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Boundary value problems for a class of first order quasilinear ordinary differential equations

Vincent Bouchez and Jean Mawhin

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Remembering Miguel Ramos, the man and the mathematician

Abstract. Using continuation theorems of Leray-Schauder degree theory, we obtain existence results for the first order quasilinear boundary value problem

$$\big(\phi(u)\big)' = f(t,u), \qquad u(T) = bu(0),$$

where $\phi : \mathbb{R} \to (-a,a)$ is an homeomorphism such that $\phi(0) = 0$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, a and T being positive real numbers and b some non zero real number.

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1. Introduction

We consider in this paper the first order quasilinear boundary value problem

$$\big(\phi(u)\big)' = f(t,u), \qquad u(T) = bu(0), \tag{$\mathscr{P}_b$}$$

where $\phi : \mathbb{R} \to (-a,a)$ is an homeomorphism such that $\phi(0) = 0$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, a and T are positive real number and b is a real nonzero number. We call *solution* of this problem any function $u : [0,T] \to \mathbb{R}$ such that $\phi(u)$ is continuously differentiable, satisfying the two conditions of ($\mathscr{P}_b$).

Several papers have been recently devoted to the study of the second order version of this problem

$$\big(\phi(u')\big)' = f(t,u,u'), \qquad l(u,u') = 0, \tag{$\mathscr{Q}$}$$

where $l(u, u') = 0$ denotes the periodic, Neumann or Dirichlet boundary conditions. For positive classical or non classical solutions, $\phi(s) = s/\sqrt{1+s^2}$ (for which the left-member of $(\mathscr{Q})$ is the curvature of the graph of u) and Dirichlet conditions, one can consult [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [16], [17], [18], [19], [20], [21]. In [1] and [2], the scalar case with ϕ like in $(\mathscr{P}_b)$ and f continuous is considered, and it is observed that the non-surjectivity of ϕ on $\mathbb{R}$ leads to some difficulties, with respect to the classical case $\phi : \mathbb{R} \to \mathbb{R}$ and to the relativistic case $\phi : (-a, a) \to \mathbb{R}$. Among others ones the following theorems are proved in [1], [2], where a *solution* of problem $(\mathscr{Q})$ is any function $u : [0, T] \to \mathbb{R}^N$ of class C^1 such that $\phi(u')$ is absolutely continuous, which satisfies the two conditions of $(\mathscr{Q})$ a.e. on $[0, T]$.

Theorem A. *If the function f satisfies the following conditions:*

(1) *There exists $c \in C([0, T])$ such that $\|c^-\|_1 < \frac{a}{2}$ and $f(t, u, v) \geq c(t)$ for all $(t, u, v) \in [0, T] \times \mathbb{R}^2$, where $c- := \max\{-c, 0\}$.*

(2) *There exists $R > 0$ and $\varepsilon \in \{\pm 1\}$ such that for all $u \in C([0, T])$,*

$$\varepsilon \int_0^T f(t, u(t), u'(t))\, dt > 0 \quad \text{if } \min u \geq R, \quad \|u'\|_\infty \leq M,$$

$$\varepsilon \int_0^T f(t, u(t), u'(t))\, dt < 0 \quad \text{if } \max u \leq -R, \quad \|u'\|_\infty \leq M,$$

where $M := \max\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)|\}$,

problem $(\mathscr{Q})$ with periodic or Neumann boundary conditions has at least one solution.

Theorem B. *If the function f satisfies the condition*

$$\exists c > 0,\ \forall (t, u, v) \in [0, T] \times \mathbb{R}^2 : |f(t, u, v)| \leq c < \frac{a}{2T},$$

problem $(\mathscr{Q})$ with Dirichlet boundary conditions has at least one solution.

Inspired by those results, we study the problem $(\mathscr{P}_b)$ by using similar topological methods based upon Leray-Schauder degree [14]. Such a problem does not seem to have been considered in the literature.

In this paper, we denote by $C([0, T])$ the space of continuous functions from $[0, T]$ to $\mathbb{R}$ equiped by the supremum norm $\|\cdot\|_\infty$. For $u \in C([0, T])$, we write

$$u_m := \min_{[0, T]} u, \qquad u_M := \max_{[0, T]} u.$$

If Ω is an open bounded set of $C([0,T])$ (resp. $\mathbb{R}^n$) and if $\chi : \overline{\Omega} \to C([0,T])$ (resp. $f : \overline{\Omega} \to \mathbb{R}^n$) is a completely continuous operator (resp. continuous mapping) without fixed-point (resp. zero) on $\partial\Omega$, we denote by $\deg_{\mathrm{LS}}[id - \chi, \Omega]$ (resp. $\deg_{\mathrm{B}}[f, \Omega]$) the Leray-Schauder (resp. Brouwer) degree of the operator $id - \chi$ (resp. f) on Ω at 0. If $h \in C([0,T])$, $\bar{h}$ denotes the mean value $T^{-1}\int_0^T h$ of h, and h^+, h^- respectively $\max\{h, 0\}$ and $\max\{-h, 0\}$. At last, if $t \in \mathbb{R}_0$ and $\phi : \mathbb{R} \to (-a, a)$ is an homeomorphism, $\operatorname{sgn}(t)$ and $\sigma(\phi)$ denote respectively the function equal to 1 if $t > 0$ and -1 if $t < 0$, and to 1 if ϕ is increasing and -1 if ϕ is decreasing. Finally, let

$$B_{\phi,b} : \mathbb{R} \to \mathbb{R}, \qquad u \to \phi(bu) - \phi(u). \tag{1}$$

The main result of this paper is the following one, and can be seen as a variant of Theorem A above for problem $(\mathscr{P}_b)$.

Theorem 1. *If the function f satisfies the following conditions* (i) (*resp.* (i′)) *and* (ii) (*resp.* (ii′))*:*

(i) *There exists $M_1 < M_2$ such that for all $u \in C([0,T])$,*

$$\int_0^T f\big(t, u(t)\big)\,\mathrm{d}t > \sup B_{\phi,b} \quad \text{if } u_m \geq M_2,$$
$$\int_0^T f\big(t, u(t)\big)\,\mathrm{d}t < \inf B_{\phi,b} \quad \text{if } u_M \leq M_1,$$

(i′) *There exists $M_1 < M_2$ such that for all $u \in C([0,T]$,*

$$\int_0^T f\big(t, u(t)\big)\,\mathrm{d}t > \sup B_{\phi,b} \quad \text{if } u_M \leq M_1,$$
$$\int_0^T f\big(t, u(t)\big)\,\mathrm{d}t < \inf B_{\phi,b} \quad \text{if } u_m \geq M_2,$$

(ii) $|\phi(M_2) + \sigma(\phi)\int_0^T f^+(t, M_2)\,\mathrm{d}t| < a$, $|\phi(M_1) - \sigma(\phi)\int_0^T f^-(t, M_1)\,\mathrm{d}t| < a$,

(ii′) $|\phi(M_2) + \sigma(\phi)\int_0^T f^-(t, M_2)\,\mathrm{d}t| < a$, $|\phi(M_1) - \sigma(\phi)\int_0^T f^+(t, M_1)\,\mathrm{d}t| < a$,

problem $(\mathscr{P}_b)$ with $b > 0$ has at least one solution.

To obtain the *a priori* estimates required by Leray-Schauder method, we use a technique introduced in [15] for the problem

$$u' = f(t, u), \qquad u(T) = u(0), \tag{$\mathscr{R}$}$$

combining the classical condition:

$$\text{if } u \text{ is solution of problem } (\mathscr{R}) \text{ then } \int_0^T f\big(t, u(t)\big)\,\mathrm{d}t = 0,$$

with the technique of guiding functions, to obtain existence results with the topological method. We adapt it here for more general boundary conditions than just periodic ones, by using the boundedness of ϕ, or, more precisely, by using the property:

$$\begin{aligned}&\text{if } u \text{ is solution of problem } (\mathscr{P}_b) \text{ then,}\\ &\text{for all } 0 \le t_1 < t_2 \le 1 : \left| \int_{t_1}^{t_2} f\big(t, u(t)\big)\,\mathrm{d}t \right| < 2a. \end{aligned} \tag{2}$$

The study of problem $(\mathscr{P}_b)$ is completed by two more results. The first one can be seen as a variant of Theorem B above for problem $(\mathscr{P}_b)$. The second one replaces the integral condition on f in Theorem 1 by a point-wise one.

Theorem 2. *If there exists a function $c \in L^1([0,T], \mathbb{R}^+)$ such that $\|c\|_1 < a$ and*

$$\forall t \in [0,T],\ \forall u \in \mathbb{R} : |f(t,u)| \le c(t),$$

then problem $(\mathscr{P}_b)$ with $b < 0$ has at least one solution.

