Portugal. Math. (N.S.)
Vol. 71, Fasc. 1, 2014, 1–24
DOI 10.4171/PM/1938

Portugaliae Mathematica

Global L_2-solvability of a problem governing two-phase fluid motion without surface tension

Irina V. Denisova

Abstract. The paper deals with an interface problem for the Navier–Stokes system governing the motion of two incompressible fluids in a container, one liquid being inside another one. We prove unique solvability of the problem in an infinite time interval provided that the data are small enough, surface tension effect being neglected on the interface between the fluids. The norms of the solution are shown to decay exponentially at infinity with respect to time. The proof is based on an exponential estimate of a generalized energy and on a local existence theorem of the problem in anisotropic Sobolev–Slobodetskiĭ spaces.

We give also the main steps of the proof of the local theorem for the problem with and without including surface tension.

Mathematics Subject Classification (2010). Primary 35Q30, 35R35; Secondary 76D05, 76D03.

Keywords. Incompressible fluids, Navier–Stokes system, two-phase problem with unknown interface, Lagrangian coordinates, Sobolev–Slobodetskiĭ spaces.

1. Statement of the problem and the main result

We are concerned here with unsteady motion of a viscous incompressible drop inside another fluid. The both liquids are located in a container with a solid boundary S. The flow is governed by the Navier–Stokes equations with non-slip condition on the outer boundary and with continuity conditions for the velocity and the normal stresses on the interface Γ_t. This problem is classified as a free boundary problem because the interface between the liquids is unknown and to be defined by the solving process.

In the case of the whole space, existence and uniqueness theorem for the system was proved in a finite time interval whose magnitude is determined by the norms of the data: initial velocity and external forces. This result was stated in [3], [6]. It was obtained in several steps by considering modal linear problems [5], [8], [4]. In these papers, it was also studied a system where surface tension forces were taken into account. In this case, the length of solvability interval depends, in addition, on the curvature of the initial interface Γ_0. Local existence theorem for bounded domains is proved in a similar way.

We mention also some other results of 1990s on this subject. They are of Y. Giga and Sh. Takahashi [10], [20], and also of A. Nouri, F. Poupaud, and Y. Demay [13], [12]. These papers dealt with the existence of global weak solutions for the Stokes and Navier-Stokes equations governing the motion of two (or several) immiscible fluids without including surface tension.

We note that recently the interest to two-phase fluids is increasing. New researchers are studying the problem on two liquid flow by new mathematical methods in different functional spaces. As this problem is enough complicated, it has many aspects for discussion. In particular, H. Abels found conditions when there exist weak solutions to the non-linear problem without surface tension but he was not able to describe the interface in this case. On the contrary, in the case with the presence of surface tension, he only estimated the Hausdorff measure of the interface leaving open the existence of generalized solutions [1]. Next, Yo. Shibata and S. Shimizu investigated the problems with surface tension by operator methods in the anisotropic Sobolev spaces $W^{2,1}_{q,p}$, $2 \le n < q < \infty$, $2 < p < \infty$, $\Omega^{\pm} \subset \mathbb{R}^n$. But they proved only the solvability of model diffraction problems for the Stokes system [15]. Ja. Prüss with the co-authors studied behaviour of solutions for the two-phase Navier–Stokes equations also taking surface tension into account but in absence of external forces. They proved that the problem was locally uniquely solvable in L_p-setting, $p > n+3$, when the interface was close to a flat configuration for the whole space [14], and for bounded domains [11], the solution becoming instantaneously real analytic. Global (in time) classical solvability of the problems with and without including surface tension forces was analyzed by the author alone [7] and together with V. A. Solonnikov [9], respectively. There it was applied the method of an exponential estimate of a solution in terms of the data.

Here, we prove global solvability in the Sobolev–Slobodetskiĭ functional classes $W_2^{l,l/2}$ in a similar way by assuming an exponential decrease of the mass forces and ignoring surface tension. We give also main steps of the proof for local solvability for the problem with non-negative surface tension coefficient which was not published earlier in detail. We remark that in the case without surface tension we don't need require additional smoothness for the initial interface Γ_0. It is enough to suppose only natural regularity for it, the same as for the surface S.

Note that in the Sobolev spaces, the existence of a global solution for a problem governing the motion of a drop of single fluid in vacuum was obtained by V. A. Solonnikov in [17], local existence theorem for the problem being proven by him in [19].

Now we give a mathematical setting of the two-phase problem.

Let, at the initial moment $t = 0$, a fluid with viscosity $\nu^+ > 0$ and density $\rho^+ > 0$ fill a bounded domain $\Omega_0^+ \subset \mathbb{R}^3$. Let a fluid with viscosity $\nu^- > 0$ and density $\rho^- > 0$ be situated in a domain Ω_0^- surrounding Ω_0^+. We denote $\partial\Omega_0^+$ by Γ_0. The boundary $S \equiv \partial(\Omega_0^+ \cup \Gamma_0 \cup \Omega_0^-)$ is a given closed surface, $S \cap \Gamma_0 = \emptyset$.

For every $t > 0$, it is necessary to find the interface Γ_t between the domains Ω_t^+ and Ω_t^-, as well as the velocity vector field $\mathbf{v}(x, t) = (v_1, v_2, v_3)$ and the pressure function p which satisfy the following initial-boundary value problem for the Navier–Stokes system:

$$\begin{aligned}
&\mathscr{D}_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu^\pm\nabla^2\mathbf{v} + \frac{1}{\rho^\pm}\nabla p = \mathbf{f},\\
&\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega_t^- \cup \Omega_t^+, \qquad t > 0,\\
&\mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega_0^- \cup \Omega_0^+, \qquad \mathbf{v}|_{S_T} = 0 \qquad (S_T = S \times (0, T)),\\
&[\mathbf{v}]|_{\Gamma_t} \equiv \lim_{\substack{x\to x_0\in\Gamma_t\\ x\in\Omega_t^+}} \mathbf{v}(x) - \lim_{\substack{x\to x_0\in\Gamma_t\\ x\in\Omega_t^-}} \mathbf{v}(x) = 0, \qquad [\mathbb{T}\mathbf{n}]|_{\Gamma_t} = 0.
\end{aligned} \tag{1.1}$$

Here $\mathscr{D}_t = \partial/\partial t$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\nu^\pm$, $\rho^\pm$ are step functions of viscosity and density, respectively, $\mathbf{v}_0$ is the initial distribution of the velocity, the stress tensor is $\mathbb{T}(\mathbf{v}, p) \equiv -p\mathbb{I} + \mu^\pm\mathbb{S}(\mathbf{v})$, where $\mathbb{S}(\mathbf{v})$ is twice the strain tensor with the elements

$$S_{ik} = \partial v_i/\partial x_k + \partial v_k/\partial x_i, \qquad i, k = 1, 2, 3;$$

$\mu^\pm = \nu^\pm\rho^\pm$, $\mathbb{I}$ is the unit matrix, $\mathbf{n}$ is the outward normal to Ω_t^+. We suppose that a Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. The centered dot denotes the Cartesian scalar product.

Summation is implied over the repeated indices from 1 to 3 if they are denoted by Latin letters and from 1 to 2 if they are done by Greek letters. We mark the vectors and the vector spaces by boldface letters.

Moreover, to exclude the mass transportation through Γ_t, we assume that the liquid particles do not leave Γ_t. This means that Γ_t consists of points $x(\xi, t)$ such that the corresponding vector $\mathbf{x}(\xi, t)$ solves the Cauchy problem

$$\mathscr{D}_t\mathbf{x} = \mathbf{v}\big(x(\xi, t)\big), \qquad \mathbf{x}|_{t=0} = \boldsymbol{\xi},\ \xi \in \Gamma_0,\ t > 0. \tag{1.2}$$

Hence $\Gamma_t = \{x(\xi, t) \mid \xi \in \Gamma_0\}$, $\Omega_t^\pm = \{x(\xi, t) \mid \xi \in \Omega_0^\pm\}$.

We prove unique solvability for problem (1.1), (1.2) in Sobolev–Slobodetskiĭ spaces for all $t > 0$, provided that the initial data are smooth and small enough. Our proof is based on a local existence theorem for the problem in Lagrangian coordinates.

We pass from Eulerian to Lagrangian coordinates by the formula

$$\mathbf{x} = \boldsymbol{\xi} + \int_0^t \mathbf{u}(\xi, \tau)\, d\tau \equiv \mathbf{X}_{\mathbf{u}}(\xi, t). \tag{1.3}$$

(Here $\mathbf{u}(\xi, t)$ is the velocity vector field in the Lagrangian coordinates.)

As a result of transformation (1.3) and of projecting the last boundary condition in (1.1) onto the tangent planes first to Γ_t, then to Γ_0, we arrive at the problem for $\mathbf{u}$, $q = p(X_{\mathbf{u}}, t)$ with the given interface $\Gamma \equiv \Gamma_0$. If the angle between $\mathbf{n}$ and the exterior normal $\mathbf{n}_0$ to Γ is acute, this system is equivalent to the following one:

$$\begin{aligned}
&\mathscr{D}_t\mathbf{u} - \nu^{\pm}\nabla_{\mathbf{u}}^2\mathbf{u} + \frac{1}{\rho^{\pm}}\nabla_{\mathbf{u}}q = \mathbf{f}, \qquad \nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \qquad \text{in } Q_T^{\pm} = \Omega_0^{\pm} \times (0, T),\\
&\mathbf{u}|_{t=0} = \mathbf{v}_0 \qquad \text{in } \Omega_0^- \cup \Omega_0^+, \qquad \mathbf{u}|_{S_T} = 0, \qquad (S_T \equiv S \times (0, T))\\
&[\mathbf{u}]|_{G_T} = 0, \qquad [\mu^{\pm}\Pi_0\Pi\mathbb{S}_{\mathbf{u}}(\mathbf{u})\mathbf{n}]|_{G_T} = 0 \qquad (G_T \equiv \Gamma \times (0, T)),\\
&[\mathbf{n}_0 \cdot \mathbb{T}_{\mathbf{u}}(\mathbf{u}, q)\mathbf{n}]|_{G_T} = 0.
\end{aligned} \tag{1.4}$$

Here we have used the notation: $\nabla_{\mathbf{u}} = \mathbb{A}\nabla$, $\mathbb{A}$ is the matrix of co-factors A_{ij} to the elements

$$a_{ij}(\xi, t) = \delta_j^i + \int_0^t \frac{\partial u_i(\xi, t')}{\partial \xi_j}\, dt'$$

of the Jacobian matrix of transformation (1.3), δ_j^i is the Kronecker symbol, $i, j = 1, 2, 3$, the vector $\mathbf{n}$ is connected with $\mathbf{n}_0$ by the relation

$$\mathbf{n} = \frac{\mathbb{A}\mathbf{n}_0}{|\mathbb{A}\mathbf{n}_0|};$$

$\Pi\boldsymbol{\omega} = \boldsymbol{\omega} - \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\omega})$, $\Pi_0\boldsymbol{\omega} = \boldsymbol{\omega} - \mathbf{n}_0(\mathbf{n}_0 \cdot \boldsymbol{\omega})$ are projections of a vector $\boldsymbol{\omega}$ onto the tangent planes to Γ_t and Γ, respectively; $\mathbb{T}_{\mathbf{u}}(\mathbf{w}, q) = -q\mathbb{I} + \mu^{\pm}\mathbb{S}_{\mathbf{u}}(\mathbf{w})$, where the tensor $\mathbb{S}_{\mathbf{u}}(\mathbf{w})$ contains the elements

$$\left(\mathbb{S}_{\mathbf{u}}(\mathbf{w})\right)_{ij} = A_{ik}\partial w_j/\partial \xi_k + A_{jk}\partial w_i/\partial \xi_k.$$

Let Ω be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be the multi-index of order $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$ with integer non-negative components α_i, $i = 1, \ldots, n$. We denote the generalized derivative of a function u by $\mathscr{D}_x^{\boldsymbol{\alpha}}u = \frac{\partial^{|\boldsymbol{\alpha}|}u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$.

