [1] went on to show that the problem of minimizing a submodular integrally convex function over a bounded discrete rectangle can be solved in polynomial time.

The concept of M-convexity was introduced by Murota [7]. Then Murota and Shioura [10] introduced $\mathrm{M}^\natural$-convexity. Due to some relations between these two classes, all theorems stated for M-convexity are true for $\mathrm{M}^\natural$-convexity and vice versa. An $\mathrm{M}^\natural$-convex function is supermodular, but the converse does not hold (see Murota [9]:145–146). More information about the problem of local minimum for these two types of functions can be found in [9]. Integral convexity of $\mathrm{M}^\natural$-convex functions were studied by Murota and Shioura [11]:170.

Murota [8] introduced another form of well-behaved discrete convex functions, namely L-convex functions, as a collection of submodular functions with an extra condition. $\mathrm{L}^\natural$-convex functions were introduced by Fujishige and Murota [2] as a variant of L-convexity. More information about local minima and other properties of this class can be found in Murota [9].

Murota and Shioura [11] compared the class of M-convex (L-convex) functions with other kinds of discrete convexity. They also studied the behavior of theses classes under some operations.

Kiselman [3], and Kiselman and Samieinia [5] showed by some examples the importance of these three problems in the discrete case. Kiselman [3], [4] limited his work to $\mathbb{Z}^2$ and introduced a new definition for two-dimensional integrally convex functions, and as a result of this definition, it is evident that the class of two-dimensional integrally convex functions is closed under addition. Example 4.4 in [11] shows that the closedness under addition is not true in general. His definition is based on the differences of functions on the integer points and then he compared it with the old definition of integrally convex functions on $\mathbb{Z}^2$. He went on to show the equivalent relations between these two definitions. We shall look at the notion of convexity which was defined in [3], [4]. We generalize the class to the n-dimensional case by choosing a special set of points. Then we shall show that for functions with finite values, we can check the properties for a smaller number of points.

We compare this new class with the class of integrally convex functions, and see that the result of equivalence in the two-dimensional case is not true for higher dimensions. To be able to compare these two types of convexity we use a statement which is similar to that of Favati and Tardella [1], but with a smaller number of points. In subsection 3 we shall show that this new class of functions is integrally convex. An example shows that the converse does not hold. It is also clear that we have the property of local minimum for this new class of functions.

2. Lateral convexity

Kiselman [3], [4] defined a notion of convexity which is based on differences of values of the functions at certain points. He showed that this class is equivalent to the class of integrally convex functions in the two-dimensional case. Using his definition of difference operators, we shall generalize the class of functions introduced by him to higher dimensions. First we define this class of functions when they have $\mathbb{R}_! = \mathbb{R} \cup \{-\infty, +\infty\}$ as their codomain. Then we shall show that for codomain $\mathbb{R}$ we need only a smaller number of points to check.

2.1. Background. For a given $a \in \mathbb{Z}^n$, a difference operator D_a on functions $f : \mathbb{Z}^n \to \mathbb{R}$ is defined by

$$(D_a f)(x) = f(x + a) - f(x), \qquad x \in \mathbb{Z}^n.$$

It is an operator from $\mathbb{R}^{\mathbb{Z}^n}$ to $\mathbb{R}^{\mathbb{Z}^n}$. This operator can be defined also for functions $F : \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$, and it operates also on functions $f : \mathbb{Z}^n \to \mathbb{Z}$, $a \in \mathbb{Z}^n$, keeping the integers as values.

