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Normal forms for symplectic matrices

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Abstract. We give a self contained and elementary description of normal forms for symplectic matrices, based on geometrical considerations. The normal forms in question are expressed in terms of elementary Jordan matrices and integers with values in $\{-1, 0, 1\}$ related to signatures of quadratic forms naturally associated to the symplectic matrix.

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1. Introduction

Let V be a real vector space of dimension $2n$ with a non degenerate skewsymmetric bilinear form Ω. The symplectic group $\mathrm{Sp}(V, \Omega)$ is the set of linear transformations of V which preserve Ω:

$$\mathrm{Sp}(V, \Omega) = \{A : V \to V \mid A \text{ linear and } \Omega(Au, Av) = \Omega(u, v) \text{ for all } u, v \in V\}.$$

A *symplectic basis* of the symplectic vector space (V, Ω) of dimension $2n$ is a basis $\{e_1, \ldots, e_{2n}\}$ in which the matrix representing the symplectic form is $\Omega_0 = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$. In a symplectic basis, the matrix A' representing an element $A \in \mathrm{Sp}(V, \Omega)$ belongs to

$$\mathrm{Sp}(2n, \mathbb{R}) = \{A' \in \mathrm{Mat}(2n \times 2n, \mathbb{R}) \mid A'^{\tau}\Omega_0 A' = \Omega_0\}$$

where $(\cdot)^{\tau}$ denotes the transpose of a matrix.

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Given an element A in the symplectic group $\mathrm{Sp}(V,\Omega)$, we want to find a symplectic basis of V in which the matrix A' representing A has a distinguished form; to give a *normal form* for matrices in $\mathrm{Sp}(2n,\mathbb{R})$ means to describe a distinguished representative in each conjugacy class. In general, one cannot find a symplectic basis of the complexified vector space for which the matrix representing A has Jordan normal form.

The normal forms considered here are expressed in terms of elementary Jordan matrices and matrices depending on an integer $s \in \{-1,0,1\}$. They are closely related to the forms given by Long in [9], [8]; the main difference is that, in those references, some indeterminacy was left in the choice of matrices in each conjugacy class, in particular when the matrix admits 1 as an eigenvalue. We speak in this case of *quasi-normal forms*. Other constructions can be found in [16], [5], [6], [15], [12] but they are either quasi-normal or far from Jordan normal forms. Closely related are the constructions of normal forms for real matrices that are selfadjoint, skewadjoint or unitary with respect to an indefinite inner product where sign characteristics are introduced; they have been studied in many sources; for instance-mainly for selfadjoint and skewadjoint matrices-in the monograph of I. Gohberg, P. Lancaster and L. Rodman [2], and for unitary matrices in the papers [1], [3], [10], [13]. Normal forms for symplectic matrices have been given by C. Mehl in [11] and by V. Sergeichuk in [14]; in those descriptions, the basis producing the normal form is not required to be symplectic.

We construct here normal forms using elementary geometrical methods.

The choice of representatives for normal (or quasi normal) forms of matrices depends on the application one has in view. Quasi normal forms were used by Long to get precise formulas for indices of iterates of Hamiltonian orbits in [7]. The forms obtained here were useful for us to give new characterisations of Conley-Zehnder indices of general paths of symplectic matrices [4]. We have chosen to give a normal form in a symplectic basis. The main interest of our description is the natural interpretation of the signs appearing in the decomposition, and the description of the decomposition for matrices with 1 as an eigenvalue. It also yields an easy natural characterization of the conjugacy class of an element in $\mathrm{Sp}(2n,\mathbb{R})$. We hope it can be useful in other situations.

Assume that V decomposes as a direct sum $V = V_1 \oplus V_2$ where V_1 and V_2 are Ω-orthogonal A-invariant subspaces. Suppose that $\{e_1,\dots,e_{2k}\}$ is a symplectic basis of V_1 in which the matrix representing $A|_{V_1}$ is $A' = \begin{pmatrix} A'_1 & A'_2 \\ A'_3 & A'_4 \end{pmatrix}$. Suppose also that $\{f_1,\dots,f_{2l}\}$ is a symplectic basis of V_2 in which the matrix representing $A|_{V_2}$ is $A'' = \begin{pmatrix} A''_1 & A''_2 \\ A''_3 & A''_4 \end{pmatrix}$. Then $\{e_1,\dots,e_k,f_1,\dots,f_l,e_{k+1},\dots,e_{2k},f_{l+1},\dots,f_{2l}\}$ is a symplectic basis of V and the matrix representing A in this basis is

$$\begin{pmatrix} A_1' & 0 & A_2' & 0 \\ 0 & A_1'' & 0 & A_2'' \\ A_3' & 0 & A_4' & 0 \\ 0 & A_3'' & 0 & A_4'' \end{pmatrix}.$$

The notation $A' \diamond A''$ is used in Long [7] for this matrix. It is "a direct sum of matrices with obvious identifications". We call it the *symplectic direct sum* of the matrices A' and A''.

We $\mathbb{C}$-linearly extend Ω to the complexified vector space $V^{\mathbb{C}}$ and we $\mathbb{C}$-linearly extend any $A \in \mathrm{Sp}(V, \Omega)$ to $V^{\mathbb{C}}$. If v_λ denotes an eigenvector of A in $V^{\mathbb{C}}$ of the eigenvalue λ, then $\Omega(Av_\lambda, Av_\mu) = \Omega(\lambda v_\lambda, \mu v_\mu) = \lambda\mu\Omega(v_\lambda, v_\mu)$, thus $\Omega(v_\lambda, v_\mu) = 0$ unless $\mu = \frac{1}{\lambda}$. Hence the eigenvalues of A arise in "quadruples"

$$[\lambda] := \left\{\lambda, \frac{1}{\lambda}, \bar{\lambda}, \frac{1}{\bar{\lambda}}\right\}. \tag{1}$$

We find a symplectic basis of $V^{\mathbb{C}}$ so that A is a symplectic direct sum of block-upper-triangular matrices of the form

$$\begin{pmatrix} J(\lambda,k)^{-1} & 0 \\ 0 & J(\lambda,k)^{\tau} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & D(k,s) \\ 0 & \mathrm{Id} \end{pmatrix},$$

or

$$\begin{pmatrix} J(\bar{\lambda},k)^{-1} & & & 0 \\ & J(\lambda,k)^{-1} & & \\ & & J(\bar{\lambda},k)^{\tau} & \\ 0 & & & J(\lambda,k)^{\tau} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 & 0 & D(k,s) \\ & \mathrm{Id} & D(k,s) & 0 \\ & & \mathrm{Id} & 0 \\ 0 & & & \mathrm{Id} \end{pmatrix},$$

or

$$\begin{pmatrix} J(\bar{\lambda},k)^{-1} & & & 0 \\ & J(\lambda,k+1)^{-1} & & \\ & & J(\bar{\lambda},k)^{\tau} & \\ 0 & & & J(\lambda,k+1)^{\tau} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 & 0 & S(k,s,\lambda) \\ & \mathrm{Id} & S(k,s,\lambda)^{\tau} & 0 \\ & & \mathrm{Id} & 0 \\ 0 & & & \mathrm{Id} \end{pmatrix}.$$

Here, $J(\lambda,k)$ is the elementary $k \times k$ Jordan matrix corresponding to an eigenvalue λ, $D(k,s)$ is the diagonal $k \times k$ matrix

$$D(k,s) = \mathrm{diag}(0, \ldots 0, s),$$

and $S(k,s,\lambda)$ is the $k\times(k+1)$ matrix defined by

$$S(k,s,\lambda) := \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & \frac{1}{2}is & \lambda is \end{pmatrix},$$

with s an integer in $\{-1,0,1\}$. Each $s\in\{\pm 1\}$ is called a sign and the collection of such signs appearing in the decomposition of a matrix A is called the sign characteristic of A.

More precisely, on the real vector space V, we shall prove:

Theorem 1.1 (Normal forms for symplectic matrices). *Any symplectic endomorphism A of a finite dimensional symplectic vector space (V,Ω) is the direct sum of its restrictions $A_{|V_{[\lambda]}}$ to the real A-invariant symplectic subspace $V_{[\lambda]}$ whose complexification is the direct sum of the generalized eigenspaces of eigenvalues $\lambda, \frac{1}{\lambda}, \bar{\lambda}$ and $\frac{1}{\bar{\lambda}}$:*

$$V^{\mathbb{C}}_{[\lambda]} := E_\lambda \oplus E_{1/\lambda} \oplus E_{\bar{\lambda}} \oplus E_{1/\bar{\lambda}}.$$

We distinguish three cases: $\lambda\notin S^1$, $\lambda=\pm 1$ and $\lambda\in S^1\setminus\{\pm 1\}$.

Normal form for $A_{|V_{[\lambda]}}$ for $\lambda\notin S^1$:

Let $\lambda\notin S^1$ be an eigenvalue of A. Let $k := \dim_{\mathbb{C}}\operatorname{Ker}(A-\lambda\,\mathrm{Id})$ (on $V^{\mathbb{C}}$) and q be the smallest integer so that $(A-\lambda\,\mathrm{Id})^q$ is identically zero on the generalized eigenspace E_λ.

- *If λ is a real eigenvalue of A ($\lambda\notin S^1$ so $\lambda\neq\pm 1$), there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form*

$$\begin{pmatrix} J(\lambda,q_j)^{-1} & 0 \\ 0 & J(\lambda,q_j)^{\tau} \end{pmatrix}$$

with $q=q_1\geq q_2\geq\cdots\geq q_k$ and $J(\lambda,m)$ is the elementary $m\times m$ Jordan matrix associated to λ

$$J(\lambda,m) = \begin{pmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & & 0 & \\ & & \lambda & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & \lambda & 1 & \\ & 0 & & & & \lambda & 1 \\ & & & & & & \lambda \end{pmatrix}.$$

This decomposition is unique, when λ has been chosen in $\{\lambda, \lambda^{-1}\}$. It is determined by the chosen λ and by the dimension $\dim\big(\mathrm{Ker}(A - \lambda\,\mathrm{Id})^r\big)$ for each $r > 0$.

- *If $\lambda = re^{i\phi} \notin (S^1 \cup \mathbb{R})$ is a complex eigenvalue of A, there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form*

$$\begin{pmatrix} J_{\mathbb{R}}(re^{-i\phi}, 2q_j)^{-1} & 0 \\ 0 & J_{\mathbb{R}}(re^{-i\phi}, 2q_j)^{\tau} \end{pmatrix}$$

with $q = q_1 \geq q_2 \geq \cdots \geq q_k$ and $J_{\mathbb{R}}(re^{i\phi}, 2m)$ is the $2m \times 2m$ block upper triangular matrix defined by

$$J_{\mathbb{R}}(re^{i\phi}, 2m) := \begin{pmatrix} R(re^{i\phi}) & \mathrm{Id} & & & & & \\ & R(re^{i\phi}) & \mathrm{Id} & & & 0 & \\ & & R(re^{i\phi}) & \mathrm{Id} & & & \\ & & & \ddots & \ddots & & \\ & & & & R(re^{i\phi}) & \mathrm{Id} & \\ & 0 & & & & R(re^{i\phi}) & \mathrm{Id} \\ & & & & & & R(re^{i\phi}) \end{pmatrix}$$

with $R(re^{i\phi}) = \begin{pmatrix} r\cos\phi & -r\sin\phi \\ r\sin\phi & r\cos\phi \end{pmatrix}$.

This decomposition is unique, when λ has been chosen in $\{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$. It is determined by the chosen λ and by the dimension $\dim\big(\mathrm{Ker}(A - \lambda\,\mathrm{Id})^r\big)$ for each $r > 0$.