We define the Sobolev–Slobodetskiĭ space $W_2^m(\Omega)$ for $m > 0$ as the space of functions u with finite norm

$$\|u\|_{W_2^m(\Omega)} = \Big(\sum_{|\boldsymbol{\alpha}|<m} \|\mathscr{D}_x^{\boldsymbol{\alpha}} u\|_\Omega^2 + \|u\|_{\dot{W}_2^m(\Omega)}^2\Big)^{1/2},$$

where $\|\cdot\|_\Omega$ is the norm of $L_2(\Omega)$, and

$$\|u\|_{\dot{W}_2^m(\Omega)}^2 = \begin{cases} \sum\limits_{|\boldsymbol{\alpha}|=m} \|\mathscr{D}_x^{\boldsymbol{\alpha}} u\|_\Omega^2 & \text{for } m \in \mathbb{N}, \\ \sum\limits_{|\boldsymbol{\alpha}|=[m]} \int_\Omega \int_\Omega \frac{|\mathscr{D}_x^{\boldsymbol{\alpha}} u(x) - \mathscr{D}_y^{\boldsymbol{\alpha}} u(y)|^2}{|x-y|^{n+2(m-[m])}} \,\mathrm{d}x\,\mathrm{d}y & \text{for } m \notin \mathbb{N}, \end{cases}$$

$[m]$ being the integal part of m.

The anisotropic space $W_2^{m,m/2}(Q_T)$ consists of functions defined in the cylinder $Q_T = \Omega \times (0, T)$, $0 < T \leq \infty$, and having finite norm

$$\|u\|_{W_2^{m,m/2}(Q_T)} = \Big(\int_0^T \|u\|_{W_2^m(\Omega)}^2 \,\mathrm{d}t + \int_\Omega \|u\|_{W_2^{m/2}(0,T)}^2 \,\mathrm{d}x\Big)^{1/2}.$$

We say that a vector field belongs to a certain space if each of its components belongs to this space and we define its norm as the sum of the norms of its components. The same is valid for a tensor-valued function. The numeration of constants is individual for each section. Different constants may be denoted by c without any index.

We will also need the following norms. Let $u \in W_2^{l,l/2}(Q_T)$, $l \in (0, 1)$. We define

$$\|u\|_{Q_T}^{(l,l/2)} = (\|u\|_{W_2^{l,l/2}(Q_T)}^2 + T^{-l}\|u\|_{Q_T}^2)^{1/2},$$
$$(\|u\|_{Q_T}^{(0,l/2)})^2 = \int_\Omega \|u\|_{W_2^{l/2}(0,T)}^2 \,\mathrm{d}x + T^{-l}\|u\|_{Q_T}^2.$$

The equivalent normalization of $W_2^{2+l,1+l/2}(Q_T)$ is as follows

$$(\|u\|_{Q_T}^{(2+l,1+l/2)})^2 = \|u\|_{W_2^{2+l,1+l/2}(Q_T)}^2 + T^{-l}\Big\{\|\mathscr{D}_t u\|_{Q_T}^2 + \sum_{|\boldsymbol{\alpha}|=2} \|\mathscr{D}_x^{\boldsymbol{\alpha}} u\|_{Q_T}^2\Big\}$$
$$+ \sup_{t \leq T} \|u(\cdot, t)\|_{W_2^{1+l}(\Omega)}^2.$$

For $\alpha, \beta \in (0, 1)$, we will consider the following Hölder norms of u in Q_T:

$$|u|_{Q_T}^{(0,\alpha)} = \sup_{Q_T}|u| + \sup_{x\in\Omega}\sup_{t,\tau\le T}\frac{|u(x,t)-u(x,\tau)|}{|t-\tau|^{\alpha}},$$

$$|u|_{Q_T}^{(1,\beta)} = \sup_{Q_T}|u| + \sup_{0\le t\le T}\sup_{x,y\in\Omega}\frac{|u(x,t)-u(y,t)|}{|x-y|} + \sup_{x\in\Omega}\sup_{t,\tau\le T}\frac{|u(x,t)-u(x,\tau)|}{|t-\tau|^{\beta}}.$$

For a bounded domain Ω and $T<\infty$, it is evident that

$$\|u\|_{Q_T}^{(l,l/2)} \le c(1+T^{1/2-l/2})|u|_{Q_T}^{(1,\beta)}, \qquad \beta\in[l/2,1).$$

Let be $T\in(0,\infty]$, $t,\tau>0$. We introduce the notation:

$$\Omega \equiv \Omega_t^- \cup \overline{\Omega_t^+}, \qquad Q_T=\Omega\times(0,T), \qquad D_T = Q_T^-\cup Q_T^+, \qquad Q_T^{\pm}=\Omega_0^{\pm}\times(0,T),$$
$$Q^{\pm}_{(t,t+\tau)} = \Omega_t^{\pm}\times(t,t+\tau), \qquad D_{(t,t+\tau)} = \bigcup Q^{\pm}_{(t,t+\tau)}, \qquad G_{(t,t+\tau)} = \Gamma_t\times(t,t+\tau).$$

For a function u defined in the domain $\bigcup_{i=\pm}\Omega_0^i$, we set

$$\|u\|_{W_2^m(\bigcup_{i=\pm}\Omega_0^i)} = \|u\|_{W_2^m(\Omega_0^-)} + \|u\|_{W_2^m(\Omega_0^+)},$$

and for a function u defined in D_T, we put

$$\|u\|_{W_2^{m,m/2}(D_T)} = \|u\|_{W_2^{m,m/2}(Q_T^-)} + \|u\|_{W_2^{m,m/2}(Q_T^+)}.$$

Now we state a local existence theorem for a bounded domain.

Theorem 1.1 (Local existence theorem). *Assume that for some* $l\in(1/2,1)$, *we have* $\Gamma\in W_2^{3/2+l}$, $\mathbf{f}\in\mathbf{W}_2^{l,l/2}(Q_T)$, $0<T<\infty$, $\mathbf{f}(\cdot,t)$, $\nabla\mathbf{f}(\cdot,t)\in \mathrm{Lip}(\Omega)$ *for* $\forall t\in[0,T]$, $\mathbf{f}(\xi,\cdot)$, $\nabla\mathbf{f}(\xi,\cdot)\in\mathbf{C}^{\beta}(0,T)$ *for* $\forall\xi\in\Omega$ *with some* $\beta\in[1/2,1)$. *In addition, let the initial velocity vector field* $\mathbf{v}_0\in\mathbf{W}_2^{1+l}(\bigcup_{i=\pm}\Omega_0^i)$ *satisfy the compatibility conditions*

$$\begin{aligned} \nabla\cdot\mathbf{v}_0 = 0 \quad &\text{in } \Omega_0^-\cup\Omega_0^+, \qquad \mathbf{v}_0|_S = 0,\\ [\mathbf{v}_0]|_\Gamma = 0, \quad &[\mu^{\pm}\Pi_0\mathbb{S}(\mathbf{v}_0)\mathbf{n}_0]|_\Gamma = 0. \end{aligned} \tag{1.5}$$

Under these assumptions, there exists a constant $T_0\in(0,T]$ *such that problem* (1.4) *is uniquely solvable on the interval* $(0,T_0]$, *and its solution* $(\mathbf{u},q)$ *has the properties:* $\mathbf{u}\in\mathbf{W}_2^{2+l,1+l/2}(D_{T_0})$, $q\in W_2^{l,l/2}(D_{T_0})$, $\nabla q\in\mathbf{W}_2^{l,l/2}(D_{T_0})$, $[q]|_\Gamma\in W_2^{l+1/2,l/2+1/4}(G_{T_0})$ *and*

$$\begin{aligned} &\|\mathbf{u}\|_{D_{T_0}}^{(2+l,1+l/2)} + \|\nabla q\|_{D_{T_0}}^{(l,l/2)} + \|q\|_{D_{T_0}}^{(l,l/2)} + \|[q]|_\Gamma\|_{W_2^{l+1/2,l/2+1/4}(G_{T_0})}\\ &\quad\le c_1(c_2+T_0^{(1-l)/2}\|\mathbf{v}_0\|_{\mathbf{W}_2^1(\Omega)})\{|\mathbf{f}|_{Q_{T_0}}^{(1,\beta)} + |\nabla\mathbf{f}|_{Q_{T_0}}^{(0,\beta)} + \|\mathbf{v}_0\|_{\mathbf{W}_2^{1+l}(\bigcup_i\Omega_0^i)}\}. \end{aligned} \tag{1.6}$$

The value T_0 *depends on the norms of* $\mathbf{f}$ *and* $\mathbf{v}_0$.

The following theorem gives us the existence of a global solution of system (1.1), (1.2). This is the main result of the paper.

Theorem 1.2 (Global existence theorem). *Let for some* $l \in (1/2, 1)$ *the interface* $\Gamma \in W_2^{3/2+l}$, *the vector field* $\mathbf{f} \in \mathbf{W}_2^{l,l/2}(Q_\infty)$, $\mathbf{f}(\cdot, t)$, $\nabla\mathbf{f}(\cdot, t) \in \mathrm{Lip}(\Omega)$ *for* $\forall t \in [0, \infty]$, $\mathbf{f}(\xi, \cdot)$, $\nabla\mathbf{f}(\xi, \cdot) \in \mathbf{C}^\beta(0, T)$ *for* $\forall \xi \in \Omega$ *with some* $\beta \in [1/2, 1)$. *We also suppose that the initial velocity vector field* $\mathbf{v}_0 \in \mathbf{W}_2^{1+l}(\bigcup_{i=\pm}\Omega_0^i)$ *satisfies conditions* (1.5) *and together with the mass forces is small enough, i.e.,*

$$\|\mathbf{v}_0\|_{\mathbf{W}_2^{1+l}(\bigcup_i \Omega_0^i)} + \|\mathbf{f}\|_{\mathbf{W}_2^{l,l/2}(Q_\infty)} + \int_0^\infty \mathrm{e}^{bt}\|\mathbf{f}\|_\Omega \,\mathrm{d}t + |\mathrm{e}^{bt}\mathbf{f}|_{Q_\infty}^{(1,\beta)} + |\mathrm{e}^{bt}\nabla\mathbf{f}|_{Q_\infty}^{(0,\beta)} \le \varepsilon \ll 1. \tag{1.7}$$

(*Here* $b = \min\{\nu^+, \nu^-\}/(2c_0)$, *where* c_0 *is the constant from inequality* (3.3).)

Then problem (1.1), (1.2) *is uniquely solvable for all positive moments of time t, and solution* $(\mathbf{v}, p)$ *possesses the properties:* $\mathbf{v} \in \mathbf{W}_2^{2+l,1+l/2}$, $p \in W_2^{l,l/2}$, $\nabla p \in \mathbf{W}_2^{l,l/2}$, $[p]|_{\Gamma_t} \in W_2^{l+1/2,l/2+1/4}$, $\Gamma_t \in W_2^{3/2+l}$, *the pressure being defined up to a bounded function of time. This means that for any* $t_0 \in (0, \infty)$, *the solution* $(\mathbf{u}, q)$ *in the Lagrangian coordinates and its derivatives belong to the corresponding Sobolev spaces over* $D_{(t_0, t_0+\tau)}$ *for a sufficiently small time interval* $(t_0, t_0 + \tau)$. *Moreover, there holds the estimate*

$$\|\mathbf{u}\|_{D_{(t_0,t_0+\tau)}}^{(2+l,1+l/2)} + \|\nabla q\|_{D_{(t_0,t_0+\tau)}}^{(l,l/2)} + \|q\|_{D_{(t_0,t_0+\tau)}}^{(l,l/2)} + \|[q]|_\Gamma\|_{W_2^{l+1/2,l/2+1/4}(G_{(t_0,t_0+\tau)})} \le c_3 \mathrm{e}^{-bt_0}\varepsilon, \tag{1.8}$$

where c_3 *is independent of* t_0.