In this paper, we shall also use the second-order operator $D_b D_a$ which is given by

$$(D_b D_a f)(x) = f(x + a + b) - f(x + a) - f(x + b) + f(x).$$

We also shall work with functions with codomain $\mathbb{R}_!$. In order to calculate with infinities we need to use an extension of the operation of addition $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to the extended real numbers $\mathbb{R}_! = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$ by defining upper addition. It is defined as commutative and associative operation $\mathbb{R}_! \times \mathbb{R}_! \to \mathbb{R}_!$ which satisfy

$$(+\infty) \dot{+} x = +\infty \qquad \text{for all } x \in \mathbb{R}_!,$$
$$(-\infty) \dot{+} x = -\infty \qquad \text{for all } x \in [-\infty, +\infty[.$$

Using the upper addition, D_a for functions $f : \mathbb{Z}^n \to \mathbb{R}_!$, can be extended to $(D_a)_! : (\mathbb{R}_!)^{\mathbb{Z}^n} \to (\mathbb{R}_!)^{\mathbb{Z}^n}$ by

$$(D_a)_! f(x) = f(x + a) \dot{+} \big(-f(x)\big) \qquad x, a \in \mathbb{Z}^n.$$

The extension of D_bD_a to functions with infinite value is as follows:

$$(D_bD_a)_! f(x) = f(x+a+b) \dot{+} \big(-f(x+a)\big) \dot{+} \big(-f(x+b)\big) \dot{+} f(x).$$

The notion of A-lateral convexity was introduced in [5] as follows.

Definition 2.1. Given any subset A of $\mathbb{Z}^n \times \mathbb{Z}^n$ we shall say that a function $f : \mathbb{Z}^n \to \mathbb{R}$ is *A-laterally convex* if $D_bD_af \geq 0$ for all $(a,b) \in A$. For a function f with infinite values, $f : \mathbb{Z}^n \to \mathbb{R}_!$, we shall say that it is *A-laterally convex* if $(D_bD_a)_! f \geq 0$ for all $(a,b) \in A$.

We define the notion of A-laterally convex set by using the indicator function $\mathrm{Ind}_S : \mathbb{Z}^n \to \{0, +\infty\}$,

$$\mathrm{Ind}_S(x) = \begin{cases} 0, & x \in S, \\ +\infty, & x \notin S. \end{cases}$$

Thus a set $S \subseteq \mathbb{Z}^n$ is said to be A-laterally convex if its indicator function Ind_S is an A-laterally convex function.

Let us consider the set B defined by

$$\left\{ \Big(\sum_{j\in V} e_j, \sum_{j \in V} e_j + \sum_{i \notin V} a_i e_i \Big) \,\Big|\, \emptyset \neq V \subseteq [1,n]_{\mathbb{Z}} \text{ and } a_i \in [-1,1]_{\mathbb{Z}} \right\}.$$

The set B consists of pairs $(a,b) \in \mathbb{Z}^n \times \mathbb{Z}^n$ such that for an arbitrary and non-empty set $V \subseteq \{1, \ldots, n\}$, a has 1 in its i-th coordinate, $i \in V$, and 0 on the other coordinates. On the other hand the point b has the same coordinate i as a for $i \in V$, and the other coordinates have values $a_j = -1, 0, 1$, $j \notin V$. For $a_j = 0$ the positivity of D_bD_af implies the convex extensibility of the function along the axis of the variable x_i, $i \in V$. In the two-dimensional case the $a_j = -1, 1$ needs to prove the convexity of canonical extension which is an equivalent condition to integral convexity. Assuming the set B smaller than this case, yields difficulty in the convexity of canonical extension. The set B in the n-dimensional case plays the same role as we had for the dimension 2. Choosing a smaller number of points in the set B for functions with infinite values does not imply the convexity of canonical extension. However, for real-valued functions, B-lateral convexity can be checked using a smaller number of points. We shall show this in Theorem 2.2. Choosing a smaller number of points for the set C in the finite valued functions will be also impossible in order to have the integral convexity.