Theorem 3. *Let us distinguish the six cases*

$$\begin{array}{llllll} (a_1) & b < -1, & (a_2) & -1 < b < 0, & (a_3) & b = -1,\\ (b_1) & 0 < b < 1, & (b_2) & 1 < b, & (b_3) & b = 1, \end{array}$$

and the six following assumptions:

$$\begin{array}{llll} (A_1) & \exists M > 0, & \forall t \in [0,T] : & \sigma(\phi)\cdot f(t,-M) > 0 > \sigma(\phi)\cdot f(t,M),\\ (A_2) & \exists M > 0, & \forall t \in [0,T] : & \sigma(\phi)\cdot f(t,-M) < 0 < \sigma(\phi)\cdot f(t,M),\\ (A_3) & \exists M > 0, \exists \varepsilon \in \{\pm 1\}, & \forall t \in [0,T] : & \varepsilon\cdot f(t,-M) < 0 < \varepsilon\cdot f(t,M),\\ (B_1) & \exists M_1 < 0 < M_2, & \forall t \in [0,T] : & \sigma(\phi)\cdot f(t,M_1) < 0 < \sigma(\phi)\cdot f(t,M_2),\\ (B_2) & \exists M_1 < 0 < M_2, & \forall t \in [0,T] : & \sigma(\phi)\cdot f(t,M_1) > 0 > \sigma(\phi)\cdot f(t,M_2),\\ (B_3) & \exists M_1 < M_2, \exists \varepsilon \in \{\pm 1\}, & \forall t \in [0,T] : & \varepsilon\cdot f(t,M_1) < 0 < \varepsilon\cdot f(t,M_2). \end{array}$$

If the assumption (A_i) (resp. (B_i)) is satisfied, then problem $(\mathscr{P}_b)$ in the case (a_i) (resp. (b_i)) has at least one solution with values in $(-M, M)$ (resp. (M_1, M_2)) $(i = 1, 2, 3)$.

In this paper we work directly with problem $(\mathscr{P}_b)$ and do not use the reformulation

$$u' = f(t,u) \cdot \big(\phi'(u)\big)^{-1} =: g(t,u), \qquad u(T) = bu(0),$$

when ϕ is a diffeomorphism, or

$$v' = f\big(t, \phi^{-1}(v)\big) =: h(t,u), \qquad \phi^{-1}\big(v(T)\big) = b\phi^{-1}\big(v(0)\big),$$

with $v := \phi(u)$ when ϕ is a homeomorphism. Those problems are more classical but, in the second approach, we have complicated nonlinear boundary conditions (except in the periodic case), and in the first one, working with g depending of f and ϕ' does not seem easier than dealing directly with f and ϕ.

The paper is organized as follows. We start with the search of equivalent fixed point problems in Section 2. Theorem 1 is proved in Section 3. We proceed then to the proof of Theorems 2 and 3 in Section 4. Section 5 deals with examples and numerical experiments. An appendix gives some information about the numerical tools.

2. Fixed point formulations

2.1. The forced problem. We call *forced problem* the special case of problem $(\mathscr{P}_b)$ where the right member is a continuous function h independent u:

$$\big(\phi(u)\big)' = h(t), \qquad u(T) = bu(0). \tag{$\mathscr{P}_{b,h}$}$$

Because of the equivalences

$$\begin{aligned} u(T) = bu(0) &\iff \phi\big(u(T)\big) = \phi\big(bu(0)\big) \iff \phi\big(u(0)\big) + \int_0^T h(s)\,\mathrm{d}s = \phi\big(bu(0)\big) \\ &\iff B_{\phi,b}\big(u(0)\big) = \int_0^T h(s)\,\mathrm{d}s \end{aligned}$$

we see that problem $(\mathscr{P}_{b,h})$ is solvable if and only if the scalar equation in $v \in \mathbb{R}$

$$B_{\phi,b}(v) = \int_0^T h(t)\,dt$$

has a solution, where $B_{\phi,b}(u)$ is defined in (1).

When $b < 0$, the injectivity of $B_{\phi,b}$ is ensured by the fact that $\phi(b\cdot)$ and $-\phi$ are simultaneously increasing or decreasing. In this case, the unique solution of $(\mathscr{P}_{b,h})$ satisfies the identity

$$\phi\big(u(t)\big) = \phi\bigg(B_{\phi,b}^{-1}\Big(\int_0^T h(s)\,\mathrm{d}s\Big)\bigg) + \int_0^t h(s)\,\mathrm{d}s,$$

from which we deduce necessary and sufficient conditions on h for $(\mathscr{P}_{b,h})$ to have a solution

$$\int_0^T h(s)\,\mathrm{d}s \in \operatorname{im} B_{\phi,b}, \qquad \forall t \in [0,T] : \left|\phi\Big(B_{\phi,b}^{-1}\Big(\int_0^T h(s)\,\mathrm{d}s\Big)\Big) + \int_0^t h(s)\,\mathrm{d}s\right| < a$$

Let us observe that $B_{\phi,b}$ being bounded, continuous and defined on $\mathbb{R}$, $\operatorname{im} B_{\phi,b}$ is a bounded interval. In particular,

$$\forall b < 0 : \operatorname{im} B_{\phi,b} = (-2a, 2a) \tag{3}$$

because

$$\sup B_{\phi,b} = \lim_{u\to-\infty} B_{\phi,b}(u) = a - (-a) = 2a$$
$$\inf B_{\phi,b} = \lim_{u\to+\infty} B_{\phi,b}(u) = -a - a = -2a$$

if ϕ (and so $-B_{\phi,b}$) is increasing, and

$$\sup B_{\phi,b} = \lim_{u\to+\infty} B_{\phi,b}(u) = a - (-a) = 2a$$
$$\inf B_{\phi,b} = \lim_{u\to-\infty} B_{\phi,b}(u) = -a - a = -2a$$

if ϕ (and so $-B_{\phi,b}$) is decreasing.

When $b > 0$, $B_{\phi,b}$ may be non-injective, and the necessary and sufficient conditions on h to have a solution of $(\mathscr{P}_{b,h})$ become

$$\int_0^T h(s)\,\mathrm{d}s \in \operatorname{im} B_{\phi,b}, \qquad \exists r \in B_{\phi,b}^{-1}\Big(\int_0^T h(s)\,\mathrm{d}s\Big),\ \forall t \in [0,T] : \left|\phi(r) + \int_0^t h(s)\,\mathrm{d}s\right| < a.$$

Excepted for $b = 1$ where $B_{\phi,b} \equiv 0$, $\operatorname{im} B_{\phi,b}$ depends upon ϕ but we can nevertheless have an idea of the graph of $B_{\phi,b}$. If $b \in (0,\infty)\backslash\{1\}$, $B_{\phi,b}$ vanishes only at 0 and has opposite signs on the left and on the right of 0. Indeed,

$$B_{\phi,b}(u) = 0 \iff \phi(u) = \phi(bu) \iff u = bu \iff u = 0$$

and, if $A := (0,1)$ when ϕ is increasing, $A := (1,\infty)$ when ϕ is decreasing, and A^c denotes $[(0,\infty)\backslash\{1\}]\backslash A$, then, for $b \in A$,

$$u \cdot B_{\phi,b}(u) < 0 \iff u \cdot [\phi(bu) - \phi(u)] < 0$$
$$\iff \phi(u) > \phi(bu) \quad \text{ for } u > 0, \qquad \phi(u) < \phi(bu) \quad \text{ for } u < 0,$$

and, for $b \in A^c$,

$$u \cdot B_{\phi,b}(u) > 0 \iff u \cdot [\phi(bu) - \phi(u)] > 0$$
$$\iff \phi(u) < \phi(bu) \quad \text{for } u > 0, \qquad \phi(u) > \phi(bu) \quad \text{for } u < 0.$$

Moreover, we have

$$\forall b > 0 : \|B_{\phi,b}\|_\infty < a,$$

and

$$\lim_{u \to \pm\infty} B_{\phi,b}(u) = \lim_{u \to \pm\infty} \phi(bu) - \phi(u) = 0,$$

which combined with Dini's Theorem ensures that

$$\lim_{b \to 1} \|B_{\phi,b}\|_\infty = 0.$$

Example 2.1. If $b = -1$, i.e., if $u(T) = -u(0)$, $(\mathscr{P}_b)$ is called *antiperiodic problem*. In this case, because of (3) and the injectivity of $B_{\phi,-1}$, h must satisfy condition

$$\left|\int_0^T h(s)\,ds\right| < 2a, \qquad \forall t \in [0,T] : \left|B_{\phi,-1}^{-1}\left(\int_0^T h(s)\,\mathrm{d}s\right) + \int_0^t h(s)\,\mathrm{d}s\right| < a \tag{4}$$

in order to have a solution of $(\mathscr{P}_{b,h})$, which will be given by

$$u(t) = \phi^{-1}\left(B_{\phi,-1}^{-1}\left(\int_0^T h(s)\,\mathrm{d}s\right) + \int_0^t h(s)\,\mathrm{d}s\right). \tag{5}$$

Let us remark that ϕ odd implies $B_{\phi,-1}^{-1}(v) = \phi^{-1}\left(\frac{-v}{2}\right)$. Condition (4) becomes then

$$\left|\int_0^T h(s)\,ds\right| < 2a, \qquad \forall t \in [0,T] : \left|\int_0^T G(s,t)h(s)\,\mathrm{d}s\right| < a, \tag{6}$$

where G is given by

$$G(s,t) = \begin{cases} \frac{1}{2} & \text{if } 0 \le s \le t \\ -\frac{1}{2} & \text{if } t \le s \le T \end{cases}$$

and solution (5) becomes

$$u(t) = \phi^{-1}\left(\int_0^T G(s,t)h(s)\,\mathrm{d}s\right). \tag{7}$$

Hence, (6) is ensured by $\|h\|_1 < 2a$, because we have, for all $t \in [0, T]$,

$$\Big| \int_0^T h(s)\, ds \Big| \leq \int_0^T |h(s)|\, \mathrm{d}s < 2a,$$

$$\forall t \in [0, T] : \Big| \int_0^T G(s,t)h(s)\, \mathrm{d}s \Big| \leq \int_0^T |G(s,t)h(s)|\, \mathrm{d}s = \frac{1}{2}\int_0^T |h(s)|\, \mathrm{d}s < a. \quad (8)$$