One can conclude from this theorem that the trivial solution is unique when initial velocity and mass forces are absent. The stability of this solution takes place in the sense that the solution differs a little from zero under a small deviation of the data from zero.

At the end of the paper, we give a necessary upper bound of the initial distance between the outer boundary and the fluid interface.

2. Local existence theorem for the case of non-negative surface tension

The aim of this section is to consider the main steps of the proof of Theorem 1.1 which was not published earlier in detail. It was obtained in the Ph.D. thesis of the author [4] while its statement was published in [3], [6]. More precisely, in [4] it was studied an interface problem, more general than (1.4), which was governing the motion of two incompressible liquids with including capillary effect. The system was

$$\begin{aligned}
&\mathscr{D}_t\mathbf{u} - \nu^{\pm}\nabla^2_{\mathbf{u}}\mathbf{u} + \frac{1}{\rho^{\pm}}\nabla_{\mathbf{u}}q = \mathbf{f}, \qquad \nabla_{\mathbf{u}}\cdot\mathbf{u} = 0 \quad \text{in } Q_T^{\pm} = \Omega_0^{\pm}\times(0,T),\\
&\mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega_0^-\cup\Omega_0^+, \qquad \mathbf{u}|_{S_T} = 0, \qquad (S_T \equiv S\times(0,T)),\\
&[\mathbf{u}]|_{G_T} = 0, \qquad [\mu^{\pm}\Pi_0\Pi\mathbb{S}_{\mathbf{u}}(\mathbf{u})\mathbf{n}]|_{G_T} = 0 \qquad (G_T \equiv \Gamma\times(0,T)),\\
&[\mathbf{n}_0\cdot\mathbb{T}_{\mathbf{u}}(\mathbf{u},q)\mathbf{n}]|_{G_T} = \sigma H\mathbf{n}\cdot\mathbf{n}_0,
\end{aligned} \tag{2.1}$$

where $\sigma \geq 0$ is the surface tension coefficient, $H(x,t)$ is twice the mean curvature of Γ_t ($H < 0$ at the points where Γ_t is convex toward Ω_t^-).

Theorem 2.1 (Local existence theorem for the problem with surface tension). *Suppose that* $\Gamma \in W_2^{5/2+l}$ *for some* $l \in (1/2,1)$. *Let, in addition, the assumptions of Theorem* 1.1 *be satisfied. Then there exists a constant* $T_0 \in (0,T]$ *such that problem* (2.1) *is uniquely solvable on the interval* $(0,T_0]$, *its solution* $(\mathbf{u},q)$ *has the properties:* $\mathbf{u} \in \mathbf{W}_2^{2+l,1+l/2}(D_{T_0})$, $q \in W_2^{l,l/2}(D_{T_0})$, $\nabla q \in \mathbf{W}_2^{l,l/2}(D_{T_0})$, $[q]|_{\Gamma} \in W_2^{l+1/2,l/2+1/4}(G_{T_0})$, *and the inequality*

$$\begin{aligned}
&\|\mathbf{u}\|_{D_{T_0}}^{(2+l,1+l/2)} + \|\nabla q\|_{D_{T_0}}^{(l,l/2)} + \|q\|_{D_{T_0}}^{(l,l/2)} + \|[q]|_{\Gamma}\|_{W_2^{l+1/2,l/2+1/4}(G_{T_0})}\\
&\quad \leq c_1(c_2 + T_0^{(1-l)/2}\|\mathbf{v}_0\|_{\mathbf{W}_2^1(\Omega)})\\
&\qquad \times\{|\mathbf{f}|_{Q_{T_0}}^{(1,\beta)} + |\nabla\mathbf{f}|_{Q_{T_0}}^{(0,\beta)} + \|\mathbf{v}_0\|_{\mathbf{W}_2^{1+l}(\cup_i\Omega_0^i)} + \sigma\|H_0\|_{W_2^{l+1/2}(\Gamma)}\}
\end{aligned} \tag{2.2}$$

holds; here H_0 *denotes the doubled mean curvature of* Γ. *The value* T_0 *depends on the norms of* $\mathbf{f}$, $\mathbf{v}_0$ *and the curvature of* Γ.

The base for proving Theorem 2.1 is the unique solvability of a linearized problem that was also obtained in [4] for an arbitrary finite time interval. The surface S was absent there, the domain $\Omega_0^-\cup\overline{\Omega_0^+}$ coinciding with the whole space $\mathbb{R}^3$. But this result is also valid in the case with S bounding a finite fluid volume.

We apply the well-known relation

$$H\mathbf{n} = \Delta(t)\mathbf{x} \equiv \Delta(t)\mathbf{X}_{\mathbf{u}}, \tag{2.3}$$

where $\Delta(t)$ denotes the Beltrami-Laplace operator on Γ_t.

Thus, let us consider problem (2.1) linearized on a given vector field $\mathbf{u}$:

$$
\begin{aligned}
&\mathscr{D}_t\mathbf{w}-\nu^{\pm}\nabla_{\mathbf{u}}^2\mathbf{w}+\frac{1}{\rho^{\pm}}\nabla_{\mathbf{u}}s=\mathbf{f},\qquad \nabla_{\mathbf{u}}\cdot\mathbf{w}=r\qquad \text{in } D_T,\\
&\mathbf{w}|_{t=0}=\mathbf{w}_0\qquad \text{in } \Omega_0^-\cup\Omega_0^+,\\
&[\mathbf{w}]|_{G_T}=0,\qquad \mathbf{w}|_{S_T}=0,\qquad [\mu^{\pm}\Pi_0\Pi\mathbb{S}_{\mathbf{u}}(\mathbf{w})\mathbf{n}]|_{\Gamma}=\Pi_0\mathbf{a},\\
&[\mathbf{n}_0\cdot\mathbb{T}_{\mathbf{u}}(\mathbf{w},s)\mathbf{n}]|_{\Gamma}-\sigma\mathbf{n}_0\cdot\Delta(t)\int_0^t\mathbf{w}|_{\Gamma}\,\mathrm{d}\tau=b+\sigma\int_0^t B\,\mathrm{d}\tau\qquad \text{on } G_T.
\end{aligned}
\tag{2.4}
$$

The functions in the right-hand sides of all of the equations, initial and boundary conditions are given.

The first step is the consideration of problem (2.4) with $\mathbf{u}=0$. Unique solvability of this system was obtained in [8], where $\Omega_0^-\cup\overline{\Omega_0^+}\equiv\mathbb{R}^3$. In order to prove this result for a bounded domain, we need *a priori* estimates of a solution near outer boundary. To this end, we can apply existence theorem for the Dirichlet problem for the Stokes system in a half-space [16]. Now we can state the theorem of the existence and uniqueness for the bounded domain $\Omega_0^-\cup\Omega_0^+$.

Theorem 2.2 (Existence theorem for the linear problem). *Suppose that for some* $l\in(1/2,1)$, $T<\infty$, $\Gamma\in W_2^{3/2+l}$, $\mathbf{f}\in\mathbf{W}_2^{l,l/2}(D_T)$, $r\in W_2^{1+l,1/2+l/2}(D_T)$, $r=\nabla\cdot\mathbf{R}$, $\mathbf{R}\in\mathbf{W}_2^{0,1+l/2}(D_T)$, $[\mathbf{R}\cdot\mathbf{n}]|_{G_T}=0$, $\mathbf{w}_0\in\mathbf{W}_2^{1+l}(\bigcup_{i=-,+}\Omega_0^i)$, $\mathbf{a}\in\mathbf{W}_2^{l+1/2,l/2+1/4}(G_T)$, $b\in W_2^{l+1/2,l/2+1/4}(G_T)$ *and* $B\in W_2^{l-1/2,l/2-1/4}(G_T)$. *Moreover, assume also that the compatibility conditions*

$$
\begin{gathered}
[\mathbf{w}_0]|_{\Gamma}=0,\qquad [\mu^{\pm}\Pi_0\mathbb{S}(\mathbf{w}_0)\mathbf{n}_0]|_{\Gamma}=\Pi_0\mathbf{a}|_{t=0},\qquad \mathbf{w}_0|_S=0,\\
\nabla\cdot\mathbf{w}_0=r|_{t=0}\qquad \text{in } \Omega_0^-\cup\Omega_0^+
\end{gathered}
$$

are satisfied.

Then problem (2.4) *with* $\mathbf{u}=0$ *is uniquely solvable and its solution* $(\mathbf{w},s)$ *has the properties:* $\mathbf{w}\in\mathbf{W}_2^{2+l,1+l/2}(D_T)$, $s\in W_2^{l,l/2}(D_T)$, $\nabla s\in\mathbf{W}_2^{l,l/2}(D_T)$, $[s]|_{G_T}\in W_2^{l+1/2,l/2+1/4}(G_T)$ *and*

$$
\begin{aligned}
N_T[\mathbf{w},s]\equiv{}&\|\mathbf{w}\|_{D_T}^{(2+l,1+l/2)}+\|\nabla s\|_{D_T}^{(l,l/2)}+\|\mathbf{w}\|_{D_T}^{(l,l/2)}+\|[s]|_{\Gamma}\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\
&\le c_1(T)\{\|\mathbf{f}\|_{D_T}^{(l,l/2)}+\|\mathbf{w}_0\|_{\mathbf{W}_2^{1+l}(\bigcup_i\Omega_0^i)}+\|r\|_{W_2^{1+l,0}(D_T)}+\|\mathbf{R}\|_{\mathbf{W}_2^{0,1+l/2}(D_T)}\\
&\quad+T^{-l/2}\|\mathscr{D}_t\mathbf{R}\|_{D_T}+\|\mathbf{a}\|_{\mathbf{W}_2^{l+1/2,l/2+1/4}(G_T)}+\|b\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\
&\quad+T^{-l/2}\|b\|_{W_2^{1/2,0}(G_T)}+\sigma\|B\|_{W_2^{l-1/2,l/2-1/4}(G_T)}\}\\
&\equiv c_1(T)F,
\end{aligned}
\tag{2.5}
$$

$c_1(T)$ *being a non-decreasing function of* T.

The second step is to prove the solvability of problem (2.4) in the general case. We give the statement of existence theorem for it.

Theorem 2.3 (Existence theorem for the linearized problem). *Let the hypotheses of Theorem 2.2 be satisfied and let, in addition, a vector field* $\mathbf{u} \in \mathbf{W}_2^{2+l,1+l/2}(D_T)$ *be continuous across the boundary* Γ *and satisfy the inequality*

$$T^{1/2}\|\mathbf{u}\|_{D_T}^{(2+l,1+l/2)} \le \delta, \tag{2.6}$$

with a small number δ *for some* $T < \infty$.

Then there exists a unique solution $(\mathbf{w}, s)$ *of* (2.4) *such that* $\mathbf{w} \in \mathbf{W}_2^{2+l,1+l/2}(D_T)$, $s \in W_{2,\mathrm{loc}}^{l,l/2}(D_T)$, $\nabla s \in \mathbf{W}_2^{l,l/2}(D_T)$, $[s]|_{G_T} \in W_2^{l+1/2,l/2+1/4}(G_T)$ *and inequality* (2.5) *holds for it with* $c_1(T) = c_2 + c_3 T^{(1-l)/2}\|\mathbf{u}(\cdot,0)\|_{\mathbf{W}_2^1(\Omega)}$, c_2, c_3 *being a non-decreasing functions of* T.