Theorem 2.2. *Let B be the set*

$$\left\{ \Big(\sum_{j\in V} e_j, \sum_{j \in V} e_j + \sum_{i \notin V} a_i e_i \Big) \,\Big|\, \emptyset \neq V \subseteq [1,n]_{\mathbb{Z}} \text{ and } a_i \in [-1,1]_{\mathbb{Z}} \right\},$$

and

$$C = \left\{ \left(e_j, e_j + \sum_{i \neq j} a_i e_i \right) \,\middle|\, j \in [1, n]_{\mathbb{Z}} \text{ and } a_i \in [-1, 1]_{\mathbb{Z}} \right\}. \tag{1}$$

A function $f : \mathbb{Z}^n \to \mathbb{R}$ is B-laterally convex if and only if it is C-laterally convex.

Proof. Since we have $C \subset B$, a B-laterally convex function is indeed C-laterally convex. To prove the converse, let us consider a C-laterally convex function $f : \mathbb{Z}^n \to \mathbb{R}$. We may assume that $V = \{v_1, \ldots, v_m\}$, $1 \leq m \leq n$, and that $v_i \in \{1, \ldots, n\}$. Suppose now that

$$(a, b) = \left(\sum_{j \in V} e_j, \sum_{j \in V} e_j + \sum_{i \notin V} a_i e_i \right) \in B, \qquad a_i \in [-1, 1]_{\mathbb{Z}}.$$

We shall show that $D_b D_a f \geq 0$. It is easy to verify the following calculation:

$$\begin{aligned} D_b D_a f(x) = D_b D_{e_{v_1}} f\Bigl(x + \sum_{i \in V \setminus \{v_1\}} e_i\Bigr) \\ + D_b D_{e_{v_2}} f\Bigl(x + \sum_{i \in V \setminus \{v_1, v_2\}} e_i\Bigr) + \cdots + D_b D_{e_{v_m}} f(x). \end{aligned} \tag{2}$$

This means that we have used the partial addition $(a, b) +_1 (c, b) = (a + c, b)$ as defined in [5]. We may write $b = e_{v_1} + \sum_{i \in V \setminus \{v_1\}} e_i + \sum_{i \notin V} a_i e_i$ which is equal to $b = e_{v_1} + \sum_{i \neq v_1} a_i' e_i$, for some $a_i' \in [-1, 1]_{\mathbb{Z}}$. Using the property of C-lateral convexity, we have

$$D_b D_{e_{v_1}} f\Bigl(x + \sum_{i \in V \setminus \{v_1\}} e_i\Bigr) \geq 0.$$

The same calculation for each summand of (2) implies that $D_b D_a f \geq 0$. Thus f is B-laterally convex. □

We shall now introduce an equivalent property for lateral convexity. Let us first define a metric $\delta_{k,l}$ for $k = 1, \ldots, n$ and $l \in \mathbb{R}$, $l > 0$, by

$$\delta_{k,l}(x, y) = \max\Bigl(\frac{1}{l} |x_k - y_k|, \max_{i \neq k, 1 \leq i \leq n} |x_i - y_i|\Bigr), \qquad x, y \in \mathbb{Z}^n.$$

Theorem 2.3. *A function $f : \mathbb{Z}^n \to \mathbb{R}$ is C-laterally convex, where C is defined by* (1), *if and only if we have the following property for all $1 \leq k \leq n$:*

$$f(x) + f(y) \geq f(x + e_k) + f(y - e_k), \tag{3}$$

for each two points x and y in $\mathbb{Z}^n$ where $y_k - x_k = 2$ and $\delta_{k,2}(x, y) = 1$.

Proof. Suppose that the equation (3) holds for the specific points mentioned in the statement. We shall show that a function $f : \mathbb{Z}^n \to \mathbb{R}$ is C-laterally convex. Let us consider a point $x \in \mathbb{Z}^n$. For $1 \le k \le n$, consider $(e_k, e_k + a) \in C$ where $a = \sum_{i \ne k} a_i$ and $a_i \in [-1, 1]_{\mathbb{Z}}$. We have

$$D_{e_k} D_{e_k + a} f(x) = f(x + 2e_k + a) - f(x + e_k + a) - f(x + e_k) + f(x).$$

Choose $y = x + 2e_k + a$. It is easy to see that $y_k - x_k = 2$ and $\delta_{k,2}(x, y) = 1$. Thus we are done.

