Portugal. Math. (N.S.)
Vol. 71, Fasc. 3-4, 2014, 177–182
DOI 10.4171/PM/1948

Portugaliae Mathematica

Analytical inequalities and isoperimetric constants

Michel Willem

(Communicated by Luís Sanchez and Susanna Terracini)

Dedicated to the memory of Miguel Ramos, exceptional friend and mathematician

Abstract. We use some isoperimetric constants in order estimate the best constants in some Sobolev inequalities.

Mathematics Subject Classification (2010). 26D15, 26A45.

Keywords. Isoperimetric inequalities, Sobolev inequalities, Poincaré-Sobolev inequalities, Cheeger inequality, coarea formula.

1. Introduction

We consider the relations between Sobolev inequalities and some isoperimetric constants. It was observed by Maz'ya and Federer and Fleming that the best constant in the Sobolev inequality

$$C\|u\|_{L^{N/(N-1)}(\mathbb{R}^N)} \leq \|\nabla u\|_{L^1(\mathbb{R}^N)} \tag{1}$$

is given by the classical isoperimetric constant

$$C = NV_N^{1/N}$$

where V_N is the volume of the unit open ball in $\mathbb{R}^N$. We denote by Ω an open subset of $\mathbb{R}^N$ and by $\mathscr{D}(\Omega)$ the space of functions u in $\mathscr{C}^\infty(\Omega)$ with compact support in Ω.

The optimal constant d of the Sobolev inequality on $\mathscr{D}(\Omega)$

$$d\|u\|_{L^q(\Omega)} \leq \|\nabla u\|_{L^1(\Omega)} \tag{2}$$

and c of the Poincaré-Sobolev inequality on $\mathscr{C}^\infty(\Omega)$

$$c \min_{t \in \mathbb{R}} \|u - t\|_{L^1(\Omega)} \leq \|\nabla u\|_{L^1(\Omega)} \tag{3}$$

are given in [8] (see also [7]).

We denote by $m(\omega)$ the Lebesgue measure of ω, by $p(\omega)$ the perimeter of ω and by $p_\Omega(\omega)$ the perimeter of ω relative to Ω.

Definition 1.1 ([7]). We define, for $q \geq 1$,

$$d(q, \Omega) = \inf\{p(\omega)/m(\omega)^{1/q} : \omega \in \mathscr{A}(\Omega)\}$$

where $\mathscr{A}(\Omega)$ is the set of smooth open subset ω of Ω with compact closure in Ω.

Definition 1.2 ([4]). We define for $q \geq 1$,

$$c(q, \Omega) = \inf\{p_\Omega(\omega)/m(\omega)^{1/q} : \omega \in \mathscr{B}(\Omega)\}$$

where $\mathscr{B}(\Omega)$ is the set of open subset ω of Ω with a smooth relative boundary $\Omega \cap \partial\omega$ and such that $m(\omega) \leq m(\Omega \backslash \omega)$.

For $1 \leq q \leq N(N-1)$, the optimal constant in (2) is given by

$$d = d(q, \Omega)$$

and the optimal constant in (3) is given by

$$c = c(1, \Omega).$$

The classical isoperimetric inequality is nothing but

$$d\big(N/(N-1), \mathbb{R}^N\big) = N V_N^{1/N}.$$

Classical Cheeger's inequality (see [3]) is equivalent to

$$\frac{1}{2} c(1, \Omega) \left\| u - \fint_\Omega u \right\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}$$

where the constant is not sharp. Let us recall that

$$\fint_\Omega u = \frac{1}{m(\Omega)} \int_\Omega u(x)\, dx.$$

For $p > 1$, the generalized Cheeger inequality (see [6]) is equivalent to

$$\frac{1}{p} c(1, \Omega) \min_{t \in \mathbb{R}} \|u - t\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}, \tag{4}$$

where the constant is not sharp. The corresponding result in $\mathscr{D}(\Omega)$ (see [5]) is

$$\frac{1}{p} d(1,\Omega) \|u\|_{L^p(\Omega)} \le \|\nabla u\|_{L^p(\Omega)}. \tag{5}$$

Our goal is to generalize all the above inequalities. We define $0 \cdot (+\infty) = 0$.

Theorem 1.3. *Let Ω be an open subset of $\mathbb{R}^N$, let p, q, r be such that $1 \le p \le q$ and*

$$\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r}. \tag{6}$$

Then, for every $u \in \mathscr{D}(\Omega)$,

$$\frac{r}{q} d(r,\Omega) \|u\|_{L^q(\Omega)} \le \|\nabla u\|_{L^p(\Omega)}. \tag{7}$$

Theorem 1.4. *Let Ω be an open subset of $\mathbb{R}^N$ such that $m(\Omega) < \infty$ and let p, q, r be such that $1 \le p \le q$ and (6) is satisfied. Then, for every $u \in C^\infty(\Omega)$,*

$$\frac{r}{q} c(r,\Omega) \min_{t \in \mathbb{R}} \|u - t\|_{L^q(\Omega)} \le \|\nabla u\|_{L^p(\Omega)}. \tag{8}$$

Of course the constants in (7) and (8) are not sharp, but their geometric meaning is clear.

In Section 2, we give a self-contained proof of Theorem 1.4. The proof of Theorem 1.3 is similar, but simpler. Section 3 is devoted to some estimates.

2. Proof of Theorem 1.4

Elementary proofs of the following results are given in [8] and [9].