We solve problem (2.4) by successive approximations taking $\mathbf{w}^{(0)} = 0$, $s^{(0)} = 0$ and defining $(\mathbf{w}^{(m+1)}, s^{(m+1)})$, $m \ge 0$, as solutions to the problems

$$\begin{aligned}
&\mathscr{D}_t\mathbf{w}^{(m+1)} - \nu^{\pm}\nabla^2\mathbf{w}^{(m+1)} + \frac{1}{\rho_0^{\pm}}\nabla s^{(m+1)} = \mathbf{f} + \mathbf{l}_1(\mathbf{w}^{(m)}, s^{(m)}),\\
&\nabla\cdot\mathbf{w}^{(m+1)} = r + l_2(\mathbf{w}^{(m)}) = \nabla\cdot\big(\mathbf{R} + \mathscr{L}(\mathbf{w}^{(m)})\big) \quad \text{in } D_T,\\
&\mathbf{w}^{(m+1)}|_{t=0} = \mathbf{w}_0, \qquad \mathbf{w}^{(m+1)}|_{S_T} = 0,\\
&[\mathbf{w}^{(m+1)}]|_{G_T} = 0, \qquad [\mu^{\pm}\Pi_0\mathbb{S}(\mathbf{w}^{(m+1)})\mathbf{n}_0]|_{\Gamma} = \mathbf{l}_3(\mathbf{w}^{(m)}) + \Pi_0\mathbf{a},\\
&[\mathbf{n}_0\cdot\mathbb{T}(\mathbf{w}^{(m+1)}, s^{(m+1)})\mathbf{n}_0]|_{\Gamma} - \sigma\mathbf{n}_0\cdot\Delta(0)\int_0^t \mathbf{w}^{(m+1)}|_{\Gamma}\,\mathrm{d}\tau\\
&\qquad = l_4(\mathbf{w}^{(m)}, s^{(m)}) + b + \sigma\int_0^t \big(l_5(\mathbf{w}^{(m)}) + B\big)\,\mathrm{d}\tau, \qquad t \in (0,T).
\end{aligned} \tag{2.7}$$

Here we use the notation:

$$\begin{aligned}
\mathbf{l}_1(\mathbf{w}, s) &= \nu^{\pm}(\nabla_{\mathbf{u}}^2 - \nabla^2)\mathbf{w} + (\nabla - \nabla_{\mathbf{u}})s\\
l_2(\mathbf{w}) &= (\nabla - \nabla_{\mathbf{u}})\mathbf{w} = \nabla\cdot\mathscr{L}(\mathbf{w}), \qquad \mathscr{L}(\mathbf{w}) = (\mathbb{I} - \mathbb{A}^T)\mathbf{w},\\
\mathbf{l}_3(\mathbf{w}) &= \big[\mu^{\pm}\Pi_0\big(\mathbb{S}(\mathbf{w})\mathbf{n}_0 - \Pi_{\mathbf{u}}\mathbb{S}_{\mathbf{u}}(\mathbf{w})\mathbf{n}\big)\big]|_{\Gamma},\\
l_4(\mathbf{w}, s) &= \big[\mathbf{n}_0\cdot\big(\mathbb{T}(\mathbf{w}, s)\mathbf{n}_0 - \mathbb{T}_{\mathbf{u}}(\mathbf{w}, s)\mathbf{n}\big)\big]|_{\Gamma},\\
l_5(\mathbf{w}) &= \mathbf{n}_0\cdot\mathscr{D}_t\Big\{\big(\Delta(t) - \Delta(0)\big)\int_0^t \mathbf{w}|_{\Gamma}\,\mathrm{d}\tau\Big\}\\
&= \mathbf{n}_0\cdot\Big\{\big(\Delta(t) - \Delta(0)\big)\mathbf{w}|_{\Gamma} + \dot{\Delta}(t)\int_0^t \mathbf{w}|_{\Gamma}\,\mathrm{d}\tau\Big\},
\end{aligned} \tag{2.8}$$

where $\Delta(0)$ is the Beltrami–Laplace operator on Γ, $\dot{\Delta}(t)$ is the derivative of $\Delta(t)$ with respect to time.

The operators $\mathbf{l}_1, \dots, l_5$ and $\mathscr{L}$ were considered in [19]. The estimates obtained there imply the lemma as follows:

Lemma 2.1. *If* $\mathbf{u}$ *and* $\mathbf{u}'$ *satisfy inequality* (2.6) *and* $[\mathbf{u}]|_\Gamma = [\mathbf{u}']|_\Gamma = 0$, *then*

$$\begin{aligned}
&\|\mathbf{l}_1(\mathbf{w}, s) - \mathbf{l}_1'(\mathbf{w}, s)\|_{D_T}^{(l,l/2)} + \|l_2(\mathbf{w}) - l_2'(\mathbf{w})\|_{W_2^{1+l,(1+l)/2}(D_T)} \\
&\quad + \|\mathbf{l}_3(\mathbf{w}) - \mathbf{l}_3'(\mathbf{w})\|_{W_2^{1/2+l,1/4+l/2}(G_T)} + \|l_5(\mathbf{w}) - l_5'(\mathbf{w})\|_{G_T}^{(l-1/2,l/2-1/4)} \\
&\quad \le c_{17}\sqrt{T}\|\mathbf{u} - \mathbf{u}'\|_{D_T}^{(2+l,1+l/2)}\{\|\mathbf{w}\|_{D_T}^{(2+l,1+l/2)} + \|\nabla s\|_{D_T}^{(l,l/2)}\}, \\
&\|\mathscr{D}_t(\mathscr{L}(\mathbf{w}) - \mathscr{L}'(\mathbf{w}))\|_{D_T}^{(0,l/2)} \\
&\quad \le c_{18}\{\sqrt{T}\|\mathbf{u} - \mathbf{u}'\|_{D_T}^{(2+l,1+l/2)} \\
&\qquad + T^{(1-l)/2}\|\mathbf{u}(\cdot,0) - \mathbf{u}'(\cdot,0)\|_{\mathbf{W}_2^1(\Omega)}\}\|\mathbf{w}\|_{D_T}^{(2+l,1+l/2)}, \\
&\|l_4(\mathbf{w}, s) - l_4'(\mathbf{w}, s)\|_{W^{1/2+l,1/4+l/2}(G_T)} \\
&\quad \le c_{19}\sqrt{T}\|\mathbf{u} - \mathbf{u}'\|_{D_T}^{(2+l,1+l/2)} \\
&\qquad \times \{\|\mathbf{w}\|_{D_T}^{(2+l,1+l/2)} + \|\nabla s\|_{D_T}^{(l,l/2)} + \|s\|_{D_T}^{(l,l/2)} + \|s\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\}.
\end{aligned} \tag{2.9}$$

Here the operators $\mathbf{l}_1', \dots, l_5'$ *and* $\mathscr{L}'$ *are calculated according to formulas* (2.8), *where vector* $\mathbf{u}$ *is replaced by* $\mathbf{u}'$. *If* $\mathbf{w}|_{t=0} = 0$, *then inequality* (2.9) *is valid without* $T^{(1-l)/2}\|\mathbf{u}(\cdot,0) - \mathbf{u}'(\cdot,0)\|_{\mathbf{W}_2^1(\Omega)}$ *on the right-hand side.*

From Lemma 2.1 it follows the lemma.

Lemma 2.2. *If* $\mathbf{u}$ *satisfies inequality* (2.6) *and* $[\mathbf{u}]|_\Gamma = 0$, *then*

$$\begin{aligned}
&\|\mathbf{l}_1(\mathbf{w}, s)\|_{D_T}^{(l,l/2)} + \|l_2(\mathbf{w})\|_{W_2^{1+l,(1+l)/2}(D_T)} + \|\mathbf{l}_3(\mathbf{w})\|_{W_2^{1/2+l,1/4+l/2}(G_T)} \\
&\quad + \|l_4(\mathbf{w}, s)\|_{W^{1/2+l,1/4+l/2}(G_T)} + \|l_5(\mathbf{w})\|_{G_T}^{(l-1/2,l/2-1/4)} \\
&\quad \le c_{20}\delta\{\|\mathbf{w}\|_{D_T}^{(2+l,1+l/2)} + \|\nabla s\|_{D_T}^{(l,l/2)} + \|s\|_{D_T}^{(l,l/2)} + \|s\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\}, \\
&\|\mathscr{D}_t\mathscr{L}(\mathbf{w})\|_{D_T}^{(0,l/2)} \le c_{21}\{\delta + T^{(1-l)/2}\|\mathbf{u}(\cdot,0)\|_{\mathbf{W}_2^1(\Omega)}\}\|\mathbf{w}\|_{D_T}^{(2+l,1+l/2)}.
\end{aligned} \tag{2.10}$$

If $\mathbf{w}(\cdot,0) = 0$ *in* Ω, *the term with* $\|\mathbf{u}(\cdot,0)\|_{\mathbf{W}_2^1(\Omega)}$ *may be dropped in the last inequality.*

For the difference

$$\mathbf{f}(X_{\mathbf{u}},t)-\mathbf{f}(X_{\mathbf{u}'},t)=\sum_{k=1}^{3}\int_0^1 \partial\mathbf{f}(X_{\mathbf{u}_s},t)/\partial x_k\,\mathrm{d}s\int_0^t (u_k-u_k')\,\mathrm{d}\tau,$$

where $\mathbf{u}_s=\mathbf{u}'+s\tilde{\mathbf{u}}$ is continuous transformation from $\mathbf{u}'$ into $\mathbf{u}$ with $\tilde{\mathbf{u}}\equiv\mathbf{u}-\mathbf{u}'$, the following lemma was proven in [19].

Lemma 2.3. *Let* $\mathbf{f}$ *satisfy the assumptions of Theorem* 1.1, *and let vectors* $\mathbf{u}$, $\mathbf{u}'\in\mathbf{W}_2^{l,l/2}(D_T)$, $[\mathbf{u}]|_\Gamma=[\mathbf{u}']|_\Gamma=0$, *satisfy inequality* (2.6). *Then*

$$\|\mathbf{f}(X_{\mathbf{u}},t)-\mathbf{f}(X_{\mathbf{u}'},t)\|_{Q_T}^{(l,l/2)}\le c(T)\int_0^T\|\mathbf{u}-\mathbf{u}'\|_{\mathbf{W}_2^l(\Omega)}\,\mathrm{d}t.$$

Here $c(T)$ *is a power function of* T.

Proof of Theorem 2.3. Let us return to problem (2.7). Observe that the vector $\mathscr{L}(\mathbf{w}^{(m)})=(\mathbb{I}-\mathbb{A}^T)\mathbf{w}^{(m)}$ is continuous across Γ: $[\mathscr{L}(\mathbf{w}^{(m)})\cdot\mathbf{n}_0]|_\Gamma=[\mathbf{n}_0\cdot(\mathbb{I}-\mathbb{A}^T)\mathbf{w}^{(m)}]_\Gamma=[\mathbb{A}\mathbf{n}_0]|_\Gamma\cdot\mathbf{w}^{(m)}=0$. This follows from the formula for the co-factors A_{ij} to $a_{ij}=\partial x_i/\partial\xi_j$ due to the continuity of $\mathbf{x}$ and its tangent derivatives $\nabla_\Gamma\mathbf{x}=\Pi_0\nabla\mathbf{x}$: for example, for A_{1j} we have

$$[A_{1j}n_{0j}]_\Gamma=[\mathbf{n}_0\cdot(\nabla x_2\times\nabla x_3)]_\Gamma=[\mathbf{n}_0\cdot(\nabla_\Gamma x_2\times\nabla_\Gamma x_3)]_\Gamma=0.$$

Hence, we can apply Theorem 2.2 to (2.7) and conclude by Lemma 2.2 that $(\mathbf{w}^{(m+1)},s^{(m+1)})$, $m\in\mathbb{N}$, are uniquely defined, $(\mathbf{w}^{(1)},s^{(1)})$ being a solution of (2.4) with $\mathbf{u}=0$ and satisfying inequality (2.5); $\mathbf{w}^{(0)}=0$, $s^{(0)}=0$.