We shall now prove the other direction. Let $f : \mathbb{Z}^n \to \mathbb{R}$ be C-laterally convex. Consider two points x and y in $\mathbb{Z}^n$ with $y_k - x_k = 2$ and $\delta_{k,2}(x, y) = 1$. Let $a_i = y_i - x_i$ for $1 \le i \ne k \le n$. Since $\delta_{k,2}(x, y) = 1$, $a_i \in [-1, 1]_{\mathbb{Z}}$. We can write equation (3) as

$$f(x) + f(y) - f(x + e_k) - f(y - e_k) \ge 0,$$

which is equal to $D_{e_k} D_{e_k + \sum_{i \ne k} a_i e_i} f(x)$. This, using the property of C-lateral convexity, will give the result. □

3. A characterization of integral convexity

In this section, we investigate relations between lateral and integral convexity. We first mention the definition of integral convexity as well as its characterization by using certain points. We then try to use a smaller number of points in order to make the investigation of the relation between lateral and integral convexity easier.

We state below the definition of integrally convex functions introduced by Favati and Tardella [1]. To define this class, they start with the canonical extension of f which comes as follows:

Definition 3.1. Given a function $f : \mathbb{Z}^n \to \mathbb{R}$, we define its *canonical extension* by $\mathrm{can}(f) : \mathbb{R}^n \to \mathbb{R}$ as the convex envelope of f on each unit hypercube $a + [0, 1]^n$, $a \in \mathbb{Z}^n$.

Then they went on to introduce integral convexity:

Definition 3.2. A function $f : \mathbb{Z}^n \to \mathbb{R}$ is called integrally convex if its canonical extension $\mathrm{can}(f)$ is convex.

They also characterize integral convexity by considering a special property for certain points.

Proposition 3.3 (Favati and Tardella [1]). *For a given function $f : \mathbb{Z}^n \to \mathbb{R}$ the following properties are equivalent:*

(i) *f is integrally convex.*

(ii) *For every x and y in $\mathbb{Z}^n$ with $\|x - y\|_\infty = 2$ we have*

$$(\operatorname{can}(f))\Big(\frac{1}{2}x + \frac{1}{2}y\Big) \le \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

In the next proposition we improve [1] by considering the same property for a smaller number of points.

Proposition 3.4. *A function $f : \mathbb{Z}^n \to \mathbb{R}$ is integrally convex if and only if, for all k with $1 \le k \le n$, f satisfies*

$$(\operatorname{can}(f))\Big(\frac{1}{2}x + \frac{1}{2}y\Big) \le \frac{1}{2}f(x) + \frac{1}{2}f(y). \tag{4}$$

for each two points x and y in $\mathbb{Z}^n$ with $y_k - x_k = 2$ and $\delta_{k,2}(x, y) = 1$.

Proof. Let $I(a,p) = \prod_{j=1}^n [a_j, a_j + p_j]$ and $I_{\mathbb{Z}^n}(a,p) = \prod_{j=1}^n [a_j, a_j + p_j]_{\mathbb{Z}^n}$ where $a \in \mathbb{Z}^n$ and $p \in \mathbb{N}^n$. By definition, $\operatorname{can}(f)$ is convex in every $I(a,p)$ where for all j, $p_j = 1$. We now assume that the inequality (4) is true for all points x and y in $\mathbb{Z}^n$, where $y_k - x_k = 2$ and $\delta_{k,2}(x,y) = 1$ for all $1 \le k \le n$ and this means that the inequality is valid for all points in $I_{\mathbb{Z}^n}(a, p + e_k)$ where $p_j = 1$ for all $1 \le j \le n$, and e_k has 1 in its k-coordinate, and 0 on the other coordinates. We shall prove that $\operatorname{can}(f)$ is convex in every $I(a,p)$, $a \in \mathbb{Z}^n$, $p \in \mathbb{N}^n$. To do this, we show the convexity of $\operatorname{can}(f)$ in the following steps:

(i) The convexity of $\operatorname{can}(f)$ in every $I(a, p + e_k)$ where $p_j = 1$ for $1 \le j \le n$. We shall prove this in the following two parts:

(i.a) First we show that the canonical extension $\operatorname{can}(f)$ is convex along the line segment $[x, y]$ for two points $x, y \in I_{\mathbb{Z}^n}(a, p + e_k)$ where $p_j = 1$, $1 \le j \le n$.

(i.b) Then we prove the same result for two points x, y where $x \in I_{\mathbb{Z}^n}(a, p + e_k)$, $y \in I(a, p + e_k) \backslash I_{\mathbb{Z}^n}(a, p + e_k)$.

(ii) Using induction on p_k, we show the convexity of $\operatorname{can}(f)$ in every $I(a,p)$, $p_k \ge 2$ for one k, and $p_j = 1$ for all $j \ne k$.

(iii) Then using induction on p_j, $1 \le j \le n$ and $j \ne k$, we shall see the convexity of $\operatorname{can}(f)$ in $I(a,p)$ for all $p \in \mathbb{N}^n$.

Lets us start with the proof of all steps:

(i.a) Let $x, y \in I_{\mathbb{Z}^n}(a, p + e_k)$. We shall show that, for $0 \le \alpha \le 1$,

$$(\operatorname{can}(f))(\alpha x + (1-\alpha)y) \le \alpha(\operatorname{can}(f))(x) + (1-\alpha)(\operatorname{can}(f))(y). \tag{5}$$

Because of the structure of the canonical extension, we need just to check the convexity for $\alpha = \frac{1}{2}$. The inequality (4) implies the convexity in this case.

(i.b) Without loss of generality, we shall prove the convexity of can(f) along the line segment $[\mathbf{0}, y]$ where $y \in I(a, p+e_k) \backslash I_{\mathbb{Z}^n}(a, p+e_k)$ and $y_k = 2$. Since the canonical extension is convex in each unit hypercube, it is required only to show that inequality (5) holds for $\alpha = \frac{1}{2}$.

The definition of can(f) leads us to find the points $z^i \in I_{\mathbb{Z}^n}(a, p+e_k)$ and scalars $\beta_i \in [0,1]$, for $i = 1, \ldots, m$, $m \leq 2^{n-1}$, such that

$$y = \sum_{i=1}^m \beta_i z^i, \qquad (\mathrm{can}(f))(y) = \sum_{i=1}^m \beta_i f(z^i), \qquad \sum_{i=1}^m \beta_i = 1.$$

We have

$$\frac{1}{2}\mathbf{0} + \frac{1}{2}y = \frac{1}{2}\mathbf{0} + \frac{1}{2}\sum_{i=1}^m \beta_i z^i = \sum_{i=1}^m \beta_i \Big(\frac{1}{2}\mathbf{0} + \frac{1}{2}z^i\Big) = \sum_{i=1}^m \beta_i t^i,$$

where t^i, for $i = 1, \ldots, m$, is the intersection of segment $[\mathbf{0}, z^i]$ with the hyperplane $x_1 = 1$. The points t^i, $i = 1, \ldots, m$, belong to the hypercube $\{0,1\}^n$. The convexity of canonical extension in each unit hypercube implies that

$$(\mathrm{can}(f))\Big(\frac{1}{2}\mathbf{0} + \frac{1}{2}y\Big) = (\mathrm{can}(f))\Big(\sum_{i=1}^m \beta_i t^i\Big) \leq \sum_{i=1}^m \beta_i (\mathrm{can}(f))(t^i). \tag{6}$$