Example 2.2. If $b = 1$, $(\mathscr{P}_{b,h})$ is called *periodic problem*. In this case, $B_{\phi,1} \equiv 0$ so that $B_{\phi,1}^{-1}(c) = \mathbb{R}$. Hence, h must satisfy condition

$$\int_0^T h(s)\, \mathrm{d}s = 0, \qquad \exists r \in \mathbb{R}, \forall t \in [0, T] : \Big|\phi(r) + \int_0^t h(s)\, \mathrm{d}s\Big| < a$$

in order that $(\mathscr{P}_{b,h})$ has a solution, which will be given by

$$u(t) = \phi^{-1}\Big(\phi(r) + \int_0^t h(s)\, \mathrm{d}s\Big).$$

2.2. The fixed point operators. We can write problem $(\mathscr{P}_b)$ by the abstract form

$$D_\phi(u) = N_f(u), \qquad u \in C_b([0, T]), \quad (9)$$

where

$$\begin{aligned} D_\phi : \mathrm{dom}(D_\phi) \subset C_b([0, T]) \to C([0, T]), &\quad u \mapsto \big(\phi(u)\big)', \\ N_f : C([0, T]) \to C([0, T]), &\quad u \mapsto f\big(\cdot, u(\cdot)\big), \\ C_b([0, T]) = \{u \in C([0, T]) : u(T) = bu(0)\}&, \\ \mathrm{dom}(D_\phi) = \{u \in C_b([0, T]) : \phi(u) \in C^1([0, T])\}&. \end{aligned}$$

If $b < 0$, D_ϕ has an inverse given by

$$\begin{aligned} &D_\phi^{-1} : \mathrm{dom}(D_\phi^{-1}) \subset C([0, T]) \to C_b([0, T]), \\ &h \mapsto \phi^{-1}\Big(\phi\Big(B_{\phi,b}^{-1}\Big(\int_0^T h(s)\, \mathrm{d}s\Big)\Big) + \int_0^t h(s)\, \mathrm{d}s\Big) \end{aligned}$$

and (9) is equivalent to $u = D_\phi^{-1} N_f(u)$ with $u \in C([0, T])$. Hence our problem is finding a fixed point of the operator

$$\chi_1 := D_\phi^{-1} N_f : \mathrm{dom}(D_\phi^{-1} N_f) \subseteq C([0, T]) \to C_b([0, T]) \subset C([0, T]).$$

Let us remark that the operator χ_1 is not defined on all $C([0, T])$.

In the case where $b > 0$, we cannot do the same because the operator D_ϕ is not injective. Inspired by [14], we consider the operators

$$P : C([0,T]) \to C([0,T]), \qquad u \mapsto u(0),$$
$$Q : C([0,T]) \to C([0,T]), \qquad h \mapsto \bar{h} = T^{-1}\int_0^T h(s)\,\mathrm{d}s,$$
$$H : C([0,T]) \to C^1([0,T]) \subset C([0,T]), \qquad u \mapsto \int_0^{\cdot} u(s)\,\mathrm{d}s.$$

Let us consider the operator

$$\chi_2 : \mathrm{dom}(\chi_2) \subset C([0,T]) \to C([0,T]),$$
$$u \mapsto QN_f(u) - T^{-1}B_{\phi,b}(Pu) + \phi^{-1}\big(\phi(Pu) + H(id - Q)N_f(u) + T^{-1}\mathbf{t}B_{\phi,b}(Pu)\big)$$

where $\mathbf{t}$ denotes the function which sends t on t.

Lemma 2.3. *u is a solution of problem $(\mathscr{P}_b)$ if and only if u is a fixed-point of χ_2.*

Proof. Let $u \in C([0,T])$, we have the following equivalences:

$$\big(\phi(u)\big)' = f(t,u), \qquad u(T) = bu(0)$$
$$\iff \big(\phi(u)\big)' = f(t,u), \qquad B_{\phi,b}\big(u(0)\big) = \int_0^T f\big(s,u(s)\big)\,\mathrm{d}s$$
$$\iff \big(\phi(u)\big)' = f(t,u) - \big[\overline{f(t,u)} - T^{-1}B_{\phi,b}\big(u(0)\big)\big],$$
$$T^{-1}\int_0^T f\big(s,u(s)\big)\,\mathrm{d}s - T^{-1}B_{\phi,b}\big(u(0)\big) = 0$$
$$\iff \phi\big(u(t)\big) = \phi\big(u(0)\big) + \int_0^t \big(f\big(s,u(s)\big) - \big[\overline{f(t,u)} - T^{-1}B_{\phi,b}\big(u(0)\big)\big]\big)\,\mathrm{d}s,$$
$$T^{-1}\int_0^T f\big(s,u(s)\big)\,\mathrm{d}s - T^{-1}B_{\phi,b}\big(u(0)\big) = 0$$
$$\iff u = \phi^{-1}\big(\phi(Pu) + H(id - Q)N_f(u) + T^{-1}\mathbf{t}B_{\phi,b}(Pu)\big),$$
$$QN_f(u) - T^{-1}B_{\phi,b}(Pu) = 0$$
$$\iff u = QN_f(u) - T^{-1}B_{\phi,b}(Pu) + \phi^{-1}\big(\phi(Pu) + H(id - Q)N_f(u) + T^{-1}\mathbf{t}B_{\phi,b}(Pu)\big).$$

□

Let us remark again that this operator is not defined on all $C([0,T])$. Let us note also that Lemma 2.3 is valid in all cases $b > 0$ and $b < 0$.

Remark 2.4. We shall also consider in the sequel the case where $\operatorname{im}\phi = \mathbb{R}$. In this case, we can define in a similar way $B_{\phi,b}$, which has the same monotonicity properties than in the case studied above if $b < 0$, with this time

$$\sup B_{\phi,b} = +\infty, \qquad \inf B_{\phi,b} = -\infty,$$

and the same sign properties if $b > 0$, but without necessarily the convergence to 0 at $\pm\infty$, because

$$\lim_{u\to\pm\infty} B_{\phi,b}(u) = (\pm\infty) - (\pm\infty).$$

Hence, in the case $b < 0$, we can again consider the operators D_ϕ, D_ϕ^{-1}, χ_1 and χ_2, which are, this time, defined everywhere.

3. The main result

In this section, we prove Theorem 1. Under condition (i) (resp. (i′)), condition (ii) (resp. (ii′)) induces the following necessary condition

$$\begin{gathered}|\phi(M_2) + \sigma(\phi)\sup B_{\phi,b}| < a, \qquad |\phi(M_1) + \sigma(\phi)\inf B_{\phi,b}| < a,\\ (\text{resp. } |\phi(M_2) + \sigma(\phi)\inf B_{\phi,b}| < a, \qquad |\phi(M_1) + \sigma(\phi)\sup B_{\phi,b}| < a).\end{gathered}$$

Because of (3), we understand why we do not consider the case $b < 0$ for this theorem.

The proof of Theorem 1 is based upon the obtention of *a priori* bound for the possible solutions of problem $(\mathscr{P}_b)$, and the study of a modified problem whose solutions will be solutions of the original one.

3.1. A priori bound for the solutions of the original problem.

Lemma 3.1. *Under the assumptions of Theorem* 1, *there exists* $K > 0$ *such that any possible solution* u *of problem* $(\mathscr{P}_b)$ *with* $b > 0$ *is such that* $\|u\|_\infty < K$.

Proof. Let us begin by the case of ϕ increasing and conditions (i)–(ii). If u is a solution of problem $(\mathscr{P}_b)$, the boundary condition leads to

$$\int_0^T f\big(s, u(s)\big)\, \mathrm{d}s = B_{\phi,b}\big(u(0)\big)$$

which implies following (i) the existence of $t_0 \in [0, T]$ verifying $M_1 < u(t_0) < M_2$. Let us consider $\tau \in [0, T]$ such that $u(\tau) \notin [M_1, M_2]$. If such a τ does not exist, the proof is finished, so let us suppose that it exists. We shall consider four configurations

$$(1)\quad u(\tau) > M_2, \quad \tau < t_0, \qquad (2)\quad u(\tau) > M_2, \quad \tau > t_0,$$
$$(3)\quad u(\tau) < M_1, \quad \tau < t_0, \qquad (4)\quad u(\tau) < M_1, \quad \tau > t_0,$$

for which we shall prove

(I_1) $\phi\big(u(\tau)\big) \le \phi(M_2) + \int_{[\tau,\sigma]^c} f(t, M_2)\,\mathrm{d}t - \sup B_{\phi,b}$,

(I_2) $\phi\big(u(\tau)\big) \le \phi(M_2) + \int_{[\sigma,\tau]} f(t, M_2)\,\mathrm{d}t + B_{\phi,b}\big(u(0)\big) - \sup B_{\phi,b}$,

(I_3) $\phi\big(u(\tau)\big) \ge \phi(M_1) + \int_{[\tau,\sigma]^c} f(t, M_1)\,\mathrm{d}t - \inf B_{\phi,b}$,

(I_4) $\phi\big(u(\tau)\big) \ge \phi(M_1) + \int_{[\sigma,\tau]} f(t, M_1)\,\mathrm{d}t + B_{\phi,b}\big(u(0)\big) - \inf B_{\phi,b}$.