Let us consider the differences $\mathbf{z}^{(m+1)}=\mathbf{w}^{(m+1)}-\mathbf{w}^{(m)}$, $g^{(m+1)}=s^{(m+1)}-s^{(m)}$, $m\in\mathbb{N}\cup\{0\}$. We have the problem for $m\in\mathbb{N}$ as follows:

$$\begin{aligned}
&\mathscr{D}_t\mathbf{z}^{(m+1)}-\nu^{\pm}\nabla^2\mathbf{z}^{(m+1)}+\frac{1}{\rho_0^-}\nabla g^{(m+1)}=\mathbf{l}_1(\mathbf{z}^{(m)},g^{(m)}),\\
&\nabla\cdot\mathbf{z}^{(m+1)}=l_2(\mathbf{z}^{(m)})=\nabla\cdot\mathscr{L}(\mathbf{z}^{(m)})\qquad\text{in } D_T,\\
&\mathbf{z}^{(m+1)}|_{t=0}=0,\qquad \mathbf{z}^{(m+1)}|_{S_T}=0,\\
&[\mathbf{z}^{(m+1)}]|_{G_T}=0,\qquad [\mu^{\pm}\Pi_0\mathbb{S}(\mathbf{z}^{(m+1)})\mathbf{n}_0]|_{G_T}=\mathbf{l}_3(\mathbf{z}^{(m)}),\\
&[\mathbf{n}_0\cdot\mathbb{T}(\mathbf{z}^{(m+1)},g^{(m+1)})\mathbf{n}_0]|_\Gamma-\sigma\mathbf{n}_0\cdot\Delta(0)\int_0^t\mathbf{z}^{(m+1)}|_\Gamma\,\mathrm{d}\tau\\
&\qquad=l_4(\mathbf{z}^{(m)},g^{(m)})+\sigma\int_0^t l_5(\mathbf{z}^{(m)})\,\mathrm{d}\tau,\qquad t\in(0,T).
\end{aligned}$$

If $m > 1$, then $\mathbf{z}^{(m)}|_{t=0} = 0$, and we deduce from (2.5) and Lemma 2.2 that

$$N_T[\mathbf{z}^{(m+1)}, g^{(m+1)}] \le c_{22}\delta N_T[\mathbf{z}^{(m)}, g^{(m)}]. \tag{2.11}$$

If $m = 1$ then in virtue of (2.10) we obtain

$$N_T[\mathbf{z}^{(2)}, g^{(2)}] \le (c_{22}\delta + c_{21}\delta_1)N_T[\mathbf{w}^{(1)}, s^{(1)}] \tag{2.12}$$

with $\delta_1 = T^{(1-l)/2}\|\mathbf{u}(\cdot, 0)\|_{\mathbf{W}_2^1(\Omega)}$ because $\mathbf{z}^{(1)}|_{t=0} \equiv \mathbf{w}^{(1)}|_{t=0} = \mathbf{w}_0 \ne 0$ in the general case.

Next, for $\Sigma_m = \sum_{j=2}^m N_T[\mathbf{z}^{(j)}, g^{(j)}]$ the following inequality

$$\Sigma_{m+1} \le c_{22}\delta\Sigma_m + N_T[\mathbf{z}^{(2)}, g^{(2)}]$$

holds due to (2.11). Let's choose δ such that $c_{22}\delta < 1$. It is obvious that

$$\Sigma_{m+1} \le (1 - c_{22}\delta)^{-1}N_T[\mathbf{z}^{(2)}, g^{(2)}].$$

In view of (2.5), (2.12) we have:

$$\begin{aligned} N_T[\mathbf{w}^{(m+1)}, s^{(m+1)}] &\le \Sigma_{m+1} + N_T[\mathbf{w}^{(1)}, s^{(1)}] \\ &\le \Big(\frac{1}{1 - c_{22}\delta} + \frac{c_{21}}{1 - c_{22}\delta}T^{(1-l)/2}\|\mathbf{u}(\cdot, 0)\|_{\mathbf{W}_2^1(\Omega)}\Big)c_1F, \end{aligned}$$

where F is the sum of the right-hand side norms in (2.5) which is independent of m. Hence, the sequence $\{\mathbf{w}^{(m+1)}, s^{(m+1)}\}$ is convergent in the norm $N_T[\cdot\,,\cdot]$ and its limit $(\mathbf{w}, s)$ is a solution of (2.4) satisfying inequality (2.5) with

$$c_1(T) \equiv c_0(T) = \frac{c_1}{1 - c_{22}\delta}\big(1 + c_{21}T^{(1-l)/2}\|\mathbf{u}(\cdot, 0)\|_{\mathbf{W}_2^1(\Omega)}\big).$$

In a similar way, we can conclude that the difference $(\mathbf{z} = \mathbf{w} - \mathbf{w}', g = s - s')$ of two solutions of (2.4) satisfies the estimate

$$N_T[\mathbf{z}, g] \le c_{22}\delta N_T[\mathbf{z}, g]$$

whence it follows $\mathbf{z} = 0,\ g = 0$. Thus, the uniqueness of the solution is also proved. □

In order to demonstrate Theorem 2.1, we apply again successive approximations, now for solving system (2.1), where we make use of the formula (2.3) continued as follows:

$$\Delta(t)\mathbf{X_u} = \Delta(t)\boldsymbol{\xi} + \Delta(t)\int_0^t \mathbf{u}\,d\tau = \Delta(0)\boldsymbol{\xi} + \int_0^t \dot{\Delta}(\tau)\boldsymbol{\xi}\,d\tau + \Delta(t)\int_0^t \mathbf{u}\,d\tau,$$

here $\dot{\Delta}(t) = \mathscr{D}_t\Delta(t)$.

We put $\mathbf{u}^{(0)} = 0$, $q^{(0)} = 0$ and define the first approximation $\mathbf{u}^{(1)}$, $q^{(1)}$ as a solution to the problem

$$\begin{aligned}
&\mathscr{D}_t\mathbf{u}^{(1)} - \nu^{\pm}\nabla^2\mathbf{u}^{(1)} + \frac{1}{\rho^{\pm}}\nabla q^{(1)} = \mathbf{f}, \qquad \nabla\cdot\mathbf{u}^{(1)} = 0 \quad \text{in } D_T,\\
&\mathbf{u}^{(1)}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega_0^- \cup \Omega_0^+, \qquad \mathbf{u}^{(1)}|_S = 0,\\
&[\mathbf{u}^{(1)}]|_{G_T} = 0, \qquad [\mu^{\pm}\Pi_0\mathbb{S}(\mathbf{u}^{(1)})\mathbf{n}_0]|_{G_T} = 0,\\
&[\mathbf{n}_0\cdot\mathbb{T}(\mathbf{u}^{(1)}, q^{(1)})\mathbf{n}_0]|_\Gamma - \sigma\mathbf{n}_0\cdot\Delta(0)\int_0^t \mathbf{u}^{(1)}\,d\tau|_\Gamma = \sigma H_0, \qquad t\in(0,T),
\end{aligned} \tag{2.13}$$

here $H_0(\xi) = \mathbf{n}_0\cdot\Delta(0)\boldsymbol{\xi}$ is twice the mean curvature of Γ. As $H_0 \in W_2^{l+1/2}(\Gamma)$, problem (2.13) is solvable by Theorem 2.2 on the interval $(0, T_1)$, $T_1 = T$, and

$$N_{T_1}[\mathbf{u}^{(1)}, q^{(1)}] \le c(T_1)\{\|\mathbf{f}\|_{Q_T}^{(l,l/2)} + \|\mathbf{v}_0\|_{\mathbf{W}_2^{1+l}(\cup_{i=\pm}\Omega^i)} + \sigma\|H_0\|_{W_2^{l+1/2}(\Gamma)}\}. \tag{2.14}$$

Let the functions $\mathbf{u}^{(m+1)}$, $q^{(m+1)}$, $m\in\mathbb{N}$, solve the problem

$$\begin{aligned}
&\mathscr{D}_t\mathbf{u}^{(m+1)} - \nu^{\pm}\nabla_m^2\mathbf{u}^{(m+1)} + \frac{1}{\rho^{\pm}}\nabla_m q^{(m+1)} = \mathbf{f}(X_m, t),\\
&\nabla_m\cdot\mathbf{u}^{(m+1)} = 0 \quad \text{in } D_T,\\
&\mathbf{u}^{(m+1)}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega_0^-\cup\Omega_0^+, \qquad \mathbf{u}^{(m+1)}|_S = 0,\\
&[\mathbf{u}^{(m+1)}]|_{G_T} = 0, \qquad [\mu^{\pm}\Pi_0\Pi_m\mathbb{S}_m(\mathbf{u}^{(m+1)})\mathbf{n}_m]|_{G_T} = 0,\\
&[\mathbf{n}_0\cdot\mathbb{T}_m(\mathbf{u}^{(m+1)}, q^{(m+1)})\mathbf{n}_m]|_\Gamma - \sigma\mathbf{n}_0\cdot\Delta_m(t)\int_0^t \mathbf{u}^{(m+1)}\,d\tau|_\Gamma\\
&\qquad = \sigma\Big(H_0(\xi) + \mathbf{n}_0\cdot\int_0^t \dot{\Delta}_m(\tau)\boldsymbol{\xi}\,d\tau\Big)|_\Gamma, \qquad t\in(0,T).
\end{aligned} \tag{2.15}$$

Here we have used the notation: $\nabla_m = \nabla_{\mathbf{u}^{(m)}}$, etc.; $\mathbf{n}_m$ is the outward normal to the surface $\Gamma_m(t) = \{x = X_m(\xi, t)\,|\,\xi\in\Gamma\}$, where X_m is calculated by (1.3) with $\mathbf{u} = \mathbf{u}^{(m)}$; Π_m is the projector onto the tangential plane to $\Gamma_m(t)$, $\Delta_m(t)$ is the Beltrami–Laplace operator on $\Gamma_m(t)$.

Since the vector $\mathbf{f}$ satisfies the inequality

$$\|\mathbf{f}(X_m, t)\|_{Q_{T_m}}^{(l,l/2)} \le c(T_m)\{|\mathbf{f}|_{Q_{T_m}}^{(1,\beta)} + |\nabla\mathbf{f}|_{Q_{T_m}}^{(0,\beta)}\},$$

where $c(T)$ is a power function of T, and since $\mathbf{n}_0 \cdot \dot{\Delta}_m(\tau)\boldsymbol{\xi} \in W_2^{l-1/2,l/2-1/4}(G_{T_m})$ if $\mathbf{u}^{(m)} \in \mathbf{W}_2^{2+l,1+l/2}(D_{T_m})$ [19], $H_0 \in W_2^{l+1/2,l/2+1/4}(G_{T_m})$, by Theorem 2.3, there exists a solution $(\mathbf{u}^{(m+1)}, q^{(m+1)})$ of (2.15) on an interval $(0, T_{m+1})$ on which the approximation $(\mathbf{u}^{(m)}, q^{(m)})$ is defined and condition (2.6) holds for $\mathbf{u}^{(m)}$ with a sufficiently small $\delta > 0$.