Normal form for $A_{|V_{[\lambda]}}$ for $\lambda = \pm 1$:

Let $\lambda = \pm 1$ be an eigenvalue of A. There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of matrices of the form

$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, s_j, \lambda) \\ 0 & J(\lambda, r_j)^{\tau} \end{pmatrix}$$

where $C(r_j, s_j, \lambda) := J(\lambda, r_j)^{-1}\,\mathrm{diag}(0, \ldots, 0, s_j)$ with $s_j \in \{0, 1, -1\}$. If $s_j = 0$, then r_j is odd. The dimension of the eigenspace of the eigenvalue λ is given by $2\,\mathrm{Card}\{j \mid s_j = 0\} + \mathrm{Card}\{j \mid s_j \neq 0\}$.

The number of s_j equal to $+1$ (resp. -1) arising in blocks of dimension $2k$ (i.e. with corresponding $r_j = k$) is equal to the number of positive (resp. negative) eigenvalues of the symmetric 2-form

$$\hat{Q}^\lambda_{2k} : \mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^{2k}\big) \times \mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^{2k}\big) \to \mathbb{R}$$
$$(v,w) \mapsto \lambda\Omega\big((A-\lambda\,\mathrm{Id})^{k}v, (A-\lambda\,\mathrm{Id})^{k-1}w\big).$$

The decomposition is unique up to a permutation of the blocks and is determined by λ*, by the dimension* $\dim\big(\mathrm{Ker}(A-\lambda\,\mathrm{Id})^r\big)$ *for each* $r \geq 1$*, and by the rank and the signature of the symmetric bilinear* 2*-form* $\hat{Q}^\lambda_{2k}$ *for each* $k \geq 1$*.*

Normal form for $A_{|V_{[\lambda]}}$ **for** $\lambda \in S^1\backslash\{\pm 1\}$:

Let $\lambda \in S^1$*,* $\lambda \neq \pm 1$ *be an eigenvalue of* A*. There exists a symplectic basis of* $V_{[\lambda]}$ *in which the matrix representing the restriction of* A *to* $V_{[\lambda]}$ *is a symplectic direct sum of* $4k_j \times 4k_j$ *matrices* $(k_j \geq 1)$ *of the form*

$$\left(\begin{array}{c|ccc} \big(J_{\mathbb{R}}(\bar{\lambda},2k_j)\big)^{-1} & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & s_j V^1_{k_j}(\phi) & s_j V^2_{k_j}(\phi) \\ \hline 0 & & \big(J_{\mathbb{R}}(\bar{\lambda},2k_j)\big)^{\tau} & \end{array}\right) \tag{2}$$

and $(4k_j+2)\times(4k_j+2)$ *matrices* $(k_j \geq 0)$ *of the form*

$$\left(\begin{array}{c|c|ccc|c} \big(J_{\mathbb{R}}(\bar{\lambda},2k_j)\big)^{-1} & s_j U^2_{k_j}(\phi) & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & \frac{s_j}{2} V^2_{k_j}(\phi) & \frac{-s_j}{2} V^1_{k_j}(\phi) & U^1_{k_j}(\phi) \\ \hline 0 & \cos\phi & 0 \;\cdots\; 0 & 1 & 0 & s_j \sin\phi \\ \hline 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \big(J_{\mathbb{R}}(\bar{\lambda},2k_j)\big)^{\tau} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & -s_j \sin\phi & 0 \;\cdots\; 0 & 0 & -s_j & \cos\phi \end{array}\right) \tag{3}$$

where $J_{\mathbb{R}}(e^{i\phi},2k)$ *is defined as above, where* $\big(V^1_{k_j}(\phi)\,V^2_{k_j}(\phi)\big)$ *is the* $2k_j \times 2$ *matrix defined by*

$$\big(V^1_{k_j}(\phi)\,V^2_{k_j}(\phi)\big) = \begin{pmatrix} (-1)^{k_j-1}R(e^{ik_j\phi}) \\ \vdots \\ R(e^{i\phi}) \end{pmatrix} \tag{4}$$

with $R(e^{i\phi}) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$*, where*

$$\big(U^1_{k_j}(\phi)\,U^2_{k_j}(\phi)\big) = \big(V^1_{k_j}(\phi)\,V^2_{k_j}(\phi)\big)\big(R(e^{i\phi})\big) \tag{5}$$

and where $s_j = \pm 1$*. The complex dimension of the eigenspace of the eigenvalue* λ *in* $V^{\mathbb{C}}$ *is given by the number of such matrices.*

The number of s_j equal to $+1$ (resp. -1) arising in blocks of dimension $2m$ in the normal decomposition given above is equal to the number of positive (resp. negative) eigenvalues of the Hermitian 2-form $\hat{Q}_m^\lambda$ defined on $\mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big)$ by:

$$\begin{aligned}\hat{Q}_m^\lambda : \mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big) \times \mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big) &\to \mathbb{C}\\ (v,w) &\mapsto \frac{1}{\lambda}\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\bar{\lambda}\,\mathrm{Id})^{k-1}\bar{w}\big) \quad \text{if } m=2k\\ (v,w) &\mapsto i\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\bar{\lambda}\,\mathrm{Id})^{k}\bar{w}\big) \quad \text{if } m=2k+1.\end{aligned}$$

This decomposition is unique up to a permutation of the blocks, when λ has been chosen in $\{\lambda,\bar{\lambda}\}$. It is determined by the chosen λ, by the dimension $\dim\big(\mathrm{Ker}(A-\lambda\,\mathrm{Id})^r\big)$ for each $r\geq 1$ and by the rank and the signature of the Hermitian bilinear 2-form $\hat{Q}_m^\lambda$ for each $m\geq 1$.

The normal form for $A_{|V_{[\lambda]}}$ is given in Theorem 3.1 for $\lambda\notin S^1$, in Theorem 4.1 for $\lambda=\pm 1$, and in Theorem 5.2 for $\lambda\in S^1\backslash\{\pm 1\}$. The characterisation of the signs is given in Proposition 4.3 for $\lambda=\pm 1$ and in Proposition 5.4 for $\lambda\in S^1\backslash\{\pm 1\}$.

A direct consequence of Theorem 1.1 is the following characterization of the conjugacy class of a matrix in the symplectic group.

Theorem 1.2. *The conjugacy class of a matrix $A\in \mathrm{Sp}(2n,\mathbb{R})$ is determined by the following data:*

- *the eigenvalues of A which arise in quadruples $[\lambda]=\{\lambda,\lambda^{-1},\bar{\lambda},\bar{\lambda}^{-1}\}$;*
- *the dimension $\dim\big(\mathrm{Ker}(A-\lambda\,\mathrm{Id})^r\big)$ for each $r\geq 1$ for one eigenvalue in each class $[\lambda]$;*
- *for $\lambda=\pm 1$, the rank and the signature of the symmetric form $\hat{Q}_{2k}^\lambda$ for each $k\geq 1$ and for an eigenvalue λ in $S^1\backslash\{\pm 1\}$ chosen in each $[\lambda]$, the rank and the signature of the Hermitian form $\hat{Q}_m^\lambda$ for each $m\geq 1$, with*

$$\begin{aligned}\hat{Q}_m^\lambda : \mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big) \times \mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big) &\to \mathbb{C}\\ (v,w) &\mapsto \frac{1}{\lambda}\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\bar{\lambda}\,\mathrm{Id})^{k-1}\bar{w}\big) \quad \text{if } m=2k\\ (v,w) &\mapsto i\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\bar{\lambda}\,\mathrm{Id})^{k}\bar{w}\big) \quad \text{if } m=2k+1. \qquad \square\end{aligned}$$

2. Preliminaries

Lemma 2.1. *Consider $A\in\mathrm{Sp}(V,\Omega)$ and let $0\neq\lambda\in\mathbb{C}$. Then $\mathrm{Ker}(A-\lambda\,\mathrm{Id})^j$ in $V^{\mathbb{C}}$ is the symplectic orthogonal complement of $\mathrm{Im}\big(A-\frac{1}{\lambda}\,\mathrm{Id}\big)^j$.*

Proof.

$$\Omega\big((A-\lambda\,\mathrm{Id})u, Av\big) = \Omega(Au, Av) - \lambda\Omega(u, Av) = \Omega(u, v) - \lambda\Omega(u, Av)$$
$$= -\lambda\Omega\left(u, \left(A - \frac{1}{\lambda}\,\mathrm{Id}\right)v\right)$$

and by induction

$$\Omega\big((A-\lambda\,\mathrm{Id})^j u, A^j v\big) = (-\lambda)^j\Omega\left(u, \left(A - \frac{1}{\lambda}\,\mathrm{Id}\right)^j v\right). \tag{6}$$

The result follows from the fact that A is invertible. □

Corollary 2.2. *If E_λ denotes the generalized eigenspace of eigenvalue λ, i.e. $E_\lambda := \{v \in V^{\mathbb{C}} \mid (A-\lambda\,\mathrm{Id})^j v = 0$ for an integer $j > 0\}$, we have*

$$\Omega(E_\lambda, E_\mu) = 0 \qquad \textit{when } \lambda\mu \neq 1.$$

Indeed the symplectic orthogonal complement of $E_\lambda = \bigcup_j \mathrm{Ker}(A-\lambda\,\mathrm{Id})^j$ is the intersection of the $\mathrm{Im}\big(A - \frac{1}{\lambda}\,\mathrm{Id}\big)^j$. By Jordan normal form, this intersection is the sum of the generalized eigenspaces corresponding to the eigenvalues which are not $\frac{1}{\lambda}$.

If $v = u + iu'$ is in $\mathrm{Ker}(A-\lambda\,\mathrm{Id})^j$ with u and u' in V then $\bar{v} = u - iu'$ is in $\mathrm{Ker}(A-\bar{\lambda}\,\mathrm{Id})^j$ so that $E_\lambda \oplus E_{\bar{\lambda}}$ is the complexification of a real subspace of V. From this remark and Corollary 2.2 the space

$$W_{[\lambda]} := E_\lambda \oplus E_{1/\lambda} \oplus E_{\bar{\lambda}} \oplus E_{1/\bar{\lambda}} \tag{7}$$

is the complexification of a real and symplectic A-invariant subspace $V_{[\lambda]}$ and

$$V = V_{[\lambda_1]} \oplus V_{[\lambda_2]} \oplus \cdots \oplus V_{[\lambda_K]} \tag{8}$$

where we denote by $[\lambda]$ the set $\big\{\lambda, \bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}}\big\}$ and by $[\lambda_1], \dots, [\lambda_K]$ the distinct such sets exhausting the eigenvalues of A.

We denote by $A_{[\lambda_i]}$ the restriction of A to $V_{[\lambda_i]}$. It is clearly enough to obtain normal forms for each $A_{[\lambda_i]}$ since A will be a symplectic direct sum of those.

We shall construct a symplectic basis of $W_{[\lambda]}$ (and of $V_{[\lambda]}$) adapted to A for a given eigenvalue λ of A. We assume that $(A-\lambda\,\mathrm{Id})^{p+1} = 0$ and $(A-\lambda\,\mathrm{Id})^p \neq 0$ on the generalized eigenspace E_λ. Since A is real, this integer p is the same for $\bar{\lambda}$. By Lemma 2.1, $\mathrm{Ker}(A-\lambda\,\mathrm{Id})^j$ is the symplectic orthogonal complement of $\mathrm{Im}\big(A-\frac{1}{\lambda}\,\mathrm{Id}\big)^j$ for all j, thus $\dim\mathrm{Ker}(A-\lambda\,\mathrm{Id})^j = \dim\mathrm{Ker}\big(A-\frac{1}{\lambda}\,\mathrm{Id}\big)^j$; hence the integer p is the same for λ and $\frac{1}{\lambda}$.

