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Existence and uniqueness of solutions for semilinear equations involving anti-selfadjoint operators

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Dedicated to Miguel Ramos, who left us much too early

Abstract. We consider the problem of the existence and uniqueness of solutions to a semilinear equation in a Hilbert space, of the type $Lu = Nu$, where the linear operator L is assumed to be anti-selfadjoint, and the nonlinear part N is controlled by two bounded selfadjoint operators A and B. As an example of application, we study the existence and uniqueness of periodic solutions for a system of transport equations. Precisely, we look for solutions which are periodic in each of their variables, the periods being determined by the forcing term.

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1. Introduction

Let H be a Hilbert space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We are interested in finding solutions of the semilinear equation

$$Lu = Nu\,, \tag{1}$$

where $L : D(L) \subseteq H \to H$ is an unbounded normal operator, and $N : H \to H$ is a continuous nonlinear function. We assume that the nonlinearity is of gradient-type, i.e., that there is a function $\eta : H \to \mathbb{K}$ such that $N = \nabla\eta$. Moreover, N will be assumed to be controlled by two bounded selfadjoint operators $A, B : H \to H$.

Systems of this type have been extensively considered in the literature. A comprehensive review on the subject can be found in [9], where the following result was obtained.

Theorem 1. *Let L be selfadjoint, and N be a gradient-type nonlinearity such that*

(i) $$N - A \text{ and } B - N \text{ are monotone}.$$

If moreover

(ii) $$0 \notin \sigma\big(L - (1-\lambda)A - \lambda B\big), \quad \text{for every } \lambda \in [0,1],$$

then equation (1) *has a unique solution, which can be obtained as the limit of the iterative process defined by*

$$Lu_{n+1} - \tfrac{1}{2}(A+B)u_{n+1} = Nu_n - \tfrac{1}{2}(A+B)u_n,$$

where $u_0 \in H$ is arbitrary.

The above theorem was motivated by a series of previous papers, see [1], [2], [3], [4], [5], [6], [7], [8], [11], [12], [14], [15], [16], [18], [19], where different kinds of selfadjoint operators had been considered.

As condition (ii) could not be so easy to verify in practice, the following variant was proposed in [10].

Proposition 2. *Let L be selfadjoint and assume that it commutes with A and B. Then, condition* (ii) *of Theorem* 1 *holds if*

(ii)$'$ $$\sigma(L) \cap \sigma\big((1-\lambda)A + \lambda B\big) = \emptyset, \quad \text{for every } \lambda \in [0,1].$$

The commutativity of L with A and B is verified in many practical cases, and applications were given in [10] to elliptic or hyperbolic systems, with several types of boundary conditions.

In this paper, we are interested in the complementary situation when L is *anti-selfadjoint*, i.e., when $L^* = -L$. In order to deal with this case, we will also need to assume that A and B commute. Here is our main result.

Theorem 3. *Let L be anti-selfadjoint, and assume that it commutes with A and B, which also commute with each other. Let N be a gradient-type nonlinearity such that*

(i) $$N - A \text{ and } B - N \text{ are monotone}.$$

If moreover

(ii)$''$ $$0 \notin \sigma\big((1-\lambda)A + \lambda B\big), \quad \text{for every } \lambda \in [0,1],$$

then the same conclusion of Theorem 1 *holds.*

The proof of Theorem 3 is given is Section 2. In Section 3 we propose an example of application to the search of periodic solutions for a system of transport equations of the type

$$\sum_{j=1}^{N} c_j \frac{\partial u}{\partial x_j} = \mathscr{F}(x, u),$$

where the nonlinear function $\mathscr{F}$ is periodic in its first variables $x_1, \dots, x_N$.

In the following, if H is a real Hilbert space, it will sometimes be convenient to extend the linear operators to the complexified space, while keeping the same notations for the extended operators.

2. Proof of the main result

We now prove Theorem 3.

First of all, we observe that there is an $\varepsilon > 0$ such that, setting $A_\varepsilon = A - \varepsilon I$ and $B_\varepsilon = B + \varepsilon I$, condition (ii)$''$ still holds for A_ε and B_ε, i.e.,

(ii)$''_\varepsilon$ $$0 \notin \sigma\big((1-\lambda)A_\varepsilon + \lambda B_\varepsilon\big), \qquad \text{for every } \lambda \in [0, 1].$$

Hence, without loss of generality, we can assume that the selfadjoint operator $S = B - A$, besides being monotone, is also invertible. We denote by $S^{1/2}$ and $S^{-1/2}$ the square roots of S and S^{-1}, respectively.