It is necessary to show that there exists T_0 such that $T_m \geq T_0 > 0$ for $\forall m \in \mathbb{N}$, $N_{T_0}[\mathbf{u}^{(m)} q^{(m)}]$ are uniformly bounded and that the sequence $\{\mathbf{u}^{(m)}, q^{(m)}\}_{m=1}^{\infty}$ converges to a solution of problem (2.1). The proof of these facts is based on Lemmas 2.1 and 2.3 applied to the right-hand sides of the systems

$$\begin{aligned}
&\mathscr{D}_t\tilde{\mathbf{w}}^{(j+1)} - \nu^{\pm}\nabla_j^2\tilde{\mathbf{w}}^{(j+1)} + \frac{1}{\rho^{\pm}}\nabla_j\tilde{s}^{(j+1)} \\
&\qquad = \mathbf{l}_1^{(j)}(\mathbf{u}^{(j)}, q^{(j)}) - \mathbf{l}_1^{(j-1)}(\mathbf{u}^{(j)}, q^{(j)}) + \mathbf{f}(X_j, t) - \mathbf{f}(X_{j-1}, t), \\
&\nabla_j \cdot \tilde{\mathbf{w}}^{(j+1)} = l_2^{(j)}(\mathbf{u}^{(j)}) - l_2^{(j-1)}(\mathbf{u}^{(j)}) \qquad \text{in } D_{T_{m+1}}, \\
&\tilde{\mathbf{w}}^{(j+1)}(\xi, 0) = 0, \qquad \xi \in \Omega_0^- \cup \Omega_0^+, \\
&[\tilde{\mathbf{w}}^{(j+1)}]|_\Gamma = 0, \qquad \tilde{\mathbf{w}}^{(j+1)}|_S = 0, \\
&[\mu^{\pm}\Pi_0\Pi_j\mathbb{S}_j(\tilde{\mathbf{w}}^{(j+1)})\mathbf{n}_j]|_\Gamma = \mathbf{l}_3^{(j)}(\mathbf{u}^{(j)}) - \mathbf{l}_3^{(j-1)}(\mathbf{u}^{(j)}), \\
&[\mathbf{n}_0 \cdot \mathbb{T}_j(\tilde{\mathbf{w}}^{(j+1)}, \tilde{s}^{(j+1)})\mathbf{n}_j]|_\Gamma - \sigma\mathbf{n}_0 \cdot \Delta_j(t)\int_0^t \tilde{\mathbf{w}}^{(j+1)}\,\mathrm{d}\tau|_\Gamma \\
&\qquad = l_4^{(j)}(\mathbf{u}^{(j)}, q^{(j)}) - l_4^{(j-1)}(\mathbf{u}^{(j)}, q^{(j)}) + \sigma\int_0^t \big(l_5^{(j)}(\mathbf{u}^{(j)}) - l_5^{(j-1)}(\mathbf{u}^{(j)})\big)\,\mathrm{d}\tau \\
&\qquad\quad + \sigma\int_0^t \mathbf{n}_0 \cdot \big(\dot{\Delta}_j(\tau) - \dot{\Delta}_{j-1}(\tau)\big)\xi\,\mathrm{d}\tau|_\Gamma, \qquad t \in (0, T_{m+1}),
\end{aligned} \tag{2.16}$$

where $\tilde{\mathbf{w}}^{(j+1)}$, $\tilde{s}^{(j+1)}$ mean the differences $\mathbf{u}^{(j+1)} - \mathbf{u}^{(j)}$, $q^{(j+1)} - q^{(j)}$, respectively, $j \leq m$; the operators $l_i^{(k)}$ are calculated by (2.8) with $\mathbf{u} = \mathbf{u}^{(k)}$, $k \leq m$; $\mathbf{u}^{(0)} = 0$.

The norms on the right-hand sides of (2.16) are estimated either by lower norms of $\tilde{\mathbf{w}}^{(j)}$ and $\tilde{s}^{(j)}$, or by the leading part of their norms but with small coefficients including δ from inequality (2.6). In particular,

$$\begin{aligned}
\big\|\mathbf{n}_0 \cdot \big(\dot{\Delta}_j(\tau) - \dot{\Delta}_{j-1}(\tau)\big)\xi\big\|_{G_{T_{m+1}}}^{(l-1/2,l/2-1/4)} &\leq c\|\nabla(\tilde{\mathbf{w}}^{(j)})\|_{G_{T_{m+1}}}^{(l-1/2,l/2-1/4)} \\
&\leq c\|\tilde{\mathbf{w}}^{(j)}\|_{D_{T_{m+1}}}^{(1+l,1/2+l/2)}.
\end{aligned}$$

In addition, by Lemma 2.3 we have

$$\|\mathbf{f}(X_j, t) - \mathbf{f}(X_{j-1}, t)\|_{Q_{T_{m+1}}}^{(l,l/2)} \leq c(T_{m+1})\int_0^{T_{m+1}} \|\tilde{\mathbf{w}}^{(j)}\|_{\mathbf{W}_2^l(\Omega)}\,\mathrm{d}t,$$

where $c(T)$ is nondecreasing function of T depending on the norms $|\mathbf{f}|^{(1,\beta)}$ and $|\nabla\mathbf{f}|^{(0,\beta)}$ in Q_T. One can deduce from this the boundedness of $\Sigma'_{m+1}(T') \equiv \sum_{j=2}^{m+1} N_{T'}[\tilde{\mathbf{w}}^{(j)}, \tilde{s}^{(j)}]$:

$$\Sigma'_{m+1}(T') \le c_1\big(c_2(T',\delta) + T'^{(1-l)/2}\|\mathbf{v}_0\|_{\mathbf{W}_2^1(\Omega)}\big)N_{T'}[\mathbf{u}^{(1)}, q^{(1)}], \qquad T' \in (0, T_{m+1}],$$

which implies the convergence of $\{\mathbf{u}^{(m)}, q^{(m)}\}_{m=1}^{\infty}$ in itself and the estimate

$$\begin{aligned} N_{T'}[\mathbf{u}^{(m+1)}, q^{(m+1)}] &\le \Sigma'_{m+1}(T') + N_{T'}[\mathbf{u}^{(1)}, q^{(1)}] \\ &\le c_{23}(T', \|\mathbf{v}_0\|_{\mathbf{W}_2^1(\Omega)})\{|\mathbf{f}|_{Q_{T'}}^{(1,\beta)} + |\nabla\mathbf{f}|_{Q_{T'}}^{(0,\beta)} \\ &\quad + \|\mathbf{v}_0\|_{\mathbf{W}_2^{1+l}(\bigcup_i \Omega_0^i)} + \sigma\|H_0\|_{W_2^{l+1/2}(\Gamma)}\} \end{aligned} \tag{2.17}$$

due to (2.14). Since the right-hand side is independent of m, and c_{23} is nondecreasing function of T', we can find such $T_0 \in (0, T_{m+1}]$ that

$$T_0^{1/2} N_{T_0}[\mathbf{u}^{(j)}, q^{(j)}] \le \delta, \qquad \forall j \in \mathbb{N}.$$

Hence, as follows from (2.17), $N_{T_0}[\mathbf{u}^{(j)}, q^{(j)}]$ are uniformly bounded and the sequence $\{\mathbf{u}^{(j)}, q^{(j)}\}_{j=1}^{\infty}$ convergent. Passing to the limit in system (2.15), we make sure that the approximations $(\mathbf{u}^{(j)}, q^{(j)})$, $j \in \mathbb{N}$, converge to a solution of problem (2.1) for which the inequality (2.2) holds.

A similar consideration for the case of a single fluid was presented in detail in [19].

Remark 2.1. If $\sigma = 0$, Theorem 2.1 holds with the initial interface $\Gamma \in W_2^{3/2+l}$. Indeed, in this case, we have the homogeneous boundary conditions in (2.15) and we do not need to calculate H_0 and $\dot{\Delta}_m(\tau)\boldsymbol{\xi}$ on Γ. It is the estimates of these functions that make us suppose Γ to belong to $W_2^{5/2+l}$ in the case of $\sigma > 0$.

Moreover, we observe that the magnitude of T_0 does not depend on the curvature of Γ if surface tension is not included into consideration.

Theorem 2.1 with Remark 2.1 implies Theorem 1.1.

3. Global solvability of the problem (1.1), (1.2)

In this section we don't take surface tension into account.

In order to prove the existence of a global solution to the nonlinear problem, we apply an exponential L_2–estimate for it with respect to time which was proven in [7].

Proposition 3.1. *Assume that a solution of problem* (1.1), (1.2) *is defined on* $[0, T]$ *and that* $\mathbf{v}_0$ *satisfies the compatibility conditions* (1.5). *Let* $\mathbf{f}(\cdot, \tau) \in L_2(\Omega)$, $t \in (0, T]$, *and* $\int_0^T e^{b\tau} \|\mathbf{f}(\cdot, \tau)\|_\Omega \, d\tau < \infty$.

Then

$$\|\mathbf{v}(\cdot, t)\|_\Omega \le e^{-bt} \left\{ \|\mathbf{v}_0\|_\Omega + \int_0^t e^{b\tau} \|\mathbf{f}(\cdot, \tau)\|_\Omega \, d\tau \right\}, \qquad t \in (0, T], \tag{3.1}$$

where $b = \min\{\nu^+, \nu^-\}/(2c_0)$ *with* c_0 *from inequality* (3.3).

Proof. We multiply the 1st equation in (1.1) by $\mathbf{v}$ and integrate by parts over $\Omega_t^- \cup \Omega_t^+$.

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}\|_\Omega^2 + \left\| \sqrt{\frac{\nu^\pm}{2}} \mathbb{S}(\mathbf{v}) \right\|_{\Omega_t^- \cup \Omega_t^+}^2 = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx. \tag{3.2}$$

First, we take into account the Korn inequality

$$\|\mathbf{v}\|_{W_2^1(\Omega_t^- \cup \Omega_t^+)} \le c_0 \|\mathbb{S}(\mathbf{v})\|_{\Omega_t^- \cup \Omega_t^+} \tag{3.3}$$

which is valid due to $\mathbf{v}|_S = 0$ [18]. It really holds in $\Omega \equiv \Omega_t^- \cup \overline{\Omega_t^+}$ because $\|\mathbf{v}\|_{W_2^1(\Omega_t^- \cup \Omega_t^+)}$ coincides with $\|\mathbf{v}\|_{W_2^1(\Omega)}$ in view of $[\mathbf{v}]|_{\Gamma_t} = 0$. Thus, c_0 is independent of t.

Next, we apply Hölder inequality to (3.2) and divide it by $\|\mathbf{v}\|_\Omega$. We arrive at

$$\frac{d}{dt} \|\mathbf{v}\|_\Omega + b \|\mathbf{v}\|_\Omega \le \|\mathbf{f}\|_\Omega$$

with $b = \min\{\nu^+, \nu^-\}/(2c_0)$. By the Gronwall lemma,

$$\|\mathbf{v}(\cdot, t)\|_\Omega \le e^{-bt} \|\mathbf{v}_0\|_\Omega + \int_0^t e^{-b(t-\tau)} \|\mathbf{f}(\cdot, \tau)\|_\Omega \, d\tau,$$

which coincides with (3.1). □

Below we use the following lemma.