We decompose $W_{[\lambda]}$ (and $V_{[\lambda]}$) into a direct sum of A-invariant symplectic subspaces. Given a symplectic subspace Z of $V_{[\lambda]}$ which is A-invariant, its orthogonal complement (with respect to the symplectic 2-form) $V' := Z^{\perp_\Omega}$ is again symplectic and A-invariant. The generalized eigenspace for A on $V'^{\mathbb{C}}$ are $E'_\mu = V'^{\mathbb{C}} \cap E_\mu$, and the smallest integer p' for which $(A - \lambda \operatorname{Id})^{p'+1} = 0$ on E'_λ is such that $p' \leq p$.

Hence, to get the decomposition of $W_{[\lambda]}$ (and $V_{[\lambda]}$) it is enough to build a symplectic subspace of $W_{[\lambda]}$ which is A-invariant and closed under complex conjugation and to proceed inductively. We shall construct such a subspace, containing a well chosen vector $v \in E_\lambda$ so that $(A - \lambda \operatorname{Id})^p v \neq 0$.

We shall distinguish three cases; first $\lambda \notin S^1$ then $\lambda = \pm 1$ and finally $\lambda \in S^1 \backslash \{\pm 1\}$.

We first present a few technical lemmas which will be used for this construction.

2.1. A few technical lemmas. Let (V, Ω) be a real symplectic vector space. Consider $A \in \operatorname{Sp}(V, \Omega)$ and let λ be an eigenvalue of A in $V^{\mathbb{C}}$.

Lemma 2.3. *For any positive integer j, the bilinear map*

$$\tilde{Q}_j : E_\lambda / \operatorname{Ker}(A - \lambda \operatorname{Id})^j \times E_{1/\lambda} / \operatorname{Ker}\left(A - \frac{1}{\lambda} \operatorname{Id}\right)^j \to \mathbb{C}$$

$$([v],[w]) \mapsto \tilde{Q}_j([v],[w]) := \Omega\big((A - \lambda \operatorname{Id})^j v, w\big) \qquad v \in E_\lambda,\ w \in E_{1/\lambda} \tag{9}$$

is well defined and non degenerate. In the formula, $[v]$ denotes the class containing v in the appropriate quotient.

Proof. The fact that $\tilde{Q}_j$ is well defined follows from equation (6); indeed, for any integer j, we have

$$\Omega\big((A - \lambda \operatorname{Id})^j u, v\big) = (-\lambda)^j \Omega\left(A^j u, \left(A - \frac{1}{\lambda} \operatorname{Id}\right)^j v\right). \tag{10}$$

The map is non degenerate because $\tilde{Q}_j([v],[w]) = 0$ for all w if and only if $(A - \lambda \operatorname{Id})^j v = 0$ since Ω is a non degenerate pairing between E_λ and $E_{1/\lambda}$, thus if and only if $[v] = 0$. Similarly, $\tilde{Q}_j([v],[w]) = 0$ for all v if and only if w is Ω-orthogonal to $\operatorname{Im}(A - \lambda \operatorname{Id})^j$, thus if and only if $w \in \operatorname{Ker}\big(A - \frac{1}{\lambda} \operatorname{Id}\big)^j$ hence $[w] = 0$. □

Lemma 2.4. *For any $v, w \in V$, any $\lambda \in \mathbb{C} \backslash \{0\}$ and any integers $i \geq 0$, $j > 0$ we have:*

$$\Omega\left((A-\lambda\,\mathrm{Id})^i v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^j w\right)$$
$$= -\frac{1}{\lambda}\Omega\left((A-\lambda\,\mathrm{Id})^{i+1} v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^j w\right)$$
$$-\frac{1}{\lambda^2}\Omega\left((A-\lambda\,\mathrm{Id})^{i+1} v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^{j-1} w\right). \tag{11}$$

In particular, if λ is an eigenvalue of A, if $v \in E_\lambda$ is such that $p \geq 0$ is the largest integer for which $(A-\lambda\,\mathrm{Id})^p v \neq 0$, we have for any integers $k, j \geq 0$:

$$\Omega\big((A-\lambda\,\mathrm{Id})^{p+k} v, w\big) = (-\lambda^2)^j \Omega\left((A-\lambda\,\mathrm{Id})^{p+k-j} v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^j w\right) \tag{12}$$

so that

$$\Omega\big((A-\lambda\,\mathrm{Id})^{p} v, w\big) = (-\lambda^2)^p \Omega\left(v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^p w\right) \tag{13}$$

and

$$\Omega\left((A-\lambda\,\mathrm{Id})^k v, \left(A-\frac{1}{\lambda}\mathrm{Id}\right)^j w\right) = 0 \quad \text{if } k+j > p. \tag{14}$$

Proof. We have:

$$\Omega\left((A-\lambda\,\mathrm{Id})^i v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^j w\right)$$
$$= -\frac{1}{\lambda}\Omega\left((A-\lambda\,\mathrm{Id}-A)(A-\lambda\,\mathrm{Id})^i v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^j w\right)$$
$$= -\frac{1}{\lambda}\Omega\left((A-\lambda\,\mathrm{Id})^{i+1} v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^j w\right)$$
$$+\frac{1}{\lambda}\Omega\left(A(A-\lambda\,\mathrm{Id})^i v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)\left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^{j-1} w\right)$$
$$= -\frac{1}{\lambda}\Omega\left((A-\lambda\,\mathrm{Id})^{i+1} v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^j w\right)$$

$$+\frac{1}{\lambda}\Omega\left((A-\lambda\,\mathrm{Id})^i v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^{j-1} w\right)$$
$$-\frac{1}{\lambda^2}\Omega\left(A(A-\lambda\,\mathrm{Id})^i v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^{j-1} w\right)$$

and formula (11) follows.

For any integers $k, j \geq 0$ and any v such that $(A-\lambda\,\mathrm{Id})^p v = 0$, we have, by (6),

$$(-\lambda)^j \Omega\left((A-\lambda\,\mathrm{Id})^{p+k+1-j} v, \left(A-\frac{1}{\lambda}\,\mathrm{Id}\right)^j w\right) = \Omega\big((A-\lambda\,\mathrm{Id})^{p+k+1} v, A^j w\big) = 0.$$

Hence, applying formula (11) with a decreasing induction on j, we get formula (12). The other formulas follow readily. □

Definition 2.5. For $\lambda \in S^1$ an eigenvalue of A and $v \in E_\lambda$ a generalized eigenvector, we define

$$T_{i,j}(v) := \frac{1}{\lambda^i \bar{\lambda}^j}\Omega\big((A-\lambda\,\mathrm{Id})^i v, (A-\bar{\lambda})^j \bar{v}\big). \tag{15}$$

We have, by equation (11):

$$T_{i,j}(v) = -T_{i+1,j}(v) - T_{i+1,j-1}(v), \tag{16}$$

and also,

$$T_{i,j}(v) = -\overline{T_{j,i}(v)}. \tag{17}$$

Lemma 2.6. *Let $\lambda \in S^1$ be an eigenvalue of A and $v \in E_\lambda$ be a generalised eigenvector such that the largest integer p so that $(A-\lambda\,\mathrm{Id})^p v \neq 0$ is odd, say, $p = 2k-1$. Then, in the A-invariant subspace E_λ^v of E_λ generated by v, there exists a vector v' generating the same A-invariant subspace $E_\lambda^{v'} = E_\lambda^v$, so that $(A-\lambda\,\mathrm{Id})^p v' \neq 0$ and so that*

$$T_{i,j}(v') = 0 \quad \textit{for all } i, j \leq k-1.$$

If λ is real (i.e. ± 1), and if v is a real vector (i.e. in V), the vector v' can be chosen to be real as well.

Proof. Observe that

$$\begin{aligned} T_{k,k-1}(v) &= -T_{k,k}(v) - T_{k-1,k}(v) \quad \text{by (11)} \\ &= -T_{k-1,k}(v) \quad \text{by (14)} \\ &= \overline{T_{k,k-1}(v)} \quad \text{by (17)} \end{aligned}$$

is real and can be put to $d = \pm 1$ by rescaling the vector. We use formulas (11) and (17) and we proceed by decreasing induction on $i + j$ as follows:

- if $T_{k-1,k-1}(v) = \alpha_1$, this α_1 is purely imaginary, we replace v by

$$v' := v - \frac{\alpha_1}{2\lambda d}(A - \lambda\,\mathrm{Id})v;$$

  clearly $E_\lambda^{v'} = E_\lambda^v$ and $T_{i,j}(v') = T_{i,j}(v)$ for $i + j \geq 2k - 1$ but now

$$T_{k-1,k-1}(v') = \alpha_1 - \frac{\alpha_1}{2d}T_{k,k-1}(v) - \frac{\overline{\alpha_1}}{2d}T_{k-1,k}(v) = 0;$$

  so we can now assume $T_{k-1,k-1}(v) = 0$; observe that if λ is real and v is in V, then $\alpha_1 = 0$ and $v' = v$;

- if $T_{k-2,k-1}(v) = \alpha_2 = -T_{k-1,k-2}(v)$, this α_2 is real and we replace v by

$$v - \frac{\alpha_2}{2\lambda^2 d}(A - \lambda\,\mathrm{Id})^2 v;$$

  the space E_λ^v does not change and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - 2$; now

$$T_{k-2,k-1}(v') = \alpha_2 - \frac{\alpha_2}{2d}T_{k,k-1}(v) - \frac{\overline{\alpha_2}}{2d}T_{k-2,k+1}(v) = 0,$$

  hence also $T_{k-1,k-2}(v') = 0$; observe that if λ is real and v is in V, then v' is in V.

- we now assume by induction to have a $J > 0$ so that $T_{i,j}(v) = 0$ for all $0 \leq i, j \leq k - 1$ so that $i + j > 2k - 1 - J$;
- if $T_{k-J,k-1}(v) = \alpha_J$, then $T_{k-J,k-1}(v) = (-1)^{J-1}T_{k-1,k-J}(v)$ so that α_J is real when J is even and is imaginary when J is odd; we replace v by

$$v - \frac{\alpha_J}{2\lambda^J d}(A - \lambda\,\mathrm{Id})^J v;$$

  the space E_λ^v does not change and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - J$; but now

$$\begin{aligned} T_{k-J,k-1}(v') &= \alpha_J - \frac{\alpha_J}{2d}T_{k,k-1}(v) - \frac{\overline{\alpha_J}}{2d}T_{k-J,k+J-1}(v) \\ &= \alpha_J - \frac{\alpha_J}{2} - (-1)^J\frac{\overline{\alpha_J}}{2} = 0. \end{aligned}$$

Hence also $T_{k-J+1,k-2}(v') = 0, \dots T_{k-1,k-J+1}(v') = 0$; so the induction proceeds. Observe that if λ is real and v is in V then v' is in V. $\square$

We shall use repeatedly that a $n \times n$ block triangular symplectic matrix is of the form

$$A' = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}) \iff \begin{cases} B = (D^\tau)^{-1} \\ C = (D^\tau)^{-1} S \text{ with } S \text{ symmetric.} \end{cases} \tag{18}$$

3. Normal forms for $A_{|V_{[\lambda]}}$ when $\lambda \notin S^1$

As before, p denotes the largest integer such that $(A - \lambda \,\mathrm{Id})^p$ does not vanish identically on the generalized eigenspace E_λ. Let us choose an element $v \in E_\lambda$ and an element $w \in E_{1/\lambda}$ such that

$$\tilde{Q}_p([v],[w]) = \Omega\big((A - \lambda \,\mathrm{Id})^p v, w\big) \neq 0.$$

Let us consider the smallest A-invariant subspace E_λ^v of E_λ containing v; it is of dimension $p+1$ and a basis is given by

$$\{a_0 := v, \dots, a_i := (A - \lambda \,\mathrm{Id})^i v, \dots, a_p := (A - \lambda \,\mathrm{Id})^p v\}.$$

Observe that $Aa_i = (A - \lambda \,\mathrm{Id})a_i + \lambda a_i$ so that $Aa_i = \lambda a_i + a_{i+1}$ for $i < p$ and $Aa_p = a_p$.