We only treat the case (1) (resp. (2)) because the case (3) (resp. (4)) is exactly the same. In case (1), there exists $\sigma \in [0, T]$ such that

$$\tau < \sigma < t_0, \qquad \tau \le t < \sigma \implies u(t) > M_2, \qquad u(\sigma) = M_2.$$

Let us consider the function

$$\xi : [0, T] \to \mathbb{R}, \qquad t \mapsto \begin{cases} M_2 & \text{if } t \notin [\tau, \sigma] \\ u(t) & \text{if } t \in [\tau, \sigma] \end{cases}. \tag{10}$$

Because $\xi \ge M_2$ we can define the following functions ξ_n on $[0, T]$ by

$$\xi_n : [0, T] \to \mathbb{R}, \qquad t \mapsto \begin{cases} M_2 & \text{if } 0 \le t \le \tau - \frac{1}{n}, \\ M_2 + n\big(u(\tau) - M_2\big)\big(t - \tau + \frac{1}{n}\big) & \text{if } \tau - \frac{1}{n} \le t \le \tau, \\ u(t) & \text{if } \tau \le t \le \sigma, \\ M_2 & \text{if } \sigma \le t \le T, \end{cases}$$

which verify

$$\xi_n \in C([0, T]), \qquad \xi_{n,m} \ge M_2, \qquad \int_0^T f\big(t, \xi_n(t)\big)\,\mathrm{d}t > \sup B_{\phi,b}.$$

Lebesgue's dominated convergence theorem leads to $\int_0^T f\big(t, \xi(t)\big)\,\mathrm{d}t \ge \sup B_{\phi,b}$ ensuring (I_1).

In case (2), if there exists $\tau < t_0' \le T$ such that $u(t_0') \in [M_1, M_2]$ we are in the previous case. Suppose that such t_0' does not exist. On the other hand, there exists $\sigma \in [0, T]$ verifying

$$0 < \sigma < \tau, \qquad 0 \le t < \sigma \implies u(t) > M_2, \qquad u(\sigma) = M_2, \tag{11}$$

or

$$0 = \sigma, \qquad u(0) = u(\sigma) \le M_2. \tag{12}$$

Let us consider a function ξ like previously, with M_2 and $u(t)$ are interverted in (10). We have again $\int_0^T f\big(t, \xi(t)\big)\,\mathrm{d}t \ge \sup B_{\phi,b}$, leading to (I_2) (by noting that $\phi\big(u(\sigma)\big) \le \phi(M_2)$ because ϕ is increasing).

By (ii), configurations (1) and (2) (resp. (3) and (4)) ensure

$$\phi\big(u(\tau)\big) \le K_2 := \phi(M_2) + \int_0^T f^+(t, M_2)\,\mathrm{d}t < a \tag{13}$$

$$(\text{resp. } \phi\big(u(\tau)\big) \ge K_1 := \phi(M_1) - \int_0^T f^-(t, M_1)\,\mathrm{d}t > -a) \tag{14}$$

for all τ such that $u(\tau) > M_2$ (resp. $u(\tau) < M_1$), leading to

$$-\infty < K_1' := \phi^{-1}[K_1] \le u(t) \le \phi^{-1}[K_2] =: K_2' < \infty,$$

for all $t \in [0, T]$ and hence to the thesis.

With conditions (i′) and (ii′), we have again four configurations:

$$\begin{array}{llllll} (1') & u(\tau) > M_2, & \tau > t_0, & (2') & u(\tau) > M_2, & \tau < t_0, \\ (3') & u(\tau) < M_1, & \tau > t_0, & (4') & u(\tau) < M_1, & \tau < t_0, \end{array}$$

that we can treat by the same reasoning as their corresponding ones. If ϕ is decreasing, we can put $\psi := -\phi$, $g := -f$, and consider problem $(\mathscr{P}_b)$ with ϕ, f replaced by ψ, g, whose solutions coincide with solutions of original problem $(\mathscr{P}_b)$, observe that $B_{\psi,b} = -B_{\phi,b}$ so that

$$\sup B_{\psi,b} = -\inf B_{\phi,b}, \qquad \inf B_{\psi,b} = -\sup B_{\phi,b},$$

that ψ is increasing, and use the same proof as previously with conditions (i′), (ii′) (resp. (i), (ii)) for this new problem if we worked with hypothesis (i), (ii) (resp. (i′), (ii′)) for the initial one. □

3.2. A modified problem. We now consider a modified problem where ϕ is replaced by a homeomorphism $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ which coincides with ϕ outside an open interval containing $(-K, K)$. Namely, define

$$\tilde{\phi} : \mathbb{R} \to \mathbb{R} : u \to \begin{cases} -\sigma(\phi)\kappa \ln(-u - A + 1) + \phi(-A) & \text{if } u \le -A \\ \phi(u) & \text{if } -A < u < A \\ \sigma(\phi)\kappa \ln(u - A + 1) + \phi(A) & \text{if } A \le u \end{cases}$$

with $A > K$, K an *a priori* bound of the solutions of $(\mathscr{P}_b)$, and κ a positive constant that we shall fix later. $\tilde{\phi}$ is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. We can study problem

$$\big(\tilde{\phi}(u)\big)' = f(t,u), \qquad u(T) = bu(0), \tag{$\tilde{\mathscr{P}}_b$}$$

by searching the fixed points of the operator

$$\tilde{\chi}_2 : C([0,T]) \to C([0,T]),$$
$$u \mapsto QN_f(u) - T^{-1}B_{\tilde{\phi},b}(Pu) + \tilde{\phi}^{-1}\big(\tilde{\phi}(Pu) + H(id - Q)N_f(u) + T^{-1}\mathbf{t}B_{\tilde{\phi},b}(Pu)\big),$$

the proof of this fact being a consequence of Remark 2.4. This operator is defined on all $C([0,T])$.

Lemma 3.2. *Under the assumptions of Theorem* 1, *and if κ is small enough, the solutions of problem $(\tilde{\mathscr{P}}_b)$ with $b > 0$ take values in $[-K,K]$ with K defined above.*

Proof. We will suppose in this proof that ϕ is increasing and that $b \in (0,1)$; the proof is similar in the other cases. If we want to apply the same idea than in the proof of Lemma 3.1, we must have

$$\operatorname{im} B_{\tilde{\phi},b} \subseteq \operatorname{im} B_{\phi,b}. \tag{15}$$

Inasmuch as

$$B_{\tilde{\phi},b}(u) = \tilde{\phi}(bu) - \tilde{\phi}(u), \qquad B_{\phi,b}(u) = \phi(bu) - \phi(u)$$

and as $\tilde{\phi}$ coincide with ϕ on $(-A,A)$, we have

$$|B_{\tilde{\phi},b}(u)| = \begin{cases} \kappa\left|\ln\left(\frac{b|u|-A+1}{|u|-A+1}\right)\right| & \text{if } |u| \geq \frac{A}{b} \\ \kappa\ln(u-A+1) + \phi(A) - \phi(bu) & \text{if } A \leq u < \frac{A}{b} \\ -[-\kappa\ln(-u-A+1) + \phi(-A) - \phi(bu)] & \text{if } -\frac{A}{b} < u \leq -A \\ |B_{\phi,b}(u)| & \text{if } |u| < A \end{cases}.$$

So, if κ is chosen small enough so that

$$\kappa\ln(u-A+1) + \phi(A) \leq \phi(u) \qquad \text{if } A \leq u < \frac{A}{b}$$

$$-\kappa\ln(-u+A+1) + \phi(-A) \geq \phi(u) \qquad \text{if } -\frac{A}{b} < u \leq -A,$$

which is possible, we have

$$|B_{\tilde{\phi},b}(u)| \le \begin{cases} \kappa B & \text{if } |u| \ge \frac{A}{b} \\ |B_{\phi,b}(u)| & \text{if } |u| < \frac{A}{b} \end{cases}$$

where

$$B := \sup\left\{ \left| \ln\left(\frac{b|u| - A + 1}{|u| - A + 1} \right) \right| : |u| \ge \frac{A}{b} \right\} < +\infty,$$

and so we obtain (15) if we still reduce κ so that $[-\kappa B, \kappa B] \subseteq \operatorname{im} B_{\phi,b}$. $\square$

Corollary 3.3. *Let $b > 0$ and $u \in C([0,T])$. Then u is a solution of problem $(\mathscr{P}_b)$ if and only if u is a solution of problem $(\tilde{\mathscr{P}}_b)$.*

3.3. Imbedding into a family of problems. Let us consider the family of problems

$$(\tilde{\phi}(u))' = \lambda f(t,u) + (1-\lambda)T^{-1}\int_0^T f(s,u(s))\,\mathrm{d}s, \qquad u(T) = bu(0), \qquad (\tilde{\mathscr{P}}_{b,\lambda})$$

corresponding to the search of the couples (λ, u) such that $u = \zeta(\lambda, u)$ with ζ defined by

$$\begin{aligned} &\zeta : [0,1] \times C([0,T]) \to C([0,T]),\ (\lambda, u) \\ &\mapsto QN_f(u) - T^{-1}B_{\tilde{\phi},b}(Pu) + \tilde{\phi}^{-1}\big(\tilde{\phi}(Pu) + \lambda H(id - Q)N_f(u) + T^{-1}\mathbf{t}B_{\tilde{\phi},b}(Pu)\big). \end{aligned}$$

Lemma 3.4. *Under the assumptions of Theorem* 1, *there exists an* a priori *bound independent of $\lambda \in [0,1]$ for the solutions of problem $(\tilde{\mathscr{P}}_{b,\lambda})$.*

Proof. The proof of this result is very close to the proof of Lemma 3.1. We shall just work in the case of ϕ increasing and with assumptions (i) and (ii).