We know that

$$\sum_{i=1}^m \beta_i (\mathrm{can}(f))(t^i) = \sum_{i=1}^m \beta_i (\mathrm{can}(f))\Big(\frac{1}{2}\mathbf{0} + \frac{1}{2}z^i\Big). \tag{7}$$

By the convexity of canonical extension which stated in the case (i.a), we get that

$$\sum_{i=1}^m \beta_i (\mathrm{can}(f))\Big(\frac{1}{2}\mathbf{0} + \frac{1}{2}z^i\Big) \leq \sum_{i=1}^m \beta_i \Big(\frac{1}{2}f(\mathbf{0}) + \frac{1}{2}f(z^i)\Big). \tag{8}$$

Hence (6), (7) and (8) after simple calculations imply that

$$\begin{aligned}(\mathrm{can}(f))\Big(\frac{1}{2}\mathbf{0} + \frac{1}{2}y\Big) &= (\mathrm{can}(f))\Big(\sum_{i=1}^m \beta_i t_i\Big)\\ &\leq \sum_{i=1}^m \beta_i (\mathrm{can}(f))(t_i) = \sum_{i=1}^m \beta_i (\mathrm{can}(f))\Big(\frac{1}{2}\mathbf{0} + \frac{1}{2}z^i\Big)\end{aligned}$$

$$\leq \sum_{i=1}^{m} \beta_i \Big(\frac{1}{2} f(\mathbf{0}) + \frac{1}{2} f(z^i)\Big) = \frac{1}{2} f(\mathbf{0}) + \frac{1}{2} \sum_{i=1}^{m} \beta_i f(z^i)$$
$$= \frac{1}{2} \big(\operatorname{can}(f)\big)(\mathbf{0}) + \frac{1}{2} \big(\operatorname{can}(f)\big)(y).$$

Thus $\operatorname{can}(f)$ is convex in every $I(a, p + e_k)$ where $p_j = 1$, $1 \leq j \leq n$.

(ii) Using induction on p_k, we shall show that $\operatorname{can}(f)$ is convex in every $I(a, p)$, $p_k \geq 2$, where $p_j = 1$ for all $j \neq k$. To do this, let us consider $p_j = 1$ for all $1 \leq j \leq n$. By the previous part we have that $\operatorname{can}(f)$ is convex in $I(a, p + e_k)$ as well as in $I(a + e_k, p)$. Since $I(a, p + e_k) \cap I(a + e_k, p)$ has a nonempty interior, $\operatorname{can}(f)$ is convex in their union which is $I(a, p + 2e_k)$. Thus we can conclude that $\operatorname{can}(f)$ is convex in every $I(a, p)$, $p_k \geq 2$.

(iii) Assume now that $\operatorname{can}(f)$ is convex in every $I(a, p)$, $p_k \geq 2$ for one k, and $p_j = 1$ for all $j \neq k$. We shall show that $\operatorname{can}(f)$ is convex in $I(a, p + e_j)$. Let us consider a line segment $[x, y]$. If $x, y \in I(a, p)$ or $x, y \in I(a + e_j, p)$, we already have the convexity. Suppose now that x is chosen in the interior of $I(a, p)$ and y in the interior of $I(a + e_j, p)$. Then there is a unique point z on the segment such that $z_j = a_j + 1$. We assume that all other $z_i \notin \mathbb{Z}$, $i \neq j$. Then we have the block $I(b, q)$ where $q_j = 2$, $b_j = a_j$, $b_i = \lfloor z_i \rfloor$ and $q_i = 1$ for $i \neq j$. We know by hypothesis that $\operatorname{can}(f)$ is convex there, and so on the segment near z. Thus $\operatorname{can}(f)$ is convex on the whole segment. This segment is one of the segments which forms a dense set in $I(a, p + e_j)$. By continuity, $\operatorname{can}(f)$ is convex on any segment in $I(a, p + e_j)$. We can now consider the convexity of $\operatorname{can}(f)$ in $I(a, p)$, $p_k \geq 2$, $p_j \geq 2$, and $p_m = 1$ for all $m \neq k, j$, and do the same to show the convexity of $\operatorname{can}(f)$ in $I(a, p + e_m)$. This implies that $\operatorname{can}(f)$ is convex in $I(a, p)$ for all $p \in \mathbb{N}^n$.