Similarly, we consider the smallest A-invariant subspace $E_{1/\lambda}^w$ of $E_{1/\lambda}$ containing w; it is also of dimension $p+1$ and a basis is given by

$$\left\{ b_0 := w, \dots, b_j := \left(A - \frac{1}{\lambda}\,\mathrm{Id}\right)^j w, \dots, b_p := \left(A - \frac{1}{\lambda}\,\mathrm{Id}\right)^p w \right\}.$$

One has

$\Omega(a_i, a_j) = 0$ and $\Omega(b_i, b_j) = 0$ because $\Omega(E_\lambda, E_\mu) = 0$ if $\lambda\mu \neq 1$;

$\Omega(a_i, b_j) = 0$ if $i + j > p$ by equation (14);

$\Omega(a_i, b_{p-i}) = \left(\frac{-1}{\lambda^2}\right)^{p-i} \Omega\big((A - \lambda \,\mathrm{Id})^p v, w\big)$ by equation (12) and is non zero by the choice of v, w.

The matrix representing Ω in the basis $\{b_p, \dots, b_0, a_0, \dots, a_p\}$ is thus of the form

$$\left(\begin{array}{ccc|ccc} 0 & & 0 & \bar{*} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 0 & * & & \bar{*} \\ \hline \bar{*} & & * & 0 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \bar{*} & 0 & & 0 \end{array}\right)$$

with non vanishing $\bar{*}$. Hence Ω is non degenerate on $E_\lambda^v \oplus E_{1/\lambda}^w$ which is thus a symplectic A-invariant subspace.

We now construct a symplectic basis $\{b'_p, \ldots, b'_0, a_0, \ldots, a_p\}$ of $E_\lambda^v \oplus E_{1/\lambda}^w$, extending $\{a_0, \ldots, a_p\}$, using a Gram-Schmidt procedure on the b_i's. This gives a normal form for A on $E_\lambda^v \oplus E_{1/\lambda}^w$.

If λ is real, we take v, w in the real generalized eigenspaces $E_\lambda^{\mathbb{R}}$ and $E_{1/\lambda}^{\mathbb{R}}$ and we obtain a symplectic basis of the real A-invariant symplectic vector space, $E_\lambda^{\mathbb{R}v} \oplus E_{1/\lambda}^{\mathbb{R}w}$. If λ is not real, one considers the basis of $E_{\bar{\lambda}}^{\bar{v}} \oplus E_{1/\bar{\lambda}}^{\bar{w}}$ defined by the conjugate vectors $\{\overline{b'_p}, \ldots, \overline{b'_0}, \overline{a_0}, \ldots, \overline{a_p}\}$ and this yields a conjugate normal form on $E_{\bar{\lambda}} \oplus E_{1/\bar{\lambda}}$, hence a normal form on $W_{[\lambda]}$ and this will induce a real normal form on $V_{[\lambda]}$.

We choose v and w such that $\Omega\Big(\big(A - \frac{1}{\lambda}\,\mathrm{Id}\big)^p w, v\Big) = 1$. We define inductively on j

$b'_p := \frac{1}{\Omega(b_p, a_0)} b_p = b_p$;
$b'_{p-j} = \frac{1}{\Omega(b_{p-j}, a_j)} \big(b_{p-j} - \sum_{k<j} \Omega(b_{p-j}, a_k) b'_{p-k}\big)$,
so that any b'_j is a linear combination of the b_r with $r \geq j$.

In the symplectic basis $\{b'_p, \ldots, b'_0, a_0, \ldots, a_p\}$ the matrix representing A is

$$\begin{pmatrix} B & 0 \\ 0 & J(\lambda, p+1)^\tau \end{pmatrix}$$

where

$$J(\lambda, m) = \begin{pmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & & 0 & \\ & & \lambda & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & \lambda & 1 & \\ & 0 & & & & \lambda & 1 \\ & & & & & & \lambda \end{pmatrix} \tag{19}$$

is the elementary $m \times m$ Jordan matrix associated to λ. Since the matrix is symplectic, B is the transpose of the inverse of $J(\lambda, p+1)^{\tau}$ by (18), so $B = J(\lambda, p+1)^{-1}$. This is the normal form for A restricted to $E_{\lambda}^{v} \oplus E_{1/\lambda}^{w}$.

If $\lambda = re^{i\phi} \notin \mathbb{R}$ we consider the symplectic basis $\{b_p', \ldots, b_0', a_0, \ldots, a_p\}$ of $E_{\lambda}^{v} \oplus E_{1/\lambda}^{w}$ as above and the conjugate symplectic basis $\{\overline{b_p'}, \ldots, \overline{b_0'}, \overline{a_0}, \ldots, \overline{a_p}\}$ of $E_{\overline{\lambda}}^{\overline{v}} \oplus E_{1/\overline{\lambda}}^{\overline{w}}$. Writing $b_j' = \frac{1}{\sqrt{2}}(u_j + iv_j)$ and $a_j = \frac{1}{\sqrt{2}}(w_j - ix_j)$ for all $0 \leq j \leq p$ with the vectors u_j, v_j, w_j, x_j in the real vector space V, we get a symplectic basis $\{u_p, v_p \ldots, u_0, v_0, w_0, x_0 \ldots, w_p, x_p\}$ of the real subspace of V whose complexification is $E_{\lambda}^{v} \oplus E_{1/\lambda}^{w} \oplus E_{\overline{\lambda}}^{\overline{v}} \oplus E_{1/\overline{\lambda}}^{\overline{w}}$. In this basis, the matrix representing A is

$$\begin{pmatrix} J_{\mathbb{R}}(\overline{\lambda}, 2(p+1))^{-1} & 0 \\ 0 & J_{\mathbb{R}}(\overline{\lambda}, 2(p+1))^{\tau} \end{pmatrix}$$

where $J_{\mathbb{R}}(re^{i\phi}, 2m)$ is the $2m \times 2m$ matrix written in terms of 2×2 matrices as

$$J_{\mathbb{R}}(re^{i\phi}, 2m) := \begin{pmatrix} R(re^{i\phi}) & \mathrm{Id} & & & & & \\ & R(re^{i\phi}) & \mathrm{Id} & & & 0 & \\ & & R(re^{i\phi}) & \mathrm{Id} & & & \\ & & & \ddots & \ddots & & \\ & & & & R(re^{i\phi}) & \mathrm{Id} & \\ & 0 & & & & R(re^{i\phi}) & \mathrm{Id} \\ & & & & & & R(re^{i\phi}) \end{pmatrix} \tag{20}$$

with $R(re^{i\phi}) = \begin{pmatrix} r\cos\phi & -r\sin\phi \\ r\sin\phi & r\cos\phi \end{pmatrix}$. By induction, we get

Theorem 3.1 (Normal form for $A_{|V_{[\lambda]}}$ for $\lambda \notin S^1$.). *Let $\lambda \notin S^1$ be an eigenvalue of A. Denote $k := \dim_{\mathbb{C}} \operatorname{Ker}(A - \lambda \mathrm{Id})$ (on $V^{\mathbb{C}}$) and p the smallest integer so that $(A - \lambda \mathrm{Id})^{p+1}$ is identically zero on the generalized eigenspace E_{λ}.*

- *If $\lambda \neq \pm 1$ is a real eigenvalue of A, there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form*

$$\begin{pmatrix} J(\lambda, p_j+1)^{-1} & 0 \\ 0 & J(\lambda, p_j+1)^{\tau} \end{pmatrix}$$

with $p = p_1 \geq p_2 \geq \cdots \geq p_k$ and $J(\lambda, k)$ defined by (19). *To eliminate the ambiguity in the choice of λ in $[\lambda] = \{\lambda, \lambda^{-1}\}$ we can consider the real eigenvalue such that $\lambda > 1$. The size of the blocks is determined knowing the dimension $\dim(\operatorname{Ker}(A - \lambda \mathrm{Id})^{r})$ for each $r \geq 1$.*

- *If $\lambda = re^{i\phi} \notin (S^1 \cup \mathbb{R})$ is a complex eigenvalue of A, there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form*

$$\begin{pmatrix} J_{\mathbb{R}}\big(re^{-i\phi}, 2(p_j+1)\big)^{-1} & 0 \\ 0 & J_{\mathbb{R}}\big(re^{-i\phi}, 2(p_j+1)\big)^{\tau} \end{pmatrix}$$

with $p = p_1 \geq p_2 \geq \cdots \geq p_k$ and $J_{\mathbb{R}}(re^{i\phi}, k)$ defined by (20). *To eliminate the ambiguity in the choice of λ in $[\lambda] = \{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$ we can choose the eigenvalue λ with a positive imaginary part and a modulus greater than* 1. *The size of the blocks is determined by the dimension $\dim_{\mathbb{C}}\big(\mathrm{Ker}(A - \lambda\,\mathrm{Id})^r\big)$ for each $r \geq 1$.*

This normal form is unique, when a choice of λ in the set $[\lambda]$ is fixed.

4. Normal forms for $A_{|V_{[\lambda]}}$ when $\lambda = \pm 1$

In this situation $[\lambda] = \{\lambda\}$ and $V_{[\lambda]}$ is the generalized real eigenspace of eigenvalue λ, still denoted—with a slight abuse of notation—E_λ. Again, p denotes the largest integer such that $(A - \lambda\,\mathrm{Id})^p$ does not vanish identically on E_λ. We consider $\tilde{Q}_p : E_\lambda/\mathrm{Ker}(A - \lambda\,\mathrm{Id})^p \times E_\lambda/\mathrm{Ker}(A - \lambda\,\mathrm{Id})^p \to \mathbb{R}$ the non degenerate form defined by $\tilde{Q}_p([v],[w]) = \Omega\big((A - \lambda\,\mathrm{Id})^p v, w\big)$. We see directly from equation (13) that $\tilde{Q}_p$ is symmetric if p is odd and antisymmetric if p is even.

4.1. If $p = 2k - 1$ is odd.

We choose $v \in E_\lambda$ such that

$$\tilde{Q}([v],[v]) = \Omega\big((A - \lambda\,\mathrm{Id})^p v, v\big) \neq 0$$

and consider the smallest A-invariant subspace E_λ^v of E_λ containing v; it is spanned by

$$\{a_p := (A - \lambda\,\mathrm{Id})^p v, \ldots, a_i := (A - \lambda\,\mathrm{Id})^i v, \ldots, a_0 := v\}.$$

We have

$\Omega(a_i, a_j) = 0$ if $i + j \geq p + 1 (= 2k)$ by equation (14);

$\Omega(a_i, a_{p-i}) \neq 0$; by equation (12) and by the choice of v.

Hence E_λ^v is a symplectic subspace because, in the basis defined by the e_i's, Ω has the triangular form $\begin{pmatrix} 0 & & \bar{*} \\ & \cdot^{\cdot^{\cdot}} & \\ \bar{*} & & * \end{pmatrix}$ and has a non-zero determinant.

We can choose v in $E_\lambda \subset V$ so that $\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\lambda\,\mathrm{Id})^{k-1} v\big) = \lambda s$ with $s = \pm 1$ by rescaling the vector and one may further assume, by Lemma 2.6, that

$$T_{i,j}(v) = \frac{1}{\lambda^i}\frac{1}{\lambda^j}\Omega\big((A-\lambda\,\mathrm{Id})^i v, (A-\lambda\,\mathrm{Id})^j v\big) = 0 \qquad \text{for all } 0 \le i, j \le k-1.$$

We now construct a symplectic basis $\{a'_p, \dots, a'_k, a_0, \dots, a_{k-1}\}$ of E^v_λ, extending $\{a_0, \dots, a_{k-1}\}$, by a Gram-Schmidt procedure, having chosen v as above. We define inductively on $0 \le j \le k-1$

$a'_p := \frac{1}{\Omega(a_p, a_0)} a_p$;

$a'_{p-j} = \frac{1}{\Omega(a_{p-j}, a_j)}\big(a_{p-j} - \sum_{k<j}\Omega(a_{p-j}, a_k) a'_{p-k}\big)$,

so that any a'_j is a linear combination of the a_r's with $r \ge j$ and in particular $a'_k = \frac{1}{s\lambda} a_k + \sum_{j=1}^{k-1} c_j a_{k+j}$.