We can now write (ii)$''$ as

$$0 \notin \sigma(A + B + \nu S), \qquad \text{for every } \nu \in [-1, 1],$$

and, since

$$A + B + \nu S = S^{1/2}\big(S^{-1/2}(A + B)S^{-1/2} + \nu I\big)S^{1/2},$$

we see that (ii)$''$ is equivalent to

$$\sigma\big(S^{-1/2}(A + B)S^{-1/2}\big) \cap [-1, 1] = \emptyset. \tag{2}$$

Define the operator $\tilde{L} : S^{1/2}\big(D(L)\big) \subseteq H \to H$ by

$$\tilde{L} = S^{-1/2}\big(L - \tfrac{1}{2}(A + B)\big)S^{-1/2}.$$

Since L, A and B commute with one another, we see that $\tilde{L}$ is a normal operator. We would like to prove that

$$\sigma(\tilde{L}) \cap \{z \in \mathbb{C} : |z| \leq \tfrac{1}{2}\} = \emptyset. \tag{3}$$

We will show indeed that $\sigma(\tilde{L})$ has no elements in the strip $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{R}$. To this aim, choose some $\mu = a + ib \in \mathbb{C}$, with $a \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $b \in \mathbb{R}$, and set $Y = -iL$. Then, we can write

$$\tilde{L} - \mu I = \tilde{X}_a + i\tilde{Y}_b$$

where

$$\tilde{X}_a = -\tfrac{1}{2} S^{-1/2}(A+B)S^{-1/2} - aI, \qquad \tilde{Y}_b = S^{-1/2} Y S^{-1/2} - bI$$

are both selfadjoint, and commute. Let $E_\mu(\xi, \eta)$ denote the spectral family of the normal operator $\tilde{L} - \mu I$ (cf. [17], Chapter 9), so that

$$\tilde{L} - \mu I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\xi + i\eta)\, dE_\mu(\xi, \eta).$$

Since $a \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, by (2) we know that $0 \notin \sigma(\tilde{X}_a)$. Then, there is a $\delta > 0$ for which

$$[-\delta, \delta] \cap \sigma(\tilde{X}_a) = \emptyset,$$

whence $E_\mu(\cdot, \eta)$ is constant in $[-\delta, \delta]$. Recalling the properties of the spectral family, we can then conclude that the spectrum of $\tilde{L} - \mu I$ has no elements in the strip $[-\delta, \delta] \times \mathbb{R}$. In particular, $0 \notin \sigma(\tilde{L} - \mu I)$, i.e., $\mu \notin \sigma(\tilde{L})$.

Having proved (3), let us now define $\tilde{N} : H \to H$ as

$$\tilde{N} v = S^{-1/2}\big(N(S^{-1/2} v) - \tfrac{1}{2}(A+B) S^{-1/2} v\big).$$

By the change of variable $v = S^{1/2} u$, equation (1) becomes

$$\tilde{L} v = \tilde{N} v,$$

which is equivalent to the fixed point problem

$$v = \tilde{L}^{-1} \tilde{N} v := \mathscr{T}(v).$$

Since $\tilde{L}$ is normal, using (3) we have that

$$\|\tilde{L}^{-1}\| = \frac{1}{d\big(0, \sigma(\tilde{L})\big)} < 2.$$

On the other hand, since $N = \nabla\eta$, we have that $\tilde{N} = \nabla\tilde{\eta}$, with

$$\tilde{\eta}(v) = \eta(S^{-1/2}v) - \tfrac{1}{4}\langle (A+B)S^{-1/2}v, S^{-1/2}v\rangle.$$

Moreover, from (i) we deduce that, for every v, w in H,

$$|\langle \tilde{N}v - \tilde{N}w, v - w\rangle| \le \tfrac{1}{2}\|v - w\|^2.$$

Following [13], we have that, for every v, w in H,

$$\|\tilde{N}v - \tilde{N}w\| \le \tfrac{1}{2}\|v - w\|,$$

so that

$$\|\mathscr{T}(v) - \mathscr{T}(w)\| \le \tfrac{1}{2}\|\tilde{L}^{-1}\|\,\|v - w\|.$$

Hence, the function $\mathscr{T} : H \to H$ is Lipschitz continuous, with Lipschitz constant $\frac{1}{2}\|\tilde{L}^{-1}\| < 1$. By the contraction mapping theorem it has a unique fixed point $v \in H$, which can be obtained as the limit of the iterative process defined by $v_{n+1} = \mathscr{T}(v_n)$, with $v_0 \in H$ arbitrary. Setting $u = S^{-1/2}v$, we have that u solves (1), and writing $u_n = S^{-1/2}v_n$ the conclusion readily follows.