Conversely, if f is integrally convex, the inequality (4) holds by the convexity of $\operatorname{can}(f)$. □

4. The relation between lateral convexity and integral convexity

Murota and Shioura [11] went on to show relations among various types of discrete convexity. By two examples they show that there is no inclusion relation between the class of discretely-convex functions and that of convex extensible function. An integrally convex function is discretely convex (see Favati and Tardella [1], p. 10). By the definition of integral convexity, it is clear that the elements of this class are convex extensible too. The converse does not hold in general (Example 3.4 in [11]). Fujishige and Murota [2] presented the relation between $\mathrm{L}^\natural$ convex function and integral convexity. They showed that a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is $\mathrm{L}^\natural$-convex if and only if it is submodular and integrally convex. Murota and Shioura [11] also studied some operations for classes of discrete convex functions. They clarified whether each class is closed under such operations. The class of discretely convex function, M$^\natural$-convex functions, as well as integrally convex functions, are not closed under addition. But in two dimensions, as already mentioned, integral convexity is equivalent to C-lateral convexity, and thus the class is closed under addition. However, the class of convex extensible and L$^\natural$-convex functions are closed under addition. Moreover, the sum of two integrally convex functions is again integrally convex for those functions which have a linear canonical extension (see Favati and Tardella [1], p. 9). Kiselman [3] showed that in the two-dimensional case, $\operatorname{can}(f+g) = \operatorname{can}(f) + \operatorname{can}(g)$ in the square $a + [0,1]^2$ if and only if $[D_1 D_2 f(a)][D_1 D_2 g(a)] \geq 0$. Hence for the two-dimensional case, the canonical extension is additive for the class of submodular integrally convex functions, as well as for supermodular integrally convex functions.

The following theorem and proposition show some relations between integrally and C-laterally convex functions.

Theorem 4.1. *A C-laterally convex function $f : \mathbb{Z}^n \to \mathbb{R}$ is integrally convex.*

Proof. Let $I = \{1, \ldots, n\}$ and $k \in I$. Consider two points $x, y \in \mathbb{Z}^n$ such that $y_k - x_k = 2$, and $\delta_{k,2}(x, y) = 1$. Since f is C-laterally convex, Theorem 2.3 implies that

$$f(x) + f(y) \geq f(x + e_k) + f(y - e_k). \tag{9}$$

We shall show that

$$f(x) + f(y) \geq 2\big(\operatorname{can}(f)\big)\Big(\frac{1}{2}x + \frac{1}{2}y\Big). \tag{10}$$

We use induction on the dimension to prove it. For the dimension two we have that the C-lateral convexity is equivalent with integral convexity. Therefore the result is obvious in this dimension. Suppose now that equation (10) holds for dimensions up to $n-1$. We shall prove it for the n-dimensional case. The two points $x + e_k$ and $y - e_k$ belong to the unit hypercube in $\mathbb{Z}^{n-1}$ which are indeed in the intersection of two unit hypercubes in $\mathbb{Z}^n$. Therefore the restriction of $\operatorname{can}(f)$ into this unit hypercube in $\mathbb{Z}^{n-1}$ is convex, and we have

$$\begin{aligned} f(x + e_k) + f(y - e_k) &\geq 2\big(\operatorname{can}(f)\big)\Big(\frac{1}{2}(x + e_k) + \frac{1}{2}(y - e_k)\Big) \\ &\geq 2\big(\operatorname{can}(f)\big)\Big(\frac{1}{2}x + \frac{1}{2}y\Big). \end{aligned}$$

Inserting this equation into (9) gives us inequality (10). Now, using Proposition 3.4, we have the desired result. □

The following example presents an integrally convex function which is not B-laterally convex.