Let us begin by the case $\lambda \ne 0$. If u is a solution of problem $(\tilde{\mathscr{P}}_{b,\lambda})$, the boundary conditions lead to

$$\int_0^T f(s,u(s))\,\mathrm{d}s = B_{\tilde{\phi},b}(u(0))$$

which implies by (i) the existence of $t_0 \in [0,T]$ verifying $M_1 < u(t_0) < M_2$. Let us consider $\tau \in [0,T]$ such that $u(\tau) \notin [M_1, M_2]$. If such a τ does not exist, the proof is finished, so let us suppose that it exists. We have again the configurations (1)–(4) mentioned in Lemma 3.1 for which we can prove

$$(I_1) \quad \tilde{\phi}\big(u(\tau)\big) \le \tilde{\phi}(M_2) + \lambda \int_{[0,\tau]\cup[\sigma,T]} f(t, M_2)\,\mathrm{d}t - (1-\lambda)B_{\tilde{\phi},b}\big(u(0)\big)(\sigma-\tau)T^{-1} - \lambda \sup B_{\tilde{\phi},b},$$

$$(I_2) \quad \tilde{\phi}\big(u(\tau)\big) \le \tilde{\phi}(M_2) + \lambda \int_{[\sigma,\tau]} f(t, M_2)\,\mathrm{d}t + B_{\tilde{\phi},b}\big(u(0)\big) - (1-\lambda)B_{\tilde{\phi},b}\big(u(0)\big)(\sigma+T-\tau)T^{-1} - \lambda \sup B_{\tilde{\phi},b},$$

$$(I_3) \quad \tilde{\phi}\big(u(\tau)\big) \ge \tilde{\phi}(M_1) + \lambda \int_{[0,\tau]\cup[\sigma,T]} f(t, M_1)\,\mathrm{d}t - (1-\lambda)B_{\tilde{\phi},b}\big(u(0)\big)(\sigma-\tau)T^{-1} - \lambda \inf B_{\tilde{\phi},b},$$

$$(I_4) \quad \tilde{\phi}\big(u(\tau)\big) \ge \tilde{\phi}(M_1) + \lambda \int_{[\sigma,\tau]} f(t, M_1)\,\mathrm{d}t + B_{\tilde{\phi},b}\big(u(0)\big) - (1-\lambda)B_{\tilde{\phi},b}\big(u(0)\big)(\sigma+T-\tau)T^{-1} - \lambda \inf B_{\tilde{\phi},b}.$$

We only consider the case (2), the other ones being similar or simpler. Like in Lemma 3.1, let us suppose that there does not exist $\tau < t_0' \le T$ such that $u(t_0') \in [M_1, M_2]$, observe that there exists $\sigma \in [0, T]$ such that (11) or (12) holds, and consider the same function ξ which verifies again

$$\int_0^T f\big(t, \xi(t)\big)\,\mathrm{d}t \ge \sup B_{\tilde{\phi},b}.$$

By definition of ξ and because $\tilde{\phi}(u\sigma)) \le \tilde{\phi}(M_2)$ (as ϕ is increasing), we deduce (I_2).

Configurations (1) and (2) (resp. (3) and (4)) ensure

$$\tilde{\phi}\big(u(\tau)\big) \le \tilde{\phi}(M_2) + \int_0^T f^+(t, M_2)\,\mathrm{d}t + \sup B_{\tilde{\phi},b} + |\inf B_{\tilde{\phi},b}| =: \tilde{K}_2 \tag{16}$$

$$(\text{resp. } \tilde{\phi}\big(u(\tau)\big) \ge \tilde{\phi}(M_1) - \int_0^T f^-(t, M_1)\,\mathrm{d}t - \sup B_{\tilde{\phi},b} - |\inf B_{\tilde{\phi},b}| =: \tilde{K}_1) \tag{17}$$

for all τ such that $u(\tau) > M_2$ (resp. $u(\tau) < M_1$), leading to

$$-\infty < \tilde{K}_1' := \tilde{\phi}^{-1}[\tilde{K}_1] \le u(t) \le \tilde{\phi}^{-1}[\tilde{K}_2] =: \tilde{K}_2' < \infty.$$

for all $t \in [0, T]$, and hence to the thesis.

For the case $\lambda = 0$, because we have

$$\big(\tilde{\phi}(u)\big)' = T^{-1} \int_0^T f\big(s, u(s)\big)\,\mathrm{d}s = T^{-1} B_{\tilde{\phi},b}\big(u(0)\big) =: c \in (-aT^{-1}, aT^{-1}),$$

and because there exists $t_0 \in [0, T]$ such that $M_1 < u(t_0) < M_2$, we can write

$$\tilde{\phi}\big(u(t)\big) = \tilde{\phi}\big(u(t_0)\big) + c(t - t_0)$$

and hence deduce

$$\tilde{\phi}(M_1) - a < \tilde{\phi}\big(u(t)\big) < \tilde{\phi}(M_2) + a.$$

for all $t \in [0, T]$, ensuring the thesis. □

3.4. Existence result for modified problem. In order to prove that problem $(\tilde{\mathscr{P}}_b)$ has at least one solution, we compute $\deg_{\mathrm{LS}}[id - \tilde{\chi}_2, B(0, r)]$, where r is strictly greater than the *a priori* bound of the last lemma. By noting that $\tilde{\chi}_2 = \zeta(1, \cdot)$ and that

$$\deg_{\mathrm{LS}}\big[id - \zeta(1, \cdot), B(0, r)\big) = \deg_{\mathrm{LS}}\big(id - \zeta(0, \cdot), B(0, r)\big],$$

we have to prove that $\deg_{\mathrm{LS}}[id - \zeta(0, \cdot), B(0, r)] \neq 0$. In order to follow this way, we need some compactness of ζ, a very classical fact that we will just recall without proof.

Lemma 3.5. *The operator* $\zeta : [0, 1] \times C([0, T]) \to C([0, T])$ *is completely continuous.*

Notice that the fixed points of $\zeta(0, \cdot)$ are the solutions of

$$\big(\tilde{\phi}(u)\big)' = T^{-1} \int_0^T f\big(s, u(s)\big)\, \mathrm{d}s, \qquad u(T) = bu(0). \tag{18}$$

To show that $\deg_{\mathrm{LS}}[id - \zeta(0, \cdot), B(0, r)] = \pm 1$ is easy in the periodic case $b = 1$, because then, a solution u of (18) must be constant and the reduction property of Leray-Schauder degree (see e.g. [14]) relates $\deg_{\mathrm{LS}}[id - \zeta(0, \cdot), B(0, r)]$ to the easy to compute Brouwer degree of some function of one variable. In the general case, we shall introduce a new homotopy:

$$\big(\tilde{\phi}(u)\big)' = \lambda T^{-1} \int_0^T f\big(s, u(s)\big)\, \mathrm{d}s, \qquad \int_0^T f\big(t, u(t)\big)\, \mathrm{d}t = \tilde{\phi}\big(bu(0)\big) - \tilde{\phi}\big(u(0)\big). \tag{19}$$

Lemma 3.6. *Under the assumptions* (i) (*resp.* (i′)) *and* (ii) (*resp.* (ii′)) *of Theorem* 1 *we have*

$$\deg_{\mathrm{LS}}[id - \zeta(0, \cdot), B(0, r)] \neq 0.$$

Proof. Let us consider the following homotopy:

$$\xi : [0,1] \times C([0,T]) \to C([0,T]) :$$
$$(\lambda, u) \mapsto QN_f(u) - T^{-1}B_{\tilde{\phi},b}(Pu) + \tilde{\phi}^{-1}\big(\tilde{\phi}(Pu) + \lambda T^{-1}\mathbf{t}B_{\tilde{\phi},b}(Pu)\big),$$

where $\xi(1,\cdot)$ coincide with the operator $\zeta(0,\cdot)$. If u is a fixed point of the operator $\xi(\lambda,\cdot)$, by evaluation of u at 0, we have $QN_f(u) = T^{-1}B_{\tilde{\phi},b}(Pu)$ and we can then easily obtain (19). To deduce the existence of a $r > 0$ such that $\|u\|_\infty < r$, we proceed like in the proof of Lemma 3.4 (in the case $\lambda = 0$).