Lemma 3.1. *Let* $v \in W_2^{2+l, 1+l/2}(Q_T)$, $T > 0$, $l \in (0, 1)$, $\theta > 0$. *Then the function* v *is subject to the inequality*

$$\|v\|_{Q_T}^{(l, l/2)} \le c \left\{ \theta \|v\|_{Q_T}^{(2+l, 1+l/2)} + \left(\frac{1}{\theta^{l/2}} + \frac{1}{T^{l/2}} \right) \|v\|_{Q_T} \right\}. \tag{3.4}$$

Proof. We use the known estimate (see, for example, [2]) for $v \in W_2^m(\Omega)$ with any $\varepsilon > 0$:

$$\|v\|_{\dot W_2^j(\Omega)} \le c(\varepsilon \|v\|_{\dot W_2^m(\Omega)} + \varepsilon^{-j/(m-j)} \|v\|_\Omega), \qquad 0 \le j \le m-1,\ m \ge 1.$$

Inequality (3.4) follows from the estimates

$$\|v\|_{\dot W_2^{l,0}(Q_T)} \le c(\theta \|v\|_{\dot W_2^{2+l,0}(Q_T)} + \theta^{-l/2} \|v\|_{Q_T}),$$
$$\|v\|_{\dot W_2^{0,l/2}(Q_T)} \le c(\theta \|v\|_{\dot W_2^{0,1+l/2}(Q_T)} + \theta^{-l/2} \|v\|_{Q_T}). \qquad \square$$

Proposition 3.2. *Let the solution of problem* (1.1), (1.2) *be defined on the interval* $(0, T]$ *and let the estimate*

$$N_{(0,T)}[\mathbf{v}, p] \equiv \|\mathbf{u}^0\|_{D_T}^{(2+l,1+l/2)} + \|\nabla q^0\|_{D_T}^{(l,l/2)} + \|q^0\|_{D_T}^{(l,l/2)} + \|q^0\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \le \mu$$

hold, where the pair $(\mathbf{u}^0, q^0)$ *is a solution of problem* (1.1), (1.2) *written in the Lagrangian coordinates.*

Then for $\forall t_0 \in (0, T]$ *and*

$$N_{(t_0-2\tau_0+\gamma,t_0)}[\mathbf{v}, p]$$
$$\equiv \|\mathbf{u}^0\|_{D'_\gamma}^{(2+l,1+l/2)} + \|\nabla q^0\|_{D'_\gamma}^{(l,l/2)} + \|q^0\|_{D'_\gamma}^{(l,l/2)} + \|q^0\|_{W_2^{l+1/2,l/2+1/4}(G_{(t_0-2\tau_0+\gamma,t_0)})},$$

we have

$$N_{(t_0-\tau_0,t_0)}[\mathbf{v}, p] \le c(\delta, \tau_0)\{|\mathbf{f}|_{Q'_0}^{(1,\beta)} + |\nabla \mathbf{f}|_{Q'_0}^{(0,\beta)} + \|\mathbf{v}\|_{Q'_0}\}, \tag{3.5}$$

where $Q'_\gamma = \Omega \times (t_0 - 2\tau_0 + \gamma, t_0)$, $D'_\gamma = D_{(t_0-2\tau_0+\gamma,t_0)}$, $\gamma \ge 0$, $\tau_0 \in (0, t_0/2)$, τ_0 *depends on* μ *and on the constant* δ *in* (3.8), $c(\delta, \tau_0)$ *is a nondecreasing function.*

Proof. We fix an arbitrary $t_0 \in (0, T]$. Let $\tau_0 \in (0, t_0/2)$, and let $\eta_\lambda(t)$ be a smooth monotone function of t such that

$$\eta_\lambda(t) = \begin{cases} 0 & \text{if } t \le t_0 - 2\tau_0 + \lambda/2, \\ 1 & \text{if } t \ge t_0 - 2\tau_0 + \lambda, \end{cases}$$

$\lambda \in (0, \tau_0]$, and for $\dot\eta_\lambda(t) \equiv \frac{d\eta_\lambda(t)}{dt}$ the inequalities

$$\sup_{\mathbb{R}} |\dot\eta_\lambda(t)| \le c\lambda^{-1}, \qquad \sup_{t,\tau\in\mathbb{R}} \frac{|\dot\eta_\lambda(t) - \dot\eta_\lambda(\tau)|}{|t-\tau|^{l/2}} \le c\lambda^{-1-l/2}$$

hold.

We consider the couple $\mathbf{w} = \mathbf{v}\eta_\lambda$, $s = p\eta_\lambda$. It satisfies the system

$$\begin{aligned}
&\mathscr{D}_t\mathbf{w} + (\mathbf{v}\cdot\nabla)\mathbf{w} - \nu^\pm\nabla^2\mathbf{w} + \frac{1}{\rho^\pm}\nabla s = \mathbf{f}\eta_\lambda + \mathbf{v}\dot\eta_\lambda,\\
&\nabla\cdot\mathbf{w} = 0 \quad \text{in } \Omega_t^-\cup\Omega_t^+, \quad t > t_0 - 2\tau_0,\\
&\mathbf{w}|_{t=t_0-2\tau_0} = 0 \quad \text{in } \bigcup\Omega' \equiv \Omega^-_{t_0-2\tau_0}\cup\Omega^+_{t_0-2\tau_0},\\
&[\mathbf{w}]|_{\Gamma_t} = 0, \quad [\mathbb{T}(\mathbf{w},s)\mathbf{n}]|_{\Gamma_t} = 0, \quad \mathbf{w}|_S = 0, \quad t > t_0 - 2\tau_0.
\end{aligned}$$

We introduce the Lagrangian coordinates according to the formula

$$\mathbf{x} = \boldsymbol{\xi}' + \int_{t_0-2\tau_0}^t \mathbf{u}(\xi',\tau)\,\mathrm{d}\tau \equiv \mathbf{X}(\xi',t), \qquad \xi'\in\bigcup\Omega',\ t > t_0 - 2\tau_0, \tag{3.6}$$

where $\mathbf{u}(\xi',t) = \mathbf{v}\big(X(\xi',t),t\big)$. The functions $\mathbf{w}$ and s written in the Lagrangian coordinates will be denoted by the same symbols. They solve the problem

$$\begin{aligned}
&\mathscr{D}_t\mathbf{w} - \nu^\pm\nabla^2_{\mathbf{u}}\mathbf{w} + \frac{1}{\rho^\pm}\nabla_{\mathbf{u}}s = \mathbf{f}(X,t)\eta_\lambda + \mathbf{u}\dot\eta_\lambda, \qquad \nabla_{\mathbf{u}}\cdot\mathbf{w} = 0 \quad \text{in } D_0',\\
&\mathbf{w}|_{t=t_0-2\tau_0} = 0 \quad \text{in } \bigcup\Omega',\\
&[\mathbf{w}]|_{\Gamma'} = 0, \quad [\mu^\pm\Pi_0'\Pi\mathbb{S}_{\mathbf{u}}(\mathbf{w})\mathbf{n}]|_{\Gamma'} = 0, \quad \mathbf{w}|_S = 0,\\
&[\mathbf{n}_0'\cdot\mathbb{T}_{\mathbf{u}}(\mathbf{w},s)\mathbf{n}]|_{\Gamma'} = 0, \quad t > t_0 - 2\tau_0.
\end{aligned} \tag{3.7}$$

Here, $\Gamma' = \Gamma_{t_0-2\tau_0}$, $\mathbf{n}_0'$ is the outward normal to Γ', Π_0' and Π are projectors onto the tangent planes to Γ' and to Γ_t, respectively. The other notation, for instance $\nabla_{\mathbf{u}}$, also corresponds to transformation (3.6).

In order to apply Theorem 2.3 to problem (3.7), we should verify its assumptions. To this end, we choose τ_0 so small that inequality (2.6) holds. It is sufficient to take τ_0 such that

$$(2\tau_0)^{1/2}\mu \le \delta. \tag{3.8}$$

The right-hand side of the first equation in (3.7) belongs to $\mathbf{W}_2^{l,l/2}(D_0')$. Hence, by (2.5)

$$\begin{aligned}
N_{(t_0-2\tau_0+\lambda,\,t_0)}[\mathbf{v},p] &\le N_{(t_0-2\tau_0,\,t_0)}[\mathbf{w},s]\\
&\le c_1(2\tau_0)\{\|\mathbf{f}(X,t)\eta_\lambda\|^{(l,l/2)}_{Q_0'} + \|\mathbf{u}\dot\eta_\lambda\|^{(l,l/2)}_{D_0'}\}.
\end{aligned}$$

We can estimate the Sobolev norm of the composite function $\mathbf{f}\big(X(\xi,t),t\big)$ as follows:

$$\|\mathbf{f}(X,t)\|^{(l,l/2)}_{Q_0'} \le \|\mathbf{f}\|^{(l,l/2)}_{Q_0'} + c\big(1 + (2\tau_0)^{1-l/2}\|\mathbf{u}\|_{\mathbf{W}_2^{l,0}(D_0')}\big)|\nabla\mathbf{f}|^{(0,\beta)}_{Q_0'}.$$

By Lemma 3.1, we conclude for $\lambda \le 1$

$$N_{(t_0-2\tau_0+\lambda,\, t_0)}[\mathbf{v}, p] \le c_2 \Big\{ \|\mathbf{f}(X,t)\|^{(l,l/2)}_{Q'_0} + \frac{1}{\lambda^{l/2}} \|\mathbf{f}(X,t)\|_{Q'_0} + \frac{1}{\lambda} \|\mathbf{u}\|^{(l,l/2)}_{D'_{\lambda/2}} + \frac{1}{\lambda^{1+l/2}} \|\mathbf{u}\|_{D'_{\lambda/2}} \Big\}$$

$$\le c_3(1+\delta) \Big\{ \frac{1}{\lambda^{l/2}} \|\mathbf{f}\|^{(l,l/2)}_{Q'_0} + |\nabla \mathbf{f}|^{(0,\beta)}_{Q'_0} + \frac{\theta}{\lambda} \|\mathbf{u}\|^{(2+l,\,1+l/2)}_{D'_{\lambda/2}} + \Big(\frac{1}{\lambda \theta^{l/2}} + \frac{1}{\lambda (2\tau_0)^{l/2}} + \frac{1}{\lambda^{1+l/2}} \Big) \|\mathbf{u}\|_{D'_{\lambda/2}} \Big\} \qquad (3.9)$$

We take now $\theta = \varepsilon\lambda$ in estimate (3.9). Then we have

$$N_{(t_0-2\tau_0+\lambda,\, t_0)}[\mathbf{v}, p] \le c_4(\delta) \Big\{ \varepsilon N_{(t_0-2\tau_0+\lambda/2,\, t_0)}[\mathbf{v}, p] + \frac{1}{\lambda^{l/2}} \|\mathbf{f}\|^{(l,l/2)}_{Q'_0} + |\nabla \mathbf{f}|^{(0,\beta)}_{Q'_0} + \frac{1}{\lambda^{1+l/2}} (\varepsilon^{-l/2} + 1) \|\mathbf{v}\|_{Q'_{\lambda/2}} \Big\}. \qquad (3.10)$$

Let us introduce the function $\Phi(\lambda) = \lambda^{1+l/2} N_{(t_0-2\tau_0+\lambda,\, t_0)}[\mathbf{v}, p]$. Then we can rewrite (3.10) in the form:

$$\Phi(\lambda) \le c_5 \varepsilon \Phi(\lambda/2) + K, \qquad (3.11)$$

where $c_5 = c_4(\delta) 2^{1+l/2}$,

$$K = c_4(\delta) \{ \|\mathbf{f}\|^{(l,l/2)}_{Q'_0} + |\nabla \mathbf{f}|^{(0,\beta)}_{Q'_0} + c(\varepsilon) \|\mathbf{v}\|_{Q'_0} \}.$$

We set $\varepsilon = \frac{1}{2c_5}$ in (3.11). By iterations with $\lambda/2, \dots, \lambda/2^k$, we deduce from inequality (3.11) in the limit as $k \to \infty$ that

$$\Phi(\lambda) \le 2K.$$

This inequality with $\lambda = \tau_0$ implies (3.5). □

Now we can prove Theorem 1.2.