3. An example of application

We are interested in finding periodic solutions of the first order system

$$\sum_{j=1}^{N} c_j \frac{\partial u}{\partial x_j} = \mathscr{F}(x, u), \tag{4}$$

where $c_1, \dots, c_N$ are nonzero real constants. Writing $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, the Carathéodory function $\mathscr{F} : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^M$ is assumed to be periodic in each of the variables $x_1, \dots, x_N$. We then look for solutions $u(x)$ which have the same type of periodicity in the variables $x_1, \dots, x_N$. Denoting by $T_1, \dots, T_N$ the given periods, the periodicity conditions then read as

$$u(x_1 + T_1, x_2, \dots, x_N) = u(x_1, x_2 + T_2, \dots, x_N) = \cdots$$
$$= u(x_1, x_2, \dots, x_N + T_N) = u(x_1, x_2, \dots, x_N). \quad (5)$$

We look for L^2-solutions, i.e., solutions in the Hilbert space $H = L^2(\mathscr{Q}, \mathbb{R}^M)$, where

$$\mathscr{Q} = [0, T_1] \times \cdots \times [0, T_N].$$

Here is our result.

Theorem 4. *Assume that* $\mathscr{F}(x,u) = \nabla_u \mathscr{H}(x,u)$, *for some function* $\mathscr{H} : \mathscr{Q} \times \mathbb{R}^M \to \mathbb{R}$. *Let* $\mathbb{A}$, $\mathbb{B}$ *be two symmetric* $M \times M$ *matrices, commuting with each other, such that*

$$\langle \mathbb{A}(v-w), v-w \rangle \le \langle \mathscr{F}(x,v) - \mathscr{F}(x,w), v-w \rangle \le \langle \mathbb{B}(v-w), v-w \rangle, \quad (6)$$

for almost every $x \in \mathscr{Q}$ *and every* $v, w \in \mathbb{R}^M$. *Let us order the eigenvalues of* $\mathbb{A}$ *and* $\mathbb{B}$ *as follows:*

$$\sigma(\mathbb{A}) = \{\alpha_1 \le \alpha_2 \le \cdots \le \alpha_M\}, \qquad \sigma(\mathbb{B}) = \{\beta_1 \le \beta_2 \le \cdots \le \beta_M\}.$$

If, for every $\ell = 1, 2, \dots, M$, *the corresponding eigenvalues* α_ℓ *and* β_ℓ *have the same sign, then problem* (4)–(5) *has a unique* L^2*-solution, which can be obtained as the* L^2*-limit of the iterative process defined by*

$$\sum_{j=1}^{N} c_j \frac{\partial u_{n+1}}{\partial x_j} = \mathscr{F}(x, u_n),$$

where $u_0 \in L^2(\mathscr{Q}, \mathbb{R}^M)$ *is arbitrary.*

Proof. We show how to apply Theorem 3. For simplicity, we assume

$$T_1 = \cdots = T_N = 2\pi. \quad (7)$$

Clearly, we can always reduce to this case with a change of variables, which will have the only effect of changing the constants $c_1, \dots, c_N$.

Let us first introduce the linear operator $L : D(L) \subset H \to H$. As a basis of H we consider, as usual, $(\phi_{m,n})_{m,n \in \mathbb{Z}}$, with

$$\phi_{m,n}(x_1, x_2) = \frac{1}{2\pi} e^{i(mx_1 + nx_2)}.$$

For any $f \in H$, we can write its Fourier series

$$f = \sum_{m,n \in \mathbb{Z}} \hat{f}_{m,n} \phi_{m,n}, \qquad \text{with } \hat{f}_{m,n} = \langle f, \phi_{m,n} \rangle,$$

and define

$$D(L) = \Big\{ u \in H : \sum_{m,n \in \mathbb{Z}} |(mc_1 + nc_2)\hat{u}_{m,n}|^2 < +\infty \Big\},$$
$$Lu = \sum_{m,n \in \mathbb{Z}} i(mc_1 + nc_2)\hat{u}_{m,n}\phi_{m,n}. \tag{8}$$

The operator L is anti-selfadjoint. This well-known fact will be proved, for the reader's convenience, in the Appendix.

Since assumption (6) implies that $\mathscr{F}$ has an at most linear growth, the Nemytzkii operator $N : H \to H$ is well defined, by setting

$$(Nu)(x) = \mathscr{F}\big(x, u(x)\big).$$

It is continuous, and maps bounded sets into bounded sets. The selfadjoint operators $A, B : H \to H$ are defined as follows:

$$(Au)(x) = \mathbb{A}u(x), \qquad (Bu)(x) = \mathbb{B}u(x).$$

Clearly enough, L, A and B commute with one another, and condition (i) follows from (6). Notice moreover that condition (ii)$''$ holds as well, since we are assuming that

$$0 \notin \bigcup_{\ell=1}^{M} [\alpha_\ell, \beta_\ell].$$

Theorem 3 then applies, to give the conclusion.