Using homotopy invariance and reduction property of Leray-Schauder degree, we obtain

$$\begin{aligned}
\deg_{\mathrm{LS}}[id - \xi(1,\cdot), B(0,r)] &= \deg_{\mathrm{LS}}[id - \xi(0,\cdot), B(0,r)] \\
&= \deg_{\mathrm{LS}}[id - QN_f + T^{-1}B_{\tilde{\phi},b}P - P, B(0,r)] \\
&= \deg_{\mathrm{B}}[T^{-1}B_{\tilde{\phi},b} - QN_f, (-r,r)] \\
&= \deg_{\mathrm{B}}\Big[T^{-1}\Big[B_{\tilde{\phi},b} - \int_0^T f(s,.)\,\mathrm{d}s\Big], (-r,r)\Big] = \pm 1,
\end{aligned}$$

where the last equality comes from the hypothesis (ii) and the fact that

$$-r < -\tilde{K}' < M_1 < M_2 < \tilde{K}' < r$$

where $\tilde{K}'$ is a bound of the solutions of $(\tilde{\mathscr{P}}_{b,\lambda})$. □

4. The proof of Theorems 2 and 3

We now prove Theorems 2 and 3, using an less sophisticated extension of ϕ than the one used in the previous section:

$$\tilde{\phi} : \mathbb{R} \to \mathbb{R}, \qquad u \mapsto \begin{cases} u + A + \phi(-A) & \text{if } u \le -A, \\ \phi(u) & \text{if } -A < u < A, \\ u - A + \phi(A) & \text{if } A \le u, \end{cases}$$

if ϕ is increasing and

$$\tilde{\phi} : \mathbb{R} \to \mathbb{R}, \qquad u \mapsto \begin{cases} -u - A + \phi(-A) & \text{if } u \le -A, \\ \phi(u) & \text{if } -A < u < A, \\ -u + A + \phi(A) & \text{if } A \le u, \end{cases}$$

if ϕ is decreasing, with $A > 0$ fixed.

4.1. The proof of Theorem 2. Under the hypothesis of the theorem, the operator χ_1 defined in Section 2 is bounded. Indeed, if $v = \chi_1(u)$ then $v(T) = bv(0)$ and so v has at least one zero t_0. We have therefore

$$v = \chi_1(u) \implies \big(\phi(v(t))\big)' = f(t,u)$$
$$\implies \phi\big(v(t)\big) = \phi\big(v(t_0)\big) + \int_{t_0}^{t} f\big(s,u(s)\big)\,\mathrm{d}s = \int_{t_0}^{t} f\big(s,u(s)\big)\,\mathrm{d}s$$

because $\phi(0) = 0$, and hence

$$\big|\phi\big(v(t)\big)\big| = \Big|\int_{t_0}^{t} f\big(s,u(s)\big)\,\mathrm{d}s\Big| \le \int_{t_0}^{t} \big|f\big(s,u(s)\big)\big|\,\mathrm{d}s \le \|c\|_1 < a$$

which implies

$$\|v\|_\infty \le \max\{|\phi^{-1}[\pm\|c\|_1]|\} := K < \infty.$$

Let us consider now the extension $\tilde{\phi}$, explained at the beginning of the section, with $A > K$. By the same reasoning as above, the operator $\tilde{\chi}_1$ (see Remark 2.4) is bounded with the same bound. This boundedness implies an *a priori* bound for fixed points of this operator, fixed points which coincide with the ones of χ_1. Because the operator $\tilde{\chi}_1$ is completely continuous and bounded, we can use Schauder's fixed point theorem to deduce the existence of at least one fixed point in $B[0,K]$. The proof is complete.

We can improve the bound of f in the previous theorem by a factor 2 by supposing that ϕ is odd and $b = -1$.

Theorem 4.1. *If ϕ is odd and if there exists $c \in L^1([0,T],\mathbb{R}^+)$ such that $\|c\|_1 < 2a$ and*

$$\forall t \in [0,T],\ \forall u \in \mathbb{R} : |f(t,u)| \le c(t),$$

then problem $(\mathscr{P}_b)$ with $b = -1$ has at least one solution.

Proof. Let us remark that equations (6), (7) and (8) of Example 2.1, jointed to the hypothesis of the theorem, imply that the operator χ_1 is defined on all $C([0,T])$ by

$$\chi_1(u) = \phi^{-1}\Big(\int_0^T G(s,.)f\big(s,u(s)\big)\,\mathrm{d}s\Big)$$

(with the function G defined in Example 2.1). χ_1 is again bounded because if $u \in C([0,T])$ then

$$\left|\int_0^T G(s,t) f\big(s,u(s)\big)\,\mathrm{d}s\right| \leq \int_0^T \left|G(s,t) f\big(s,u(s)\big)\right|\mathrm{d}s \leq \frac{\|c\|_1}{2} < a$$

and so

$$\|\chi_1(u)\|_\infty = \left\|\phi^{-1}\Big(\int_0^T G(s,.) f\big(s,u(s)\big)\,ds\Big)\right\|_\infty \leq |\phi^{-1}(\|c\|_1/2)| := K.$$

We can then conclude like in Theorem 2 without use an extension of ϕ. □

Remark 4.2. If ϕ is a diffeomorphism, we can rewrite problem $(\mathscr{P}_b)$ in the form

$$u' = f(t,u)\cdot\big(\phi'(u)\big)^{-1}, \qquad u(T) = bu(0),$$

and give an analog result as the two previous one, with the hypothesis:

$$\exists c \in L^1([0,T],\mathbb{R}^+),\ \forall t \in [0,T] : \left|\frac{f(t,u)}{\phi'(u)}\right| \leq c(t).$$

Because $\lim_{u\to\infty} \phi'(u) = 0$ this condition is restrictive; in fact, it is as restrictive than the conditions $\|c\|_1 < a$ or $\|c\|_1 < 2a$ of our theorems.

4.2. The proof of Theorem 3. In the periodic case (b_3) we consider the family of problems

$$\big(\phi(u)\big)' = \lambda f(t,u) + (1-\lambda)\int_0^T f\big(s,u(s)\big)\,ds, \qquad \lambda \in [0,1], \qquad (\mathscr{P}_{b,\lambda})$$

and in all the other cases the family of problems

$$\big(\phi(u)\big)' = \lambda f(t,u), \qquad u(T) = bu(0), \qquad \lambda \in [0,1]. \qquad (\mathscr{P}'_{b,\lambda})$$

It is not difficult to see that if u is a solution of one of this problem taking its values in $[-M,M]$ (resp. $[M_1,M_2]$), then u takes in fact its values in $(-M,M)$ (resp. (M_1,M_2)) in the cases (a_i) (resp. in the cases (b_i)). Indeed, if it is not the case, u takes its values in $[-M,M]$ (resp. $[M_1,M_2]$) and has at least one value, say $u(t_0)$, on $\partial[-M,M]$ (resp. $\partial[M_1,M_2]$), say $-M$ (resp. M_1). Let us remark that in the periodic case, $\int_0^T f\big(s,u(s)\big)\,ds = 0$ and so u verifies also the equation $\big(\phi(u)\big)' = \lambda f(t,u)$ on $[0,T]$. Let us begin with the case $\lambda \neq 0$. If $t_0 \in (0,T)$, then, by monotonicity of ϕ, we have

$$f(t_0,-M) = \lambda^{-1}\big(\phi(u)\big)'(t_0) = 0, \qquad (\text{resp. } f(t_0,M_1) = \lambda^{-1}\big(\phi(u)\big)'(t_0) = 0),$$

which is a contradiction with the hypothesis. If $t_0 \in \partial[0,T]$, then we have to consider problems separately. We will just examine the anti-periodic case (a_3) and the periodic case (b_3), the other ones being similar. In the (b_3) case, t_0 is 0 and T and so

$$\begin{aligned}\sigma(\phi)\cdot f(T,M_1) &= \lambda^{-1}\sigma(\phi)\big(\phi(u)\big)'(T) \le 0 \le \lambda^{-1}\sigma(\phi)\big(\phi(u)\big)'(0)\\ &= \sigma(\phi)\cdot f(0,M_1)\end{aligned}$$

which is a contradiction with the assumption; in the case (a_3), say $t_0 = 0$, then, by boundary conditions, we have

$$\begin{aligned}\sigma(\phi)\cdot f(0,-M) &= \lambda^{-1}\sigma(\phi)\big(\phi(u)\big)'(0) \ge 0, \sigma(\phi)\cdot f(T,M)\\ &= \lambda^{-1}\sigma(\phi)\big(\phi(u)\big)'(T) \ge 0,\end{aligned}$$

which is again a contradiction with the assumption. For the case $\lambda = 0$, u is the constant $-M$ (resp. M_1) which is a contradiction to the boundary conditions in all cases except the periodic one, but in this case we have $\int_0^T f(s,M_1)\,ds = 0$ which is a contradiction with the assumption.

We can work with the previous extension of ϕ where A is chosen such that $[-A,A]$ contains strictly $[-M,M]$ (resp. $[M_1,M_2]$), and with the correspondent problems $(\tilde{\mathscr{P}}_{b,\lambda})$ and $(\tilde{\mathscr{P}}'_{b,\lambda})$ whose solutions with values in $[-M,M]$ (resp. $[M_1,M_2]$) take in fact their values in $(-M,M)$ (resp. (M_1,M_2)) for the same reason as above, and whose solutions are also solutions of Problems $(\mathscr{P}_{b,\lambda})$ or $(\mathscr{P}'_{b,\lambda})$ by definition of $\tilde{\phi}$.