In the symplectic basis $\{a'_p, \dots, a'_k, a_0, \dots, a_{k-1}\}$ the matrix representing A is

$$A' = \begin{pmatrix} B & C \\ 0 & J(\lambda,k)^\tau \end{pmatrix}$$

with $J(\lambda, m)$ defined by (19) and with C identically zero except for the last column, and the coefficient $C^k_k = s\lambda$. Since the matrix is symplectic, B is the transpose of the inverse of $J(\lambda, p+1)^\tau$ by (18), so $B = J(\lambda,k)^{-1}$ and $J(\lambda,k)C$ is symmetric with zeroes except in the last column, hence diagonal of the form $\mathrm{diag}(0, \dots, 0, s)$. Thus

$$\begin{pmatrix} J(\lambda,k)^{-1} & J(\lambda,k)^{-1}\,\mathrm{diag}(0,\dots,0,s) \\ 0 & J(\lambda,k)^\tau \end{pmatrix},$$

with $s = \pm 1$, is the normal form of A restricted to E^v_λ. Recall that

$$s = \lambda^{-1}\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\lambda\,\mathrm{Id})^{k-1} v\big).$$

4.2. If $p = 2k$ is even.

We choose v and w in E_λ such that

$$\tilde{Q}([v],[w]) = \Omega\big((A-\lambda\,\mathrm{Id})^p v, w\big) = \lambda^p = 1$$

and we consider the smallest A-invariant subspace $E^v_\lambda \oplus E^w_\lambda$ of E_λ containing v and w. It is of dimension $4k+2$. Remark that $\Omega\big((A-\lambda\,\mathrm{Id})^p v, v\big) = 0$. We can choose v so that

$$T_{r,s}(v) = \frac{1}{\lambda^{r+s}}\Omega\big((A-\lambda\,\mathrm{Id})^r v, (A-\lambda\,\mathrm{Id})^s v\big) = 0 \qquad \text{for all } r, s.$$

Indeed, by formula (11) we have $T_{i,j}(v) = -T_{i+1,j}(v) - T_{i+1,j-1}(v)$. Observe that $T_{i,j}(v) = -T_{j,i}(v)$ so that $T_{i,i}(v) = 0$ and $T_{j,i}(v) = -T_{j,i+1}(v) - T_{j-1,i+1}(v)$. We proceed by induction, as in Lemma 2.6:

- $T_{p,0}(v) = 0$ implies $T_{p-r,r}(v) = 0$ for all $0 \leq r \leq p$ by equation (12).
- We assume by decreasing induction on J, starting from $J = p$, that we have $T_{i,j}(v) = 0$ for all $i + j \geq J$. Then we have $T_{J-1-s,s}(v) = -T_{J-1-s,s+1}(v) - T_{J-2-s,s+1}(v)$; the first term on the righthand side vanishes by the induction hypothesis, so $T_{J-1,0}(v) = (-1)^s T_{J-1-s,s}(v) = (-1)^{J-1} T_{0,J-1}(v) = (-1)^J T_{J-1,0}$.

  If $T_{J-1,0}(v) = \alpha \neq 0$, J must be even and we replace v by

$$v' = v + \frac{\alpha}{2\lambda^{p-J+1}} (A - \lambda\,\mathrm{Id})^{p-J+1} w.$$

  Then $v' \in E^v_\lambda \oplus E^w_\lambda$, $E^v_\lambda \oplus E^w_\lambda = E^{v'}_\lambda \oplus E^w_\lambda$, $\Omega\big((A - \lambda\,\mathrm{Id})^p v', w\big) = \lambda^p$ and $T_{i,j}(v') = T_{i,j}(v) = 0$ for all $i + j \geq J$ but now

$$\begin{aligned} T_{J-1,0}(v') &= T_{J-1,0}(v) + \frac{\alpha}{2\lambda^p}\Omega\big((A - \lambda\,\mathrm{Id})^p w, v\big) \\ &\quad + \frac{\alpha}{2\lambda^p}\Omega\big((A - \lambda\,\mathrm{Id})^{J-1} v, (A - \lambda\,\mathrm{Id})^{p-J+1} w\big) \\ &\quad + \frac{\alpha^2}{4\lambda^p}\Omega\big((A - \lambda\,\mathrm{Id})^p w, (A - \lambda\,\mathrm{Id})^{p-J+1} w\big) \\ &= \alpha - \frac{\alpha}{2} - \frac{\alpha}{2} = 0 \end{aligned}$$

  so that $T_{i,j}(v') = 0$ for all $i + j \geq J - 1$ and the induction proceeds.

We assume from now on that we have chosen v and w in E_λ so that $\Omega\big((A - \lambda\,\mathrm{Id})^p v, w\big) = 1$ and $\Omega\big((A - \lambda\,\mathrm{Id})^r v, \big(A - \frac{1}{\lambda}\,\mathrm{Id}\big)^s v\big) = 0$ for all r, s. We can proceed similarly with w so we can thus furthermore assume that $\Omega\big((A - \lambda\,\mathrm{Id})^j w, (A - \lambda\,\mathrm{Id})^k w\big) = 0$ for all j, k.

A basis of $E^v_\lambda \oplus E^w_\lambda$ is given by

$$\{a_p = (A - \lambda\,\mathrm{Id})^p v, \ldots, a_0 = v, b_0 = w, \ldots, b_p = (A - \lambda\,\mathrm{Id})^p w\}.$$

We have

$\Omega(a_i, a_j) = 0$ and $\Omega(b_i, b_j) = 0$ by the choice of v and w;

$\Omega(a_i, b_j) = 0$ if $i + j > p$ by equation (14);

$\Omega(a_i, b_{p-i}) \neq 0$ by equation (12) and the choice of v, w.

The matrix representing Ω has the form $\left(\begin{array}{c|c} 0 & \begin{matrix} \bar{*} & & 0 \\ & \ddots & \\ * & & \bar{*} \end{matrix} \\ \hline \begin{matrix} \bar{*} & & * \\ & \ddots & \\ 0 & & \bar{*} \end{matrix} & 0 \end{array}\right)$ hence is non singular and the subspace $E^v_\lambda \oplus E^w_\lambda$ is symplectic. We now construct a symplectic basis $\{a'_p, \dots, a'_0, b_0, \dots, b_p\}$ of $E^v_\lambda \oplus E^w_{1/\lambda}$, extending $\{b_0, \dots, b_p\}$, using a Gram-Schmidt procedure on the a_i's. We define inductively on j

$a'_p := \frac{1}{\Omega(a_p, b_0)} a_p$;

$a'_{p-j} = \frac{1}{\Omega(a_{p-j}, b_j)} \left(a_{p-j} - \sum_{k<j} \Omega(a_{p-j}, b_k) a'_{p-k}\right)$,

so that any a'_j is a linear combination of the a_k with $k \geq j$.

In the symplectic basis $\{a'_p, \dots, a'_0, b_0, \dots, b_p\}$ the matrix representing A is

$$\begin{pmatrix} B & 0 \\ 0 & J(\lambda, p+1)^\tau \end{pmatrix}.$$

Hence, the matrix

$$\begin{pmatrix} J(\lambda, p+1)^{-1} & 0 \\ 0 & J(\lambda, p+1)^\tau \end{pmatrix}$$

is a normal form for A restricted to $E^v_\lambda \oplus E^w_\lambda$. Thus we have:

Theorem 4.1 (Normal form for $A_{|V_{[\lambda]}}$ for $\lambda = \pm 1$.). *Let $\lambda = \pm 1$ be an eigenvalue of A. There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of matrices of the form*

$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, s_j, \lambda) \\ 0 & J(\lambda, r_j)^\tau \end{pmatrix}$$

where $C(r_j, s_j, \lambda) := J(\lambda, r_j)^{-1} \operatorname{diag}(0, \dots, 0, s_j)$ with $s_j \in \{0, 1, -1\}$. If $s_j = 0$, then r_j is odd. The dimension of the eigenspace of eigenvalue 1 *is given by* $2\,\mathrm{Card}\{j \mid s_j = 0\} + \mathrm{Card}\{j \mid s_j \neq 0\}$.

Definition 4.2. Given $\lambda \in \{\pm 1\}$, we define, for any integer $k \geq 1$, a bilinear form $\hat{Q}^\lambda_{2k}$ on $\mathrm{Ker}\big((A - \lambda\,\mathrm{Id})^{2k}\big)$:

$$\begin{aligned} \hat{Q}^\lambda_{2k} : \mathrm{Ker}\big((A - \lambda\,\mathrm{Id})^{2k}\big) \times \mathrm{Ker}\big((A - \lambda\,\mathrm{Id})^{2k}\big) &\to \mathbb{R} \\ (v, w) &\mapsto \lambda\Omega\big((A - \lambda\,\mathrm{Id})^k v, (A - \lambda\,\mathrm{Id})^{k-1} w\big). \end{aligned} \tag{21}$$

It is symmetric.

Proposition 4.3. *Given $\lambda \in \{\pm 1\}$, the number of positive (resp. negative) eigenvalues of the symmetric 2-form $\hat{Q}^{\lambda}_{2k}$ is equal to the number of s_j equal to $+1$ (resp. -1) arising in blocks of dimension $2k$ (i.e. with corresponding $r_j = k$) in the normal decomposition of A on $V_{[\lambda]}$ given in Theorem* 4.1.

On $V_{[\lambda]}$, we have:

$$\sum_j s_j = \sum_{k=1}^{\dim V} \mathrm{Signature}(\hat{Q}^{\lambda}_{2k}) \tag{22}$$

Proof. On the intersection of $\mathrm{Ker}\big((A - \lambda\,\mathrm{Id})^{2k}\big)$ with one of the symplectically orthogonal subspaces E^v_λ constructed above for an odd $p \neq 2k-1$, the form $\hat{Q}^{\lambda}_{2k}$ vanishes identically. On the intersection of $\mathrm{Ker}\big((A - \lambda\,\mathrm{Id})^{2k}\big)$ with a subspace E^v_λ for a v so that $p = 2k-1$ and $\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\lambda\,\mathrm{Id})^{k-1} v\big) = \lambda s$ the only non vanishing component is $\hat{Q}^{\lambda}_{2k}(v, v) = s$.

Indeed, $\mathrm{Ker}\big((A - \lambda\,\mathrm{Id})^{2k}\big) \cap E^v_\lambda$ is spanned by

$$\{(A - \lambda\,\mathrm{Id})^r v; r \geq 0 \text{ and } r + 2k > p\},$$

and $\Omega\big((A-\lambda\,\mathrm{Id})^{k+r} v, (A-\lambda\,\mathrm{Id})^{k-1+r'} v\big) = 0$ when $2k + r + r' - 1 > p$ so the only non vanishing cases arise when $r = r' = 0$ and $p = 2k-1$.

Similarly, the 2 form $\hat{Q}^{\lambda}_{2k}$ vanishes on the intersection of $\mathrm{Ker}\big((A - \lambda\,\mathrm{Id})^{2k}\big)$ with a subspace $E^v_\lambda \oplus E^w_\lambda$ constructed above for an even p. □

The numbers s_j appearing in the decomposition of A are thus invariant of the matrix.