4. Appendix

We prove here that the linear operator $L : D(L) \subset H \to H$ defined in (8) is anti-selfadjoint. To simplify the exposition, we will assume that $M = 1$ and

$N = 2$. It will be easily checked that analogous considerations hold in the general case.

Let us first show that $\sigma(L) \subseteq i\mathbb{R}$. Assume that $\lambda \notin i\mathbb{R}$. Then, for every $f \in H$,

$$(L - \lambda I)u = f \iff i(mc_1 + nc_2)\hat{u}_{m,n} - \lambda \hat{u}_{m,n} = \hat{f}_{m,n}, \qquad \text{for every } m, n \in \mathbb{Z}$$
$$\iff \hat{u}_{m,n} = \frac{\hat{f}_{m,n}}{i(mc_1 + nc_2) - \lambda}, \qquad \text{for every } m, n \in \mathbb{Z},$$

and, in that case,

$$\begin{aligned} \|u\|_2^2 &= \sum_{m,n \in \mathbb{Z}} \left| \frac{\hat{f}_{m,n}}{i(mc_1 + nc_2) - \lambda} \right|^2 \\ &\le \frac{1}{\operatorname{dist}(\lambda, i\mathbb{R})^2} \sum_{m,n \in \mathbb{Z}} |\hat{f}_{m,n}|^2 \\ &= \frac{1}{\operatorname{dist}(\lambda, i\mathbb{R})^2} \|f\|_2^2. \end{aligned}$$

So, $(L - \lambda I)^{-1} \in \mathscr{L}(H)$, i.e., $\lambda \notin \sigma(L)$.

The domain $D(L)$ is dense in H, since every $u \in H$ can be written as

$$u = \lim_{N \to +\infty} \left(\sum_{m,n=-N}^{N} \hat{u}_{m,n} \phi_{m,n} \right),$$

and each of the above finite sums belong to $D(L)$. Let us show that L is a closed operator. Indeed, let $(u_k)_k$ and $(f_k)_k$ be two sequences in $D(L)$ and in H, respectively, such that $Lu_k = f_k$, for every k, and $u_k \to u$, $f_k \to f$, for some $u \in H$ and $f \in H$. Then,

$$\langle u_k, \phi_{m,n} \rangle \to \langle u, \phi_{m,n} \rangle, \qquad \langle f_k, \phi_{m,n} \rangle \to \langle f, \phi_{m,n} \rangle,$$

for every $m, n \in \mathbb{Z}$, and, since

$$i(mc_1 + nc_2)\langle u_k, \phi_{m,n} \rangle = \langle f_k, \phi_{m,n} \rangle,$$

we conclude that $i(mc_1 + nc_2)\hat{u}_{m,n} = \hat{f}_{m,n}$, for every $m, n \in \mathbb{Z}$, i.e., $u \in D(L)$ and $Lu = f$. This shows that L is closed.

As a consequence, we know that $L = L^{**}$. Now, if $u, v \in D(L)$, then, since c_1 and c_2 are real,

$$\langle Lu, v\rangle = \sum_{m,n\in\mathbb{Z}} i(mc_1 + nc_2)\hat{u}_{m,n}\hat{v}^*_{m,n}$$
$$= -\sum_{m,n\in\mathbb{Z}} \hat{u}_{m,n}[i(mc_1 + nc_2)\hat{v}_{m,n}]^* = -\langle u, Lv\rangle,$$

so that $v \in D(L^*)$ and $L^*v = -Lv$. In particular, this shows that $D(L) \subseteq D(L^*)$. Let us now show that $D(L^*) \subseteq D(L)$, thus completing the proof. Take $v \in D(L^*)$. Since $L + I : D(L) \to H$ is invertible, there is a $u \in D(L)$ such that $(L + I)u = (L^* - I)v$. Since $D(L) \subseteq D(L^*)$, we have that $u + v \in D(L^*)$, and $L^*(u + v) = -Lu + L^*v$, so that

$$(L^* - I)(u + v) = -(L + I)u + (L^* - I)v = 0.$$

Then, recalling that $L^{**} = L$, since $L - I : D(L) \to H$ is invertible,

$$N(L^* - I) = I\big((L^* - I)^*\big)^{\perp} = I(L^{**} - I)^{\perp} = I(L - I)^{\perp} = \{0\}.$$

Therefore, it has to be $u + v = 0$, so that $v = -u \in D(L)$. The proof is thus completed.

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