We can then consider the Leray-Schauder degree of $id - \zeta(\lambda,\cdot)$ on Ω where ζ is defined by

$$\begin{aligned}&\zeta : [0,1]\times C([0,T]) \to C([0,T]),\\ &u \mapsto QN_f(u) + \tilde{\phi}^{-1}\big(\tilde{\phi}(Pu) + \lambda H(id-Q)N_f(u)\big)\end{aligned}$$

in the case (b_3), and by

$$\begin{gathered}\zeta : [0,1]\times C([0,T]) \to C([0,T]),\\ u \mapsto \lambda QN_f(u) - T^{-1}B_{\phi,b}(Pu) + \tilde{\phi}^{-1}[\tilde{\phi}(Pu) + \lambda H(id-Q)N_f(u) + T^{-1}\mathbf{t}B_{\tilde{\phi},b}(Pu)]\end{gathered}$$

in the other cases, and where Ω is the following open set:

$$\begin{aligned}&\Omega = \{u \in C([0,T]) : -M < u < M\}\\ &(\text{resp. } \Omega = \{u \in C([0,T]) : M_1 < u < M_2\}).\end{aligned}$$

By the fact that no solution of those problems belongs to $\partial\Omega$, we know that

$$\deg_{\mathrm{LS}}[id - \zeta(1,\cdot),\Omega] = \deg_{\mathrm{LS}}[id - \zeta(\cdot,0),\Omega].$$

In the (b_3) case, we conclude the proof by noticing that

$$\begin{aligned}\deg_{\mathrm{LS}}[id - \zeta(0,\cdot),\Omega] &= \deg_{\mathrm{LS}}[id - QN_f - P,\Omega] \\ &= \deg_{\mathrm{B}}\Big[-T^{-1}\int_0^T f(s,\cdot)\,ds,(M_1,M_2)\Big] = \pm 1,\end{aligned}$$

where the last equality comes from the hypothesis (B_3). In the other cases, for which we have

$$\zeta(0,\cdot) = -T^{-1}B_{\tilde{\phi},b}P + \tilde{\phi}^{-1}(\tilde{\phi}P + T^{-1}\mathbf{t}B_{\tilde{\phi},b}P),$$

we can consider the homotopy

$$\begin{aligned}&\xi : [0,1]\times C([0,T]) \to C([0,T]),\\ &(\lambda,u) \mapsto -T^{-1}B_{\tilde{\phi},b}(Pu) + \tilde{\phi}^{-1}\big(\tilde{\phi}(Pu) + \lambda T^{-1}\mathbf{t}B_{\tilde{\phi},b}(Pu)\big),\end{aligned}$$

where $\xi(1,.) = \zeta(0,.)$. It is easy to see that if u is such that $\xi(\lambda,u) = u$ then $u = 0 \notin \partial\Omega$. Hence, we can conclude that the Leray-Schauder degree of $id - \zeta(0,.)$ is equal to

$$\begin{aligned}\deg_{\mathrm{LS}}\big[id - \xi(1,.),\Omega\big) &= \deg_{\mathrm{LS}}\big(id - \xi(0,.),\Omega\big] \\ &= \deg_{\mathrm{LS}}[id + T^{-1}B_{\tilde{\phi},b}P - P,\Omega] \\ &= \deg_{\mathrm{B}}[T^{-1}B_{\tilde{\phi},b},(-M,M)\ (\text{resp. } (M_1,M_2))] = \pm 1,\end{aligned}$$

where the last equality comes from the study of $B_{\tilde{\phi},b}$ done in Section 2, and the fact that $-M < 0 < M$ (resp. $M_1 < 0 < M_2$). The proof is complete.

Remark 4.3. If ϕ is a diffeomorphism, we can rewrite Problems $(\mathscr{P}_{b,\lambda})$ and $(\mathscr{P}'_{b,\lambda})$ in the form

$$u' = \lambda f(t,u)\cdot\big(\phi'(u)\big)^{-1} =: \lambda g(t,u), \qquad u(T) = bu(0),$$

and notice that, because of the sign of ϕ', the hypothesis (A_i) or (B_i) are still true for g.

5. Examples and numerical experience

5.1. Application of Theorem 1. In the following example, we suppose $b > 0$ in order to apply Theorem 1.

Example 5.1. Let us consider the problem

$$\left(\frac{u}{\sqrt{1+u^2}}\right)' = h(t) + e^u, \qquad u(T) = bu(0). \tag{20}$$

Because ϕ is odd, it is also the case for $B_{\phi,b}$ and hence we can set

$$\alpha = \sup B_{\phi,b} = -\inf B_{\phi,b}.$$

As $b > 0$ we know that $\alpha < 1$. If we suppose that $\int_0^T h = T\bar{h} < -\alpha$, then

$$\int_0^T \left(h(t) + e^{M_1}\right) = T\bar{h} + Te^{M_1} < -\alpha, \qquad \int_0^T \left(h(t) + e^{M_2}\right) = T\bar{h} + Te^{M_2} > \alpha,$$

for

$$M_1 = \log(-\alpha T^{-1} - \bar{h}) - \varepsilon, \qquad M_2 = \log(\alpha T^{-1} - \bar{h}) + \varepsilon,$$

with any $\varepsilon > 0$, enuring hypothesis (i). Hence, if we suppose

$$\left|\phi\left(\log(\pm\alpha T^{-1} - \bar{h})\right) \pm \int_0^T [h(t) + (\pm\alpha T^{-1} - \bar{h})]^{\pm}\right| < 1, \tag{21}$$

we have assumption (ii) and the existence of at least one solution of (20).

The following considerations show that the conditions of Theorem 1 are not essential.

Example 5.2. Let us consider the periodic case ($b = 1$) of problem (20) (so $\alpha = 0$) with $T = 1$ and h given by $h_{p,q}(t) = p + q|2t - 1|$. By a simple computation we have $\bar{h}_{p,q} = \int_0^1 h_{p,q} = p + \frac{q}{2}$. Let us suppose for example that $\bar{h}_{p,q} = -1$ what is equivalent to $q = -2p - 2$, and study for which values of p condition (21)

$$\int_0^1 [h_{p,q}(t) + 1]^{\pm} < 1$$

is verified. The sign of the function $h_{p,q} + 1$ is given by the following table when $p \neq -1$,

	0		$\frac{1}{4}$		$\frac{1}{2}$		$\frac{3}{4}$		1
$p > -1$	+	+	0	−	−	−	0	+	+
$p < -1$	−	−	0	+	+	+	0	−	−

and $h_{p,q} + 1 \equiv 0$ when $p = -1$. Hence,

$$\int_0^1 [h_{p,q}(t) + 1]^{\pm}\, dt = \frac{|p+1|}{4}.$$

So we can apply Theorem 1 for $-5 < p < 3$. Let us note that if $p \in (-2, 0)$, the function $h_{p,q}$ is always negative and we can then apply Theorem 3. But in the other cases, the function $h_{p,q}$ changes its sign and we can only use Theorem 1.

Let us perform a numerical study in order to compare with theoretical results. To do that, we shall work with the form mentioned at the end of the Introduction:

$$v'(t) = h_{p,q}(t) + e^{v/\sqrt{1-v^2}},$$

where $v = \phi(u)$, and $v(0) = v(T)$ for the boundary conditions. Because the non-linearity of this equation is C^1, we can apply Cauchy's existence and uniqueness result. This leads us to work with the functions

$$V : (t,s) \mapsto V(t,s), \qquad P : s \mapsto V(1,s),$$

respectively the *flow function* which associates to (t,s) the value at t of the solution of the equation with the initial condition $v(0) = s$, and the *Poincaré function*. We can note also that the solutions cannot cross, which infers to the set of solutions v a simple structure because we are in dimension 1.

Let us begin by study the *global solutions* of the equation, i.e., solutions which live on all $[0,1]$. It is easy to see that such solutions live above (resp. below) a function α (resp. β) with values in $[-1,1]$, verifying the equation at the values of t where $\alpha(t)$ (resp. $\beta(t)$) is in $(-1,1)$ and admitting at least one value of t such that $\alpha(t)$ (resp. $\beta(t)$) is in $\{\pm 1\}$. Figure 1, obtained by Routine A.1 exposed in Appendix A, gives for each p the values of α and β on $\partial[0,1]$.