Theorem 2.1. *Let $1 \le p < \infty$ and $u \in L^p(\Omega)$. Then*

$$\|u\|_{L^p(\Omega)} \le \int_0^\infty m(\{|u| > t\})^{1/p}\, dt.$$

Theorem 2.2 (Coarea formula). *Let $u \in \mathscr{C}^\infty(\Omega)$ be such that $\nabla u \in L^1(\Omega, \mathbb{R}^N)$. Then*

$$\int_\Omega |\nabla u|\, dx = \int_0^\infty p_\Omega(\{|u| > t\})\, dt.$$

Proof of Theorem 1.4. Let $u \in \mathscr{C}^\infty(\Omega)$ be such that $\nabla u \in L^p(\Omega, \mathbb{R}^N)$. We define

$$m = \sup\{t : m(\{u > t\}) > m(\{u \le t\})$$

and $v = (u - m)^+$, $w = v^{q/r}$. By Sard's theorem, for almost every $t > 0$, $\{w > t\} \in \mathscr{B}(\Omega)$. By definition, we have

$$c(r, \Omega) m(\{w > t\})^{1/r} \le p_\Omega(\{w > t\}).$$

Using Theorems 2.1 and 2.2, we obtain, after integrating from 0 to ∞,

$$c(r, \Omega) \|w\|_{L^r(\Omega)} \le \|\nabla w\|_{L^1(\Omega)}.$$

It follows from Hölder inequality that

$$c(r, \Omega) \Big(\int_\Omega v^q \, dx \Big)^{1/r} \le \frac{q}{r} \Big(\int_\Omega v^{(q/r - 1)p'} \, dx \Big)^{1/p'} \Big(\int_\Omega |\nabla v|^p \, dx \Big)^{1/p}.$$

Using (6), we conclude that

$$\frac{r}{q} c(r, \Omega) \Big(\int_\Omega v^q \, dx \Big)^{1/q} \le \Big(\int_\Omega |\nabla v|^p \, dx \Big)^{1/p}$$

or

$$\frac{r}{q} c(r, \Omega) \Big(\int_{u > m} (u - m)^q \, dx \Big)^{1/q} \le \Big(\int_{u > m} |\nabla u|^p \, dx \Big)^{1/p}.$$

Similarly, we have that

$$\frac{r}{q} c(r, \Omega) \Big(\int_{u < m} (m - u)^q \, dx \Big)^{1/q} \le \Big(\int_{u < m} |\nabla u|^p \, dx \Big)^{1/p}.$$

Since $p \le q$,

$$\Big(\int_{u > m} |\nabla u|^p \, dx \Big)^{q/p} + \Big(\int_{u < m} |\nabla u|^p \, dx \Big)^{q/p} \le \Big(\int_\Omega |\nabla u|^p \, dx \Big)^{q/p},$$

and the proof is complete. □

3. Estimates

We denote by Ω^* the open ball $B(0, r)$ such that

$$r^N V_n = m(\Omega).$$

Proposition 3.1. (a) *Let* $\Omega_1 \subset \Omega_2$ *and* $q \geq 1$. *Then*

$$d(q, \Omega_2) \leq d(q, \Omega_1).$$

(b) *Let* $q > N/(N-1)$. *Then* $d(q, \Omega) = 0$.
(c) *Let* $1 \leq q \leq N/(N-1)$ *and* $m(\Omega) < \infty$. *Then*

$$N V_N^{1/N} m(\Omega)^{(N-1)/N - 1/q} = d(q, \Omega^*) \leq d(q, \Omega).$$

Proof. Using the definition of $d(q, \Omega)$ it is easy to prove (a) and (b). To prove (c), it suffices to use Schwarz symmetrization (see e.g. [9]). □

It is more difficult to estimate the isoperimetric constant $c(q, \Omega)$. Of course $d(q, \Omega)$ is related to the Dirichlet boundary condition and $c(q, \Omega)$ is related to the Neumann boundary condition. The monotonicity relation is not valid for $c(q, \Omega)$ in particular, if Ω is not connected, then $c(q, \Omega) = 0$. Estimates of $c(q, \Omega) = 0$ involve the geometry of $\partial\Omega$ and $c(q, \Omega) = 0$ for some connected open bounded subsets Ω of $\mathbb{R}^N$ (see [8]).

We will use the following Theorem from [1].

Theorem 3.2. *Assume that* $m(\Omega) < \infty$. *Then there exists* $\xi \in \mathbb{S}^{N-1}$ *such that*

$$m_{N-1}(\Omega \cap \xi^\perp) \leq m_{N-1}(\Omega^* \cap \xi^\perp).$$

Theorem 3.3. *Let* $1 \leq q \leq N/(N-1)$ *and let* Ω *be a centrally symmetric open subset of* $\mathbb{R}^N$ *such that* $m(\Omega) < \infty$. *Then*

$$c(q, \Omega) \leq c(q, \Omega^*) = 2^{1/q} \frac{V_{N-1}}{V_N^{(N-1)/N}} m(\Omega)^{(N-1)/N - 1/q}.$$

Proof. By Lemma 8 in [2],

$$c(q, \Omega^*) = 2^{1/q} \frac{V_{N-1}}{V_N^{1/q}} r^{N-1-N/q}.$$

Since Ω is centrally symmetric, Theorem 3.2 implies that $c(q, \Omega) \leq c(q, \Omega^*)$. □

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Received December 16, 2013; revised April 3, 2014

M. Willem, IRMP-Mathématiques, Université catholique de Louvain, chemin du Cyclotron 2, B-1348 Louvain-la-Neuve (Belgium)
E-mail: michel.willem@uclouvain.be