Example 4.2. For the set

$$S = \{(0,0,0), (1,1,0), (0,1,1), (1,2,1)\},$$

consider the function

$$f(x) = \begin{cases} 0, & x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$$

This function is integrally convex but not B-laterally convex. The reason that f is an integrally convex function is that the set S is the Minkowski sum of two L$^\natural$-convex sets, $\{(0,0,0), (1,1,0)\}$ and $\{(0,0,0), (0,1,1)\}$, therefore it is an integrally convex set. However, we have

$$(D_{\{e_2\}})_! f(1,1,1) \not\geq (D_{\{e_2\}})_! f(0,0,0),$$

thus f is not laterally convex. We would like to mention that the set S was also used by Murota and Shioura [11] in order to present an example of a sum of two L$^\natural$-convex functions which is not itself L$^\natural$.

We mention here the definition of submodular [supermodular] [modular] functions $f : \mathbb{Z}^n \to \mathbb{R}$.

Given a function $f : \mathbb{Z}^n \to \mathbb{R}$ and two points $x, y \in \mathbb{Z}^n$. Let $x \vee y = z$, where $z_i = \max\{x_i, y_i\}$ for $1 \leq i \leq n$, and $x \wedge y = z$, where $z_i = \min\{x_i, y_i\}$ for $1 \leq i \leq n$.

The function f is said to be *supermodular* if $f(x) + f(y) \leq f(x \vee y) + f(x \wedge y)$, *submodular* if $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$, and *modular* if $f(x) + f(y) = f(x \vee y) + f(x \wedge y)$.

We now define these properties locally, as follows:

We say that a function $f : \mathbb{Z}^n \to \mathbb{R}$ is *locally submodular* (resp. *locally supermodular*, *locally modular*) if f is submodular (resp. supermodular, modular) for the points in the hypercubes $a + \{0,1\}^n$, for all of $a \in \mathbb{Z}^n$.

Theorem 4.3. *Consider a unit hypercube $a + \{0,1\}^n$, $a \in \mathbb{Z}^n$. The faces of this hypercube is a set*

$$a + \sum_{i \in J} e_i,$$

where $I = \{1, \dots, n\}$ and J is a subset of I of cardinality m, $1 \le m \le n-1$. If an integrally convex function $f : \mathbb{Z}^n \to \mathbb{R}$ is modular on the faces of each unit hypercube $a + \{0,1\}^n$, $a \in \mathbb{Z}^n$, then it is C-laterally convex.

Proof. Consider an integrally convex function $f : \mathbb{Z}^n \to \mathbb{R}$. By Proposition 3.3,

$$2\big(\mathrm{can}(f)\big)\Big(\frac{1}{2}x + \frac{1}{2}y\Big) \le f(x) + f(y)$$

for $x, y \in \mathbb{Z}^n$, $y_k - x_k = 2$ and $\delta_{k,2}(x, y) = 1$. The three points $x + e_k$, $y - e_k$, and $\frac{x+y}{2}$ have same k-th coordinate, and the distance between other coordinates can be 0 or 1. Hence they belong to a face of a unit hypercube. Since f is locally modular on the faces of each unit hypercube,

$$2\big(\mathrm{can}(f)\big)\Big(\frac{1}{2}x + \frac{1}{2}y\Big) = f(x + e_k) + f(y - e_k).$$

Theorem 2.3 now gives the result. □

Conclusion. In this work we studied a class of discrete convex functions that was introduced by using differences of values of the functions at certain points. For real-valued two-dimensional functions this class is equal to the class of integrally convex functions. We showed that in higher dimensions a C-laterally convex function is integrally convex. Since a C-laterally convex function is integrally convex, a local minimum is indeed a global one.

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