Proof of the global existence theorem. By Theorem 1.1, we have a solution $(\mathbf{v}, p)$ on an interval $(0, T_0]$. We can take ε so small that T_0 will be greater than unit, for example. Moreover, according to (1.6), solution norm satisfies the inequality

$$N_{(0,T_0)}[\mathbf{v}, p] \le \mu \tag{3.12}$$

with some $\mu > 0$. Then, due to Proposition 3.2, there exists $\tau_0 < T_0/2$ such that (3.8) is satisfied and estimate (3.5) holds:

$$N_{(t_0-\tau_0, t_0)}[\mathbf{v}, p] \le c_5(\delta, \tau_0)\{|\mathbf{f}|_{Q_0'}^{(1,\beta)} + |\nabla \mathbf{f}|_{Q_0'}^{(0,\beta)} + \|\mathbf{v}\|_{Q_0'}\}$$

for $\forall t_0 \in (T_0/2, T_0]$. Next, inequalities (3.1), (1.7) imply that

$$\|\mathbf{v}\|_{Q_0'} \le \left\{\int_{t_0-2\tau_0}^{t_0} \mathrm{e}^{-2bt}\Big(\|\mathbf{v}_0\|_\Omega + \int_0^t \mathrm{e}^{b\tau}\|\mathbf{f}\|_\Omega \,\mathrm{d}\tau\Big)^2 \mathrm{d}t\right\}^{1/2} \le \mathrm{e}^{-b(t_0-2\tau_0)}\sqrt{2\tau_0}\varepsilon. \tag{3.13}$$

Thus,

$$\begin{aligned} N_{(t_0-\tau_0, t_0)}[\mathbf{v}, p] &\le c_5(\delta, \tau_0)\mathrm{e}^{-bt_0}\{|\mathrm{e}^{bt}\mathbf{f}|_{Q_0'}^{(1,\beta)} + |\mathrm{e}^{bt}\nabla \mathbf{f}|_{Q_0'}^{(0,\beta)} + \mathrm{e}^{2b\tau_0}\sqrt{2\tau_0}\varepsilon\} \\ &\le c_6(\delta, \tau_0)\mathrm{e}^{-bt_0}\varepsilon \qquad \text{for } \forall t_0 \in (T_0/2, T_0], \end{aligned} \tag{3.14}$$

here $c_5(\delta, \tau_0)$, $c_6(\delta, \tau_0)$ are nondecreasing functions of τ_0.

From embedding theorem for $W_2^{2+l, 1+l/2}(D^{\pm}_{(T_0-\tau_0, T_0)})$, it follows that

$$\|\mathbf{u}(\cdot, T_0)\|_{\mathbf{W}_2^{1+l}(\bigcup_i \Omega_0^i)} \le c_6(\delta, \tau_0)\mathrm{e}^{-bT_0}\varepsilon.$$

In addition, because of (3.1)

$$\|\mathbf{u}(\cdot, T_0)\|_\Omega = \|\mathbf{v}(\cdot, T_0)\|_\Omega \le \mathrm{e}^{-bT_0}\left\{\|\mathbf{v}_0\|_\Omega + \int_0^{T_0} \|\mathrm{e}^{b\tau}\mathbf{f}(\cdot, \tau)\|_\Omega \,\mathrm{d}\tau\right\} \le \varepsilon. \tag{3.15}$$

We apply Theorem 1.1 again and obtain a solution on an interval $(T_0, T_0 + T_1]$ with $0 < T_1 \le T_0$ corresponding to the initial data $\mathbf{v}(\cdot, T_0)$. Due to (1.6), we get

$$N_{(T_0, T_0+T_1)}[\mathbf{v}, p] \le c(T_1)\big(\varepsilon + c_6(\delta, \tau_0)\mathrm{e}^{-bT_0}\varepsilon\big) \le \mu,$$

where μ is the same as in (3.12) for a sufficiently small ε. Then by Proposition 3.2 and in view of (3.13), (3.15), we have similar to (3.14), in particular,

$$\begin{aligned} N_{(T_0+T_1-\tau_1, T_0+T_1)}[\mathbf{v}, p] &\le c_5(\delta, \tau_1)\mathrm{e}^{-b(T_0+T_1)}\{|\mathrm{e}^{bt}\mathbf{f}|_{Q_0'}^{(1,\beta)} + |\mathrm{e}^{bt}\nabla \mathbf{f}|_{Q_0'}^{(0,\beta)} + 2\mathrm{e}^{2b\tau_1}\sqrt{2\tau_1}\varepsilon\} \\ &\le 2c_6(\delta, \tau_0)\mathrm{e}^{-b(T_0+T_1)}\varepsilon, \end{aligned} \tag{3.16}$$

here we have put $Q_0' = \Omega \times (T_0 + T_1 - 2\tau_1, T_0 + T_1)$ and chosen $\tau_1 \in (0, T_1/2)$, $\tau_1 \leq \tau_0/4$. Hence, (3.16) gives us

$$\|\mathbf{u}(\cdot, T_0 + T_1)\|_{\mathbf{W}_2^{1+l}(\bigcup_i \Omega_0^i)} \leq c_6(\delta, \tau_0)\mathrm{e}^{-b(T_0+T_1)}\varepsilon,$$

and in virtue of (3.1), (3.15),

$$\begin{aligned}\|\mathbf{u}(\cdot, T_0 + T_1)\|_\Omega &\leq \mathrm{e}^{-bT_1}\left\{\|\mathbf{v}(\cdot, T_0)\|_\Omega + \int_{T_0}^{T_0+T_1} \|\mathrm{e}^{b\tau}\mathbf{f}(\cdot, \tau)\|_\Omega \,\mathrm{d}\tau\right\} \\ &\leq \mathrm{e}^{-bT_1}\left\{\mathrm{e}^{-bT_0}\|\mathbf{v}_0\|_\Omega + \mathrm{e}^{-bT_0}\int_0^{T_0} \|\mathrm{e}^{b\tau}\mathbf{f}(\cdot, \tau)\|_\Omega \,\mathrm{d}\tau\right. \\ &\qquad \left. + \int_{T_0}^{T_0+T_1} \|\mathrm{e}^{b\tau}\mathbf{f}(\cdot, \tau)\|_\Omega \,\mathrm{d}\tau\right\} \leq \varepsilon.\end{aligned}$$

Since data norms have not increased, $(\mathbf{v}, p)$ exists on $(T_0 + T_1, T_0 + 2T_1]$ and

$$N_{(T_0+T_1, T_0+2T_1)}[\mathbf{v}, p] \leq \mu.$$

Hence, inequality

$$N_{(t_0-\tau_1, t_0)}[\mathbf{v}, p] \leq c_6(\delta, \tau_0)\mathrm{e}^{-bt_0}\varepsilon \tag{3.17}$$

is valid for $\forall t_0 \in (T_0/2, T_0 + 2T_1]$ and so on. Thus, the solution of problem (1.1), (1.2) can be extended as far as one likes, estimate (1.8) holding for all positive t_0.

The uniqueness of a global solution follows from the uniqueness of local ones.

In conclusion, we estimate the expansion of the interface Γ_t. To this end, we need to evaluate the speed of interface displacement. As $l > 1/2$, $\mathbf{W}_2^{1+l}(\Omega_0^+)$ is embedded in the space of the continuous functions. Consequently, by the embedding theorem, we can deduce from inequality (3.17) the estimate

$$\max_{\Omega_0^+} |\mathbf{u}(\cdot, t)| \leq c_7\mathrm{e}^{-bt}.$$

We integrate this inequality by t from $T_0/2$ until infinity:

$$\int_{T_0/2}^{\infty} \max_{\Omega_0^+} |\mathbf{u}(\cdot, t)| \,\mathrm{d}t \leq c_8.$$

Thus, if the distance between the interface Γ and the solid boundary S at the initial moment is greater than $c_9 = \frac{T_0}{2}\sup_{Q^+_{T_0/2}}|\mathbf{u}| + c_8$, we can guarantee that these surfaces never intersect. □

References

[1] H. Abels, On generalized solutions of two-phase flows for viscous incompressible fluids. *Interfaces Free Bound.* **9** (2007), 31–65. Zbl 1124.35060 MR 2317298

[2] R. A. Adams, *Sobolev spaces*. Academic Press, New York 1975. Zbl 0314.46030 MR 0450957

[3] I. V. Denisova, Motion of a drop in the flow of a fluid. *Dinamika Sploshn. Sredy* **93/94** (1989), 32–37. MR 1089263

[4] I. V. Denisova, Problem on unsteady motion of a drop in viscous incompressible flow (in Russian). Ph.D. thesis, Leningrad Branch of Steklov Math. Institute, Academy of Sciences of USSR, Leningrad, 1990.

[5] I. V. Denisova, A priori estimates for the solution of the linear nonstationary problem connected with the motion of a drop in a liquid medium. *Trudy Mat. Inst. Steklov.* **188** (1990), 3–21; English transl. *Proc. Steklov Inst. Math.* **188** (1991), 1–24. Zbl 00010927 MR 1100535

[6] I. V. Denisova, Problem of the motion of two viscous incompressible fluids separated by a closed free interface. *Acta Appl. Math.* **37** (1994), 31–40. Zbl 0814.35093 MR 1308743

[7] I. V. Denisova, Global solvability of a problem on two fluid motion without surface tension. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **348** (2007), 19–39; English transl. *J. Math. Sci.* **152** (2008), 625–637. MR 2743013

[8] I. V. Denisova and V. A. Solonnikov, Solvability of a linearized problem on the motion of a drop in a fluid flow. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **171** (1989), 53–65. English transl. *J. Soviet Math.* **56** (1991), 2309–2316. Zbl 0796.76025 MR 1031984

[9] I. V. Denisova and V. A. Solonnikov, Global solvability of the problem of the motion of two incompressible capillary fluids in a container. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **397** (2011), 20–52; English transl. *J. Math. Sci.* **185** (2012), 668–686. Zbl 1278.35175 MR 2870107

[10] Y. Giga and S. Takahashi, On global weak solutions of the nonstationary two-phase Stokes flow. *SIAM J. Math. Anal.* **25** (1994), 876–893. Zbl 0806.35137 MR 1271315

[11] M. Köhne, J. Prüss, and M. Wilke, Qualitative behaviour of solutions for the two-phase Navier-Stokes equations with surface tension. *Math. Ann.* **356** (2013), 737–792. Zbl 06181636 MR 3048614

[12] A. Nouri and F. Poupaud, An existence theorem for the multifluid Navier-Stokes problem. *J. Differential Equations* **122** (1995), 71–88. Zbl 0842.35079 MR 1356130

[13] A. Nouri, F. Poupaud and Y. Demay, An existence theorem for the multi-fluid Stokes problem. *Quart. Appl. Math.* **55** (1997), 421–435. Zbl 0882.35091 MR 1466141

[14] J. Prüss and G. Simonett, On the two-phase Navier-Stokes equations with surface tension. *Interfaces Free Bound.* **12** (2010), 311–345. Zbl 1202.35359 MR 2727674

[15] Y. Shibata and S. Shimizu, Maximal L_p-L_q regularity for the two-phase Stokes equations; model problems. *J. Differential Equations* **251** (2011), 373–419. Zbl 1230.35089 MR 2800157

[16] V. A. Solonnikov, Estimates for solutions of nonstationary Navier–Stokes equations. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov* (*LOMI*) **38** (1973), 153–231; English transl. *J. Soviet Math.* **8** (1977), 467–529. Zbl 0404.35081 MR 0415097

[17] V. A. Solonnikov, Unsteady flow of a finite mass of a fluid, bounded by a free surface. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **152** (1986), 137–157. English transl. *J. Soviet Math.* **40** (1988), 672–686. Zbl 0614.76026 MR 869248

[18] V. A. Solonnikov, Unsteady motion of an isolated volume of a viscous incompressible fluid. *Izv. Akad. Nauk SSSR Ser. Mat.* **51** (1987), 1065–1087; *Math. USSR-Izv.* **31** (1988), 381–405. Zbl 0850.76180 MR 925094

[19] V. A. Solonnikov, Solvability of a problem on the evolution of a viscous incompressible fluid, bounded by a free surface, on a finite time interval. *Algebra i Analiz* **3** (1991), 222–257; English transl. *St. Petersburg Math. J.* **3** (1992), 189–220. Zbl 0850.76132 MR 1120848

[20] S. Takahashi, On global weak solutions of the nonstationary two-phase Navier-Stokes flow. *Adv. Math. Sci. Appl.* **5** (1995), 321–342. Zbl 0827.35101 MR 1325970

Received September 10, 2013

I. V. Denisova, Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, V. O., Bol'shoy prosp., 61, St. Petersburg, 199178, Russia

E-mail: denisovairinavlad@gmail.com