Corollary 4.4. *The normal decomposition described in Theorem* 4.1 *is determined by the eigenvalue λ, by the dimension* $\dim\big(\mathrm{Ker}(A - \lambda\,\mathrm{Id})^r\big)$ *for each $r \geq 1$, and by the rank and the signature of the symmetric bilinear 2-forms $\hat{Q}^{\lambda}_{2k}$ for each $k \geq 1$. It is unique up to a permutation of the blocks.* □

5. Normal forms for $A_{|V_{[\lambda]}}$ when $\lambda = e^{i\phi} \in S^1 \backslash \{\pm 1\}$

We denote again by p the largest integer such that $(A - \lambda\,\mathrm{Id})^p$ does not vanish identically on E_λ and we consider the non degenerate sesquilinear form

$$\hat{Q} : E_\lambda / \mathrm{Ker}(A - \lambda\,\mathrm{Id})^p \times E_\lambda / \mathrm{Ker}(A - \lambda\,\mathrm{Id})^p \to \mathbb{C}$$
$$\hat{Q}([v], [w]) = \overline{\lambda^p}\,\Omega\big((A - \lambda\,\mathrm{Id})^p v, \overline{w}\big).$$

Since $\hat{Q}$ is non degenerate, we can choose $v \in E_\lambda$ such that $\hat{Q}([v],[v]) \neq 0$ thus $(A - \lambda \,\mathrm{Id})^p v \neq 0$ and we consider the smallest A-invariant subspace, stable by complex conjugaison, and containing $v : E^v_\lambda \oplus E^{\bar v}_{\bar\lambda} \subset E_\lambda \oplus E_{\bar\lambda}$. A basis is given by

$$\{a_i := (A - \lambda \,\mathrm{Id})^i v, b_j := (A - \bar\lambda \,\mathrm{Id})^j \bar v \;\; 0 \leq i, j \leq p\}.$$

We have $a_i = \overline{b_i}$ and

- $\Omega(a_i, a_j) = 0$, $\Omega(b_i, b_j) = 0$ because $\Omega(E_\lambda, E_\lambda) = 0$;
- $\Omega(a_i, b_k) = 0$ if $i + k \geq p + 1$ by equation (14);
- $\Omega(a_i, b_k) \neq 0$ if $p = i + k$ by equation (12) and by the choice of v.

We conclude that $E^v_\lambda \oplus E^{\bar v}_{\bar\lambda}$ is a symplectic subspace.

5.1. If $p = 2k - 1$ is odd. Observe that $T_{k,k-1}(v) := \frac{1}{\lambda}\Omega\big((A - \lambda \,\mathrm{Id})^k v, (A - \bar\lambda \,\mathrm{Id})^{k-1} \bar v\big) = s$ is real and can be put to ± 1 by rescaling the vector (we could even put it to 1 exchanging if needed λ and its conjugate). One may further assume, by Lemma 2.6 that

$$T_{i,j}(v) = \frac{1}{\lambda^i}\frac{1}{\bar\lambda^j}\Omega\big((A - \lambda \,\mathrm{Id})^i v, (A - \bar\lambda \,\mathrm{Id})^j \bar v\big) = 0 \qquad \text{for all } 0 \leq i, j \leq k - 1.$$

We consider the basis $\{a_{2k-1}, \ldots, a_k, b_p, \ldots, b_k, b_0, \ldots, b_{k-1}, a_0, \ldots, a_{k-1}\}$ for such a vector v with $T_{k,k-1}(v) = s = \pm 1$ and $T_{i,j}(v) = 0$ for all $0 \leq i, j \leq k - 1$; the matrix representing Ω has the form

$$\left(\begin{array}{cc|cc}
 & & \begin{matrix} \bar{*} & & 0 \\ & \ddots & \\ * & & \bar{*} \end{matrix} & 0 \\
 \multicolumn{2}{c|}{0} & 0 & \begin{matrix} \bar{*} & & 0 \\ & \ddots & \\ * & & \bar{*} \end{matrix} \\ \hline
\begin{matrix} \bar{*} & & * \\ & \ddots & \\ 0 & & \bar{*} \end{matrix} & 0 & & \\
0 & \begin{matrix} \bar{*} & & * \\ & \ddots & \\ 0 & & \bar{*} \end{matrix} & \multicolumn{2}{c}{0}
\end{array}\right)$$

and we transform it by a Gram-Schmidt method into a symplectic basis composed of pairs of conjugate vectors, extending $\{b_0, \dots, b_{k-1}, a_0, \dots, a_{k-1}\}$ on which Ω identically vanishes. We define

$$a'_{2k-1} = \frac{1}{\Omega(a_{2k-1}, b_0)} a_{2k-1},$$
$$b'_{2k-1} = \frac{1}{\Omega(b_{2k-1}, a_0)} b_{2k-1} = \overline{a'_{2k-1}}$$

and, inductively on increasing j with $1 < j \le k$

$$a'_{2k-j} = \frac{1}{\Omega(a_{2k-j}, b_{j-1})}\Big(a_{2k-j} - \sum_{r=1}^{j-1} \Omega(a_{2k-j}, b_{r-1}) a'_{2k-r}\Big),$$
$$b'_{2k-j} = \overline{a'_{2k-j}}.$$

Any a'_{2k-j} is a linear combination of the a_{2k-i} for $1 \le i \le j$; reciprocally any a_{2k-j} can be written as a linear combination of the a'_{2k-i} for $1 \le i \le j$, and the coefficient of a'_{2k-j} is equal to $\Omega(a_{2k-j}, b_{j-1})$.

The basis $\{a'_{2k-1}, \dots, a'_k, b'_{2k-1}, \dots, b'_k, b_0, \dots, b_{k-1}, a_0, \dots, a_{k-1}\}$ is symplectic, and in that basis, since $A(a_r) = \lambda a_r + a_{r+1}$ and $A(b_r) = \bar{\lambda} b_r + b_{r+1}$ for all $r < 2k-2$, the matrix representing A is of the block upper triangular form

$$\begin{pmatrix} * & 0 & 0 & C \\ & * & \bar{C} & 0 \\ & & J(\bar{\lambda}, k)^{\tau} & 0 \\ 0 & & & J(\lambda, k)^{\tau} \end{pmatrix}$$

where C is a $k \times k$ matrix such that the only non vanishing terms are on the last column ($C^i_j = 0$ when $j < k$) and $C^k_k = \Omega(a_k, b_{k-1}) = s\lambda$. The fact that the matrix is symplectic implies that $S := J(\bar{\lambda}, k) C$ is hermitean; since $S^i_j = 0$ when $j \neq k$, we have,

$$C = J(\bar{\lambda}, k)^{-1} \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & s \end{pmatrix} = C(k, s, \bar{\lambda})$$

and the matrix of the restriction of A to the subspace $E^v_\lambda \oplus E^{\bar{v}}_{\bar{\lambda}}$ has the block triangular normal form

$$\begin{pmatrix} J(\bar{\lambda},k)^{-1} & 0 & 0 & C(k,s,\bar{\lambda}) \\ & J(\lambda,k)^{-1} & C(k,s,\lambda) & 0 \\ & & J(\bar{\lambda},k)^{\tau} & 0 \\ 0 & & & J(\lambda,k)^{\tau} \end{pmatrix}. \tag{23}$$

Writing $a'_{2k-j} = \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j})$, $b'_{2k-j} = \overline{a'_{2k-j}} = \frac{1}{\sqrt{2}}(e_{2j-1} + ie_{2j})$, as well as $a_{j-1} = \frac{1}{\sqrt{2}}(f_{2j-1} - if_{2j})$ and $b_{j-1} = \overline{a_{j-1}} = \frac{1}{\sqrt{2}}(f_{2j-1} + if_{2j})$ for $1 \le j \le k$, the vectors e_i, f_j all belong to the real subspace denoted $V^v_{[\lambda]}$ of V whose complexification is $E^v_\lambda \oplus E^{\bar{v}}_{\bar{\lambda}}$ and we get a symplectic basis

$$\{e_1, \ldots, e_{2k}, f_1, \ldots, f_{2k}\}$$

of this real subspace $V^v_{[\lambda]}$. The matrix representing A in this basis is:

$$\begin{pmatrix} \left(J_{\mathbb{R}}(\bar{\lambda},2k)\right)^{-1} & C_{\mathbb{R}}(k,s,\bar{\lambda}) \\ 0 & \left(J_{\mathbb{R}}(\bar{\lambda},2k)\right)^{\tau} \end{pmatrix} \tag{24}$$

where $J_{\mathbb{R}}(e^{i\phi},2k)$ is defined as in (20) and where $C_{\mathbb{R}}(k,s,e^{i\phi})$ is the $(p+1)\times(p+1)$ matrix written in terms of two by two matrices as

$$C_{\mathbb{R}}(k,s,e^{i\phi})^{\tau} = s\begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ (-1)^{k-1}R(e^{ik\phi}) & \cdots & -R(e^{i2\phi}) & R(e^{i\phi}) \end{pmatrix} \tag{25}$$

with $R(e^{i\phi}) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ as before and $s = \pm 1$. This is the normal form of A restricted to $V^v_{[\lambda]}$; recall that

$$s = \lambda^{-1}\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\bar{\lambda}\,\mathrm{Id})^{k-1}\bar{v}\big).$$

5.2. If $p = 2k$ is even. We observe that $\Omega\big((A-\bar{\lambda}\,\mathrm{Id})^k\bar{v}, (A-\lambda\,\mathrm{Id})^k v\big)$ is purely imaginary and we choose v so that it is $\Omega\big((A-\bar{\lambda}\,\mathrm{Id})^k\bar{v}, (A-\lambda\,\mathrm{Id})^k v\big) = si$ where $s = \pm 1$ (remark that the sign changes if one permutes λ and $\bar{\lambda}$). We can further choose the vector v so that:

$$\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\bar{\lambda}\,\mathrm{Id})^{k-1}\bar{v}\big) = \frac{1}{2}\lambda si \tag{26}$$

$$T_{i,j}(v) := \frac{1}{\lambda^i\bar{\lambda}^j}\Omega\big((A-\lambda\,\mathrm{Id})^i v, (A-\bar{\lambda}\,\mathrm{Id})^j\bar{v}\big) = 0 \qquad \text{for all } 0 \le i,j \le k-1;$$

Indeed, as before, by (11), we have $T_{i,j}(v) = -T_{i+1,j}(v) - T_{i+1,j-1}(v)$ and $T_{i,j}(v) = -\overline{T_{j,i}(v)}$ and we proceed as in Lemma 2.6 by decreasing induction on $i+j$:

- if $T_{k,k-1}(v) = \alpha_1$, since $T_{k-1,k}(v) = si - T_{k,k-1}(v)$ the imaginary part of α_1 is equal to $\frac{1}{2}si$ and we replace v by $v - \frac{\alpha_1}{2\lambda si}(A - \lambda\,\mathrm{Id})v$; it generates the same A-invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i+j \geq 2k$ but now $T_{k,k-1}(v) = \alpha_1 - \frac{\alpha_1}{2si}T_{k+1,k-1}(v) + \frac{\overline{\alpha_1}}{2si}T_{k,k}(v) = \alpha_1 - \frac{1}{2}\alpha_1 - \frac{1}{2}\overline{\alpha_1} = \frac{1}{2}si$ since $T_{k,k}(v) = -T_{k+1,k-1}(v) = -si$; so we can now assume $T_{k,k-1}(v) = \frac{1}{2}si$;
- if $T_{k-1,k-1}(v) = \alpha_2$, this α_2 is purely imaginary and we replace v by $v - \frac{\alpha_2}{2\lambda^2 si}(A - \lambda\,\mathrm{Id})^2 v$; it generates the same A-invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i+j \geq 2k-1$; now $T_{k-1,k-1}(v) = \alpha_2 - \frac{\alpha_2}{2si}T_{k+1,k-1}(v) + \frac{\overline{\alpha_2}}{2si}T_{k-1,k+1}(v) = \alpha_2 - \frac{1}{2}\alpha_2 + \frac{1}{2}\overline{\alpha_2} = 0$. We may thus assume this property to hold for v.
- if $T_{k-2,k-1}(v) = \alpha_3 = -T_{k-1,k-1}(v) - T_{k-1,k-2}(v) = \overline{T_{k-2,k-1}(v)}$, this α_3 is real and we replace v by $v - \frac{\alpha_3}{2\lambda^3 si}(A - \lambda\,\mathrm{Id})^3 v$; it generates and the same A-invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i+j \geq 2k-2$; now $T_{k-2,k-1}(v) = \alpha_3 - \frac{\alpha_3}{2si}T_{k+1,k-1}(v) + \frac{\overline{\alpha_3}}{2si}T_{k-2,k+2}(v) = 0$, since $T_{k+1,k-1}(v) = -T_{k,k}(v) = -T_{k-2,k+2}(v) = si$; hence also $T_{k-1,k-2}(v) = 0$;
- we now assume by induction to have a $J > 1$ so that $T_{i,j}(v) = 0$ for all $0 \leq i, j \leq k-1$ so that $i+j > 2k-1-J$;
- if $T_{k-J,k-1}(v) = \alpha_{J+1}$, then $T_{k-J,k-1}(v) = (-1)^{J-1}T_{k-1,k-J}(v)$ so that α_{J+1} is real when J is even and is imaginary when J is odd; we replace v by $v - \frac{\alpha_{J+1}}{2\lambda^{J+1} si}(A - \lambda\,\mathrm{Id})^{J+1} v$; it sgenerates the same A-invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i+j \geq 2k-J$, but now $T_{k-J,k-1}(v) = \alpha_{J+1} - \frac{\alpha_{J+1}}{2si}T_{k+1,k-1}(v) + \frac{\overline{\alpha_{J+1}}}{2si}T_{k-J,k+J}(v) = \alpha_{J+1} - \frac{\alpha_{J+1}}{2} + (-1)^{J+1}\frac{\overline{\alpha_{J+1}}}{2} = 0$.

  Hence also $T_{k-J+1,k-2}(v) = 0, \ldots, T_{k-1,k-J+1}(v) = 0$; so the induction step is proven.

Remark 5.1. For such a v, all $T_{i,j}(v)$ are determined inductively and we have

$$T_{i,j}(v) = 0 \quad \text{if } i+j \geq 2k+1 \quad \text{and} \quad \text{for all } 0 \leq i, j \leq k-1$$

$$T_{k-r,k+r}(v) = (-1)^{r+1} si \quad \text{for all } 0 \leq r \leq k$$

$$T_{k-r,k+m}(v) = (-1)^{r+1}\frac{si}{2}\frac{(r+m)(r-1)!}{m!(r-m)!} \quad \text{for all } 0 \leq m \leq r \leq k,\ r > 1$$

$$T_{i,j}(v) = T_{j,i}(v) \quad \text{for all } i, j.$$

With the notation $a_i = (A - \lambda\,\mathrm{Id})^i v$, $b_i = (A - \overline{\lambda}\,\mathrm{Id})^i \overline{v}$, we consider the basis

$$\{a_{2k}, \ldots, a_{k+1}, b_{2k}, \ldots, b_{k+1}, b_k; b_0, \ldots, b_{k-1}, a_0, \ldots, a_{k-1}, a_k\}$$

for such a vector v; the matrix representing Ω in this basis has the form

$$
\left(\begin{array}{c|c|c|c|c|c}
0 & 0 & 0 & \begin{matrix} \bar{*} & & 0 \\ & \ddots & \\ * & & \bar{*} \end{matrix} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \begin{matrix} \bar{*} & & 0 \\ & \ddots & \\ * & & \bar{*} \end{matrix} & 0 \\
\hline
0 & 0 & 0 & 0 & * \ \cdots \ * & si \\
\hline
\begin{matrix} \bar{*} & & * \\ & \ddots & \\ 0 & & \bar{*} \end{matrix} & 0 & 0 & 0 & 0 & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\
\hline
0 & \begin{matrix} \bar{*} & & * \\ & \ddots & \\ 0 & & \bar{*} \end{matrix} & \begin{matrix} * \\ \vdots \\ * \end{matrix} & 0 & 0 & 0 \\
\hline
0 & 0 & -si & * \ \cdots \ * & 0 & 0
\end{array}\right).
$$

We transform (by a Gram-Schmidt method) the basis above into a symplectic basis, composed of pairs of conjugate vectors (up to a factor) and extending

$$b_0, \ldots, b_{k-1}, a_0, \ldots, a_{k-1}$$

on which Ω identically vanishes. We define inductively, for increasing j with $1 \leq j \leq k-1$

$$
\begin{aligned}
a'_{2k} &:= \frac{1}{\Omega\big((A-\lambda\,\mathrm{Id})^{2k}v, \bar{v}\big)} (A-\lambda\,\mathrm{Id})^{2k} v = \frac{1}{\Omega(a_{2k}, b_0)} a_{2k} \\
b'_{2k} &:= \frac{1}{\Omega\big((A-\bar{\lambda}\,\mathrm{Id})^{2k}, \bar{v}, v\big)} (A-\bar{\lambda}\,\mathrm{Id})^{2k} \bar{v} = \frac{1}{\Omega(b_{2k}, a_0)} b_{2k} = \overline{a'_{2k}} \\
a'_{2k-j} &= \frac{1}{\Omega(a_{2k-j}, b_j)} \Big(a_{2k-j} - \sum_{r=0}^{j-1} \Omega(a_{2k-j}, b_r) a'_{2k-r} \Big) \\
b'_{2k-j} &= \frac{1}{\Omega(b_{2k-j}, a_j)} \Big(b_{2k-j} - \sum_{r=0}^{j-1} \Omega(b_{2k-j}, a_r) b'_{2k-r} \Big) = \overline{a'_{2k-j}} \\
a'_k &= a_k - \sum_{r=0}^{k-1} \Omega(a_k, b_r) a'_{2k-r} \\
b'_k &= \frac{1}{\Omega(b_k, a_k)} \Big(b_k - \sum_{r=0}^{k-1} \Omega(b_k, a_r) b'_{2k-r} \Big) = \frac{1}{is} \overline{a'_k}.
\end{aligned}
$$

Each a'_{2k-j} is a linear combination of the $(A-\lambda\,\mathrm{Id})^{2k-r}v$ for $0\le r\le j$. The basis

$$\{a'_{2k},\dots,a'_{k+1},b'_{2k},\dots,b'_{k+1},b'_k;b_0,\dots,b_{k-1},a_0,\dots,a_{k-1},a'_k\}$$

is now symplectic. Since $A(a_r)=\lambda a_r+a_{r+1}$ for all $r<2k$, and $A(a_{2k})=\lambda a_{2k}$, the matrix representing A in that basis is of the form

$$\begin{pmatrix} A_1 & 0 & 0 & \begin{pmatrix} 0 & c^{2k} & d^{2k}\\ & \vdots & \vdots\\ & c^{k+1} & d^{k+1}\end{pmatrix}\\ 0 & A_2 & \begin{pmatrix} & e^{2k}\\ & \vdots\\ 0 & e^{k+1}\\ & e^k\end{pmatrix} & 0\\ 0 & 0 & J(\bar\lambda,k)^\tau & 0\\ 0 & 0 & 0 & J(\lambda,k+1)^\tau\end{pmatrix}$$

with $A(b_{k-1})=\bar\lambda b_{k-1}+\sum_{j=0}^k e^{k+j}b'_{k+j}$, $A(a_{k-1})=\lambda a_{k-1}+a'_k+\sum_{j=1}^k c^{k+j}a'_{k+j}$ and $A(a'_k)=\lambda a'_k+\sum_{j=1}^k d^{k+j}a'_{k+j}$.

Since a matrix $\begin{pmatrix} A' & E\\ 0 & D\end{pmatrix}$ is symplectic if and only if $A'=(D^\tau)^{-1}$ and $D^\tau E$ is symmetric, we have

$$A_1=J(\bar\lambda,k)^{-1}\qquad A_2=J(\lambda,k+1)^{-1}$$

and

$$J(\bar\lambda,k)\begin{pmatrix} 0 & c^{2k} & d^{2k}\\ & \vdots & \vdots\\ & c^{k+1} & d^{k+1}\end{pmatrix}=\left(J(\lambda,k+1)\begin{pmatrix} & e^{2k}\\ & \vdots\\ 0 & e^{k+1}\\ & e^k\end{pmatrix}\right)^\tau.$$

This implies

$$J(\bar\lambda,k)\begin{pmatrix} c^{2k} & d^{2k}\\ \vdots & \vdots\\ c^{k+2} & d^{k+2}\\ c^{k+1} & d^{k+1}\end{pmatrix}=\begin{pmatrix} 0 & 0\\ \vdots & \vdots\\ 0 & 0\\ s_1 & s_2\end{pmatrix}\qquad J(\lambda,k+1)\begin{pmatrix} e^{2k}\\ \vdots\\ e^{k+2}\\ e^{k+1}\\ e^k\end{pmatrix}=\begin{pmatrix} 0\\ \vdots\\ 0\\ s_1\\ s_2\end{pmatrix}$$

so that $s_1 = \bar{\lambda} c^{k+1}$ and $s_2 = \bar{\lambda} d^{k+1}$. Now

$$A(a'_k) = A\Big(a_k + \sum_{j\ge 1} F_k^j a_{k+j}\Big) = \lambda a'_k + a_{k+1} + \sum_{j\ge 1} F_k^j a_{k+j+1}$$
$$= \lambda a'_k + a'_{k+1}\Omega(a_{k+1}, b_{k-1}) + \sum_{j\ge 1} F_k'^j a'_{k+j+1}$$

so that $d^{k+1} = \Omega(a_{k+1}, b_{k-1}) = \lambda^2 is$ and $s_2 = \lambda is$. We also have

$$A(a_{k-1}) = \lambda a_{k-1} + a_k = \lambda a_{k-1} + a'_k + \Omega(a_k, b_{k-1})a'_{k+1} + \sum_{j\ge 2} G^j a'_{k+j}$$

so that $c^{k+1} = \Omega(a_k, b_{k-1}) = \lambda \frac{1}{2} is$ and $s_1 = \frac{1}{2} is$.

We have thus shown that the matrix representing A in the chosen basis has the block upper-triangular normal form

$$\begin{pmatrix} J(\bar{\lambda}, k)^{-1} & 0 & 0 & J(\bar{\lambda}, k)^{-1} S \\ & J(\lambda, k+1)^{-1} & J(\lambda, k+1)^{-1} S^{\tau} & 0 \\ & & J(\bar{\lambda}, k)^{\tau} & 0 \\ & 0 & & J(\lambda, k+1)^{\tau} \end{pmatrix} \tag{27}$$

where S is the $k \times (k+1)$ matrix defined by

$$S = S(k, d, \lambda) := \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & \frac{1}{2} is & \lambda is \end{pmatrix}. \tag{28}$$

We write $a'_{2k+1-j} = \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j})$, $b'_{2k+1-j} = \overline{a'_{2k+1-j}} = \frac{1}{\sqrt{2}}(e_{2j-1} + ie_{2j})$, as well as $a_{j-1} = \frac{1}{\sqrt{2}}(f_{2j-1} - if_{2j})$ and $b_{j-1} = \overline{a_{j-1}} = \frac{1}{\sqrt{2}}(f_{2j-1} + if_{2j})$ for $1 \le j \le k$, and $a'_k = \frac{1}{\sqrt{2}}(e_{2k+1} + \mathrm{id}\, f_{2k+1})$, $b'_k = -\mathrm{id}\,\overline{a'_k} = \frac{1}{\sqrt{2}}(-f_{2k+1} - \mathrm{id}\, e_{2k+1})$. The vectors e_i, f_j all belong to the real subspace $V^{v}_{[\lambda]}$ of V whose complexification is $E^{v}_{\lambda} \oplus E^{\bar{v}}_{\bar{\lambda}}$ and we get a symplectic basis

$$\{e_1, \dots, e_{2k+1}, f_1, \dots, f_{2k+1}\}$$

of $V^{v}_{[\lambda]}$. In this basis, the matrix representing A is:

$$\left(\begin{array}{c|c|ccc|c}
\left(J_{\mathbb{R}}(\bar{\lambda},2k)\right)^{-1} & sU^2(\phi) & \begin{matrix}0 & \cdots & 0\\ \vdots & & \vdots \\ 0 & \cdots & 0\end{matrix} & \frac{s}{2}V^2(\phi) & \frac{-s}{2}V^1(\phi) & U^1(\phi)\\ \hline
0 & \cos\phi & 0\ \cdots\ 0 & 1 & 0 & s\sin\phi\\ \hline
0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \multicolumn{3}{c|}{\left(J_{\mathbb{R}}(\bar{\lambda},2k)\right)^{\tau}} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ \hline
0 & -s\sin\phi & 0\ \cdots\ 0 & 0 & -s & \cos\phi
\end{array}\right)$$

where $s=\pm 1$, $U^1(\phi)$, $U^2(\phi)$, $V^1(\phi)$ and $V^2(\phi)$ are real $2k\times 1$ column matrices such that

$$\left(V^1(\phi)V^2(\phi)\right)=\begin{pmatrix}(-1)^{k-1}R(e^{ik\phi})\\ \vdots\\ R(e^{i\phi})\end{pmatrix}$$

$$\left(U^1(\phi)U^2(\phi)\right)=\begin{pmatrix}(-1)^{k-1}R(e^{i(k+1)\phi})\\ \vdots\\ R(e^{i2\phi})\end{pmatrix}=\left(V^1(\phi)V^2(\phi)\right)\left(R(e^{i\phi})\right).$$

This is the normal form of A restricted to $V^{v}_{[\lambda]}$. Recall that

$$s=i\Omega\left((A-\lambda\,\mathrm{Id})^k v,(A-\bar{\lambda}\,\mathrm{Id})^k\bar{v}\right).$$

Theorem 5.2 (Normal form for $A_{|V_{[\lambda]}}$ for $\lambda\in S^1\backslash\{\pm 1\}$.). *Let $\lambda\in S^1\backslash\{\pm 1\}$ be an eigenvalue of A. There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of $4k_j\times 4k_j$ matrices $(k_j\geq 1)$ of the form*

$$\left(\begin{array}{c|ccc}
\left(J_{\mathbb{R}}(\bar{\lambda},2k_j)\right)^{-1} & \begin{matrix}0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{matrix} & s_jV^1_{k_j}(\phi) & s_jV^2_{k_j}(\phi)\\ \hline
0 & \multicolumn{3}{c}{\left(J_{\mathbb{R}}(\bar{\lambda},2k_j)\right)^{\tau}}
\end{array}\right) \tag{29}$$

and $(4k_j+2)\times(4k_j+2)$ matrices $(k_j\geq 0)$ of the form

$$\begin{pmatrix} \left(J_{\mathbb{R}}(\overline{\lambda},2k_j)\right)^{-1} & s_j U^2_{k_j}(\phi) & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & \frac{s_j}{2} V^2_{k_j}(\phi) & \frac{-s_j}{2} V^1_{k_j}(\phi) & U^1_{k_j}(\phi) \\ 0 & \cos\phi & 0 \ \cdots \ 0 & 1 & 0 & s_j \sin\phi \\ 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \left(J_{\mathbb{R}}(\overline{\lambda},2k_j)\right)^{\tau} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 & -s_j \sin\phi & 0 \ \cdots \ 0 & 0 & -s_j & \cos\phi \end{pmatrix} \tag{30}$$

where $J_{\mathbb{R}}(e^{i\phi},2k)$ is defined as in (20), *where $\left(V^1_{k_j}(\phi)V^2_{k_j}(\phi)\right)$ is the $2k_j\times 2$ matrix defined by*

$$\left(V^1_{k_j}(\phi)V^2_{k_j}(\phi)\right) = \begin{pmatrix} (-1)^{k_j-1}R(e^{ik_j\phi}) \\ \vdots \\ R(e^{i\phi}) \end{pmatrix} \tag{31}$$

with $R(e^{i\phi}) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, where

$$\left(U^1_{k_j}(\phi)U^2_{k_j}(\phi)\right) = \left(V^1_{k_j}(\phi)V^2_{k_j}(\phi)\right)\left(R(e^{i\phi})\right) \tag{32}$$

and where $s_j=\pm1$. The complex dimension of the eigenspace of eigenvalue λ in $V^{\mathbb{C}}$ is given by the number of such matrices.

Definition 5.3. Given $\lambda\in S^1\backslash\{\pm1\}$, we define, for any integer $m\geq 1$, a Hermitian form $\hat{Q}^\lambda_m$ on $\mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big)$ by:

$$\begin{aligned} \hat{Q}^\lambda_m : \mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big)\times\mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big) &\to \mathbb{C} \\ (v,w) &\mapsto \frac{1}{\lambda}\Omega\big((A-\lambda\,\mathrm{Id})^k v,(A-\overline{\lambda}\,\mathrm{Id})^{k-1}\overline{w}\big) \quad \text{if } m=2k \\ (v,w) &\mapsto i\Omega\big((A-\lambda\,\mathrm{Id})^k v,(A-\overline{\lambda}\,\mathrm{Id})^{k}\overline{w}\big) \quad \text{if } m=2k+1. \end{aligned}$$

Proposition 5.4. *For $\lambda\in S^1\backslash\{\pm1\}$, the number of positive* (*resp. negative*) *eigenvalues of the Hermitian 2-form $\hat{Q}^\lambda_m$ is equal to the number of s_j equal to $+1$* (*resp.* -1) *arising in blocks of dimension $2m$ in the normal decomposition of A on $V_{[\lambda]}$ given in Theorem* 5.2.

Proof. On the intersection of $\mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big)$ with one of the symplectically orthogonal subspaces $E^v_\lambda\oplus E^{\bar{v}}_{\bar{\lambda}}$ constructed above from a v such that $(A-\lambda\,\mathrm{Id})^p v\neq 0$

and $(A - \lambda \operatorname{Id})^{p+1} v = 0$, the form $\hat{Q}_m^\lambda$ vanishes identically, except if $p = m - 1$ and the only non vanishing component is $\hat{Q}_m^\lambda(v, v) = s$.

Indeed, $\operatorname{Ker}\big((A - \lambda \operatorname{Id})^m\big) \cap E_\lambda^v$ is spanned by

$$\{(A - \lambda \operatorname{Id})^r v; r \geq 0 \text{ and } r + m > p\},$$

and $\hat{Q}_m^\lambda\big((A - \lambda \operatorname{Id})^r v, (A - \lambda \operatorname{Id})^{r'} v\big) = 0$ when $m + r + r' - 1 > p$ so the only non vanishing cases arise when $r = r' = 0$ and $m = p + 1$ so for $\hat{Q}_m^\lambda(v, v)$. This is equal to $\frac{1}{\lambda}\Omega\big((A - \lambda \operatorname{Id})^k v, (A - \bar{\lambda} \operatorname{Id})^{k-1} \bar{v}\big) = \frac{1}{\lambda}\lambda s = s$ if $m = 2k$, and to $i\Omega\big((A - \lambda \operatorname{Id})^k v, (A - \bar{\lambda} \operatorname{Id})^k \bar{v}\big) = i(-is) = s$ if $m = 2k + 1$. □

The numbers s_j appearing in the decomposition are thus invariant of the matrix.

Corollary 5.5. *The normal decomposition described in Theorem* 5.2 *is unique up to a permutation of the blocks when the eigenvalue* λ *has been chosen in* $\{\lambda, \bar{\lambda}\}$*, for instance by specifyng that its imaginary part is positive. It is completely determined by this chosen* λ*, by the dimension* $\dim_{\mathbb{C}}\big(\operatorname{Ker}(A - \lambda \operatorname{Id})^r\big)$ *for each* $r \geq 1$ *and by the rank and the signature of the Hermitian bilinear* 2*-forms* $\hat{Q}_m^\lambda$ *for each* $m \geq 1$. □

References

[1] Y.-H. Au-Yeung, C.-K. Li, and L. Rodman, H-unitary and Lorentz matrices: a review. *SIAM J. Matrix Anal. Appl.* **25** (2004), 1140–1162. Zbl 1063.15021 MR 2081135

[2] I. Gohberg, P. Lancaster, and L. Rodman, *Indefinite linear algebra and applications*. Birkhäuser Verlag, Basel 2005. Zbl 1084.15005 MR 2186302

[3] I. Gohberg and B. Reichstein, On H-unitary and block-Toeplitz H-normal operators. *Linear and Multilinear Algebra* **30** (1991), 17–48. Zbl 0746.15016 MR 1119467

[4] J. Gutt, Generalized Conley–Zehnder index. *Ann. Fac. Sci. Toulouse Math.* To appear.

[5] A. J. Laub and K. Meyer, Canonical forms for symplectic and Hamiltonian matrices. *Celestial Mech.* **9** (1974), 213–238. Zbl 0316.15005 MR 0339273

[6] W.-W. Lin, V. Mehrmann, and H. Xu, Canonical forms for Hamiltonian and symplectic matrices and pencils. *Linear Algebra Appl.* **302/303** (1999), 469–533. Zbl 0947.15004 MR 1733547

[7] Y. Long, Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics. *Adv. Math.* **154** (2000), 76–131. Zbl 0970.37013 MR 1780096

[8] Y. Long, *Index theory for symplectic paths with applications*. Progr. Math. 207, Birkhäuser Verlag, Basel 2002. Zbl 1012.37012 MR 1898560

[9] Y. Long and D. Dong, Normal forms of symplectic matrices. *Acta Math. Sin.* (Engl. Ser.) **16** (2000), 237–260. Zbl 0959.15008 MR 1778705

[10] C. Mehl, Essential decomposition of polynomially normal matrices in real indefinite inner product spaces. *Electron. J. Linear Algebra* **15** (2006), 84–106. Zbl 1095.15011 MR 2201775

[11] C. Mehl, On classification of normal matrices in indefinite inner product spaces. *Electron. J. Linear Algebra* **15** (2006), 50–83. Zbl 1095.15010 MR 2201774

[12] D. Müller and C. Thiele, Normal forms of involutive complex Hamiltonian matrices under the real symplectic group. *J. Reine Angew. Math.* **513** (1999), 97–114. Zbl 0932.15006 MR 1713321

[13] L. Rodman, Similarity vs unitary similarity and perturbation analysis of sign characteristics: complex and real indefinite inner products. *Linear Algebra Appl.* **416** (2006), 945–1009. Zbl 1098.15023 MR 2242476

[14] V. V. Sergeĭchuk, Classification problems for systems of forms and linear mappings. *Izv. Akad. Nauk SSSR Ser. Mat.* **51** (1987), 1170–1190; English transl. *Math. USSR-Izv.* **31** (1988), 481–501. Zbl 04110848 MR 933960

[15] E. Spence, *m*-symplectic matrices. *Trans. Amer. Math. Soc.* **170** (1972), 447–457. Zbl 0281.15007 MR 0311684

[16] H. K. Wimmer, Normal forms of symplectic pencils and the discrete-time algebraic Riccati equation. *Linear Algebra Appl.* **147** (1991), 411–440. Zbl 0722.15012 MR 1088671

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