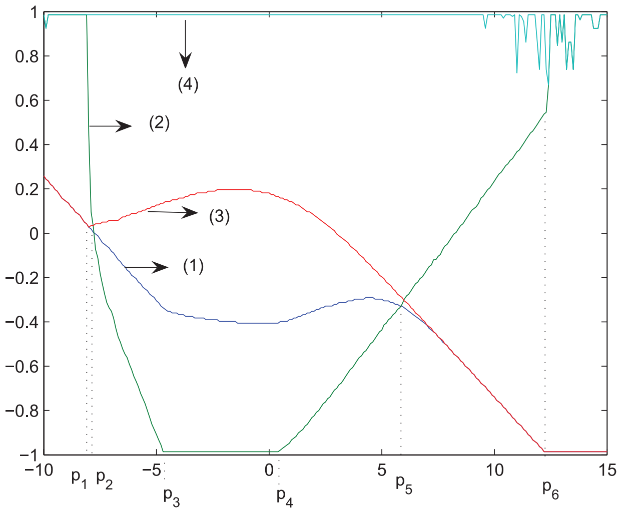

Figure 1. (1) $\alpha(0)$, (2) $\alpha(1)$, (3) $\beta(0)$, (4) $\beta(1)$.

p_1	p_2	p_3	p_4	p_5	p_6
−7.98	−7.83	−4.7	0.45	5.85	12.34

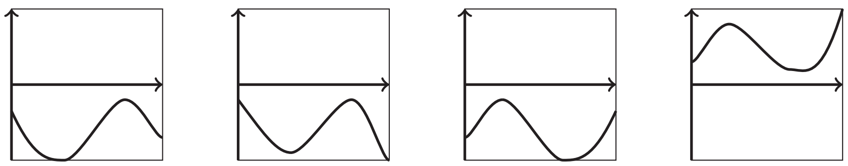

Figure 2. Pictures of α and β

We see on this figure that there exists reals p_1 and p_6 such that for all $p < p_1$ or $p > p_6$, configurations of $\alpha(0)$, $\beta(0)$, $\alpha(1)$, $\beta(1)$ imply the non existence of global solutions. Let us consider the 3 cases

$$(a)\;\; p \in (p_1, p_3), \qquad (b)\;\; p \in (p_3, p_4), \qquad (c)\;\; p \in (p_4, p_6),$$

for which the graph of α is given by Figure 2. Indeed, in the case (a), the graph of α reaches -1 in $(0, 1)$ at the first root r_1 of $h_{p,q}$, this is explained by the equation whose the right member is $h_{p,q}(r_1) + 0$ and by the sign of $h_{p,q}$. In the case (b), the graph of α reaches -1 at $t = 1$. Finally, in the case (c), the graph of α reaches -1

at the second root r_2 of $h_{p,q}$, this is explained like in the case (a). In all cases, the graph of β is given by the figure 2 (d) (lets us note that the equation ensures that $\beta(1) = 1$, so the right part of the curve $\beta(1)$ on Figure 1 comes from numerical errors) up to the convexity of the beginning of the graph which depends of p. By observing the evolution following p of the graphs of α and β, we see how the global solutions disappear out of (p_1, p_6): by superposition of α and β. Let us note that by using condition (2) with t_1 and t_2 given by r_1 and r_2 for $p < -2$ et by 0 and r_1 or r_2 and 1 for $p > 0$ (i.e. working in a connected part of the domain of $h^+_{p,q}$), we are theoretically sure that global solutions do not exist out of

$$(-6 - 2\sqrt{6}, 8 + 8\sqrt{2}) \simeq (-10.89, 19.31).$$

If $s_1 \leq s_2$, we know that for each $t \in [0,1]$ we have

$$h_{p,q}(t) + e^{s_1} \leq h_{p,q}(t) + e^{s_2},$$

hence we deduce easily that the Poincaré function P is increasing, and even that the function $P - id$ giving at s the increase of the solution $V(\cdot, s)$, is increasing. Hence, for the existence of a periodic solution, we need to have the following configuration:

$$\alpha(1) - \alpha(0) < 0, \qquad \beta(1) - \beta(0) > 0.$$

The second condition being always true, Figure 1 ensures that there exists one and only one periodic solution for $p \in (p_2, p_5)$. The theoretically sufficient bounds for p was -5 and 3, and the observed numerically bounds are -7.83 and 5.85.

The graph of the periodic solution v_p of our equation is given at Figure 3 for some values of p in the three cases

$$\text{(i)}\ p < -1, \qquad \text{(ii)}\ p = -1, \qquad \text{(iii)}\ p > -1.$$

The case (i) (resp. (iii)) concerns functions with the same behavior as Figure 2(a) (resp. (c)), and the case (ii) is the constant solution 0. Those graphs are obtained by Routine A.3 given in Appendix A. Let us try to explain the shape of the graph of v_p with the equation. Let us discuss what happens for $p > 0$. The shape of $h_{p,q}$ is given by

0		$\frac{1}{4}+\frac{1}{2q}$		$\frac{1}{4}$		$\frac{1}{2}$		$\frac{3}{4}$		$\frac{3}{4}-\frac{1}{2q}$		1
	$+$	0	$-$	-1	$-$	min	$-$	-1	$+$	0	$+$	

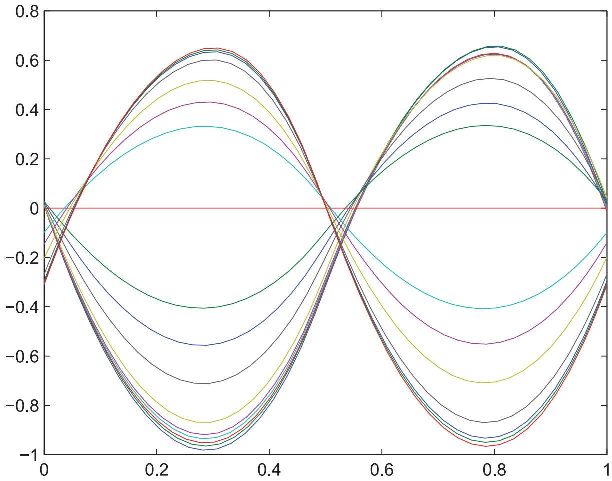

Figure 3. In the case (a), $-7.7 \leq p \leq -7.3$ by step of 0.1 and $-7 \leq p \leq -4$ by step of 1, and in the case (c), $2 \leq p \leq 5$ by step of 1 and $5.4 \leq p \leq 5.7$ by step of 0.1

When $t = r_1$, the equation ensures that $v_p' > 0$. This is to compare with the fact that the numerical experiments seem to ensure that v_p reaches its maximum after $\frac{1}{4}$ which is closer and closer to r_1 when p increases. When $t = \frac{1}{2}$, the function v_p seems to be close to 0 which, combined with equation, ensures that

$$v_p' \simeq e^0 + p + q = -p - 1.$$

Hence, when p increases, we see that v_p' is more and more negative. When $t = \frac{3}{4}$, $v_p < 0$ which, combined with the equation, ensures that $v_p' < 0$. When $t = r_2$, the equation ensures that $v_p' > 0$. So, v_p reaches its minimum between $\frac{3}{4}$ and r_2 which are closer and closer when p increases. When $t > r_2$, the equation ensures that $v_p' > 0$ which corresponds to Figure 3.

This description is not applicable for $-2 < p < 0$ because for those values of p, the function $h_{p,q}$ is always negative. For $p = -1$, the function v_p is identically 0, what is consistent with the equation. For $p < -2$, Figure 3 shows a similar behavior for v_p but with an inversion of the t where v_p reaches its minimum and maximum, what is clear with the equation because of the change of the sign-behavior of $h_{p,q}$. But despite this kind of differences, we can do the same type of description for v_p.

A discussion about the possibly numerical errors for the graphs of v_p can be found at the end of Appendix A.

5.2. Applications of Theorems 2 and 4.1. Let us begin with an application of Theorems 2, 3 and 4.1 for problem $(\mathscr{P}_b)$ with $b < 0$.

Example 5.3. Let us consider the problem

$$\left(\frac{-\gamma_1 u^- + \gamma_2 u^+}{1 + \gamma_1 u^- + \gamma_2 u^+}\right)' = \alpha \cdot \frac{\sin(\beta t + u)}{\sqrt{t} + e^{-|u|}}, \qquad u(T) = bu(0), \tag{22}$$

where $\alpha, \beta \in \mathbb{R}$ and $\gamma_1, \gamma_2 > 0$. Because

$$|f(t,u)| = \left|\alpha \cdot \frac{\sin(\beta t + u)}{\sqrt{t} + e^{-|u|}}\right| \leq \frac{|\alpha|}{\sqrt{t} + e^{-|u|}} \leq \frac{|\alpha|}{\sqrt{t}},$$

Theorem 2 ensures that if

$$\left\|\frac{|\alpha|}{\sqrt{t}}\right\|_1 = 2|\alpha|\sqrt{T} < 1$$

then problem (22) has at least one solution. If moreover $\gamma_1 = \gamma_2$ (ϕ odd) and $b = -1$ (antiperiodic problem), Theorem 4.1 ensures that condition $|\alpha|\sqrt{T} < 1$ suffices for the existence of at least one solution of Problem (22).

Let us give now an application of Theorem 3 for problem $(\mathscr{P}_b)$ with $b > 0$ or $b < 0$.

Example 5.4. Let us consider the problem

$$\left(\frac{u}{\sqrt{1+u^2}}\right)' = h(t) + e^{|u|}\sin(u), \qquad u(T) = bu(0), \tag{23}$$

where $h \in C([0,T])$. We can choose $k > 0$ large enough so that for all $t \in [0,T]$,

$$h(t) + e^{|-M|}\sin(-M) < 0 < h(t) + e^{|M|}\sin(M)$$

with $M = \frac{\pi}{2} + 2k\pi$, and

$$h(t) + e^{|-M'|}\sin(-M') > 0 > h(t) + e^{|-M'|}\sin(M')$$

with $M' = \frac{3\pi}{2} + 2k\pi$. So, problem (23) has at least one solution.

Appendix A. Routines for numerical experiences

The routine used to obtain Figure 1 is, for example for the curve of $\alpha(0)$, the following

Routine A.1.

$$a := -0.99, \qquad b := 0.99, \qquad c := 0, \qquad \varepsilon := 2^{-7},$$

while $(b-a) > \varepsilon$

(1) v_s *is the solution of the Cauchy problem* $v' = h_{p,q}(t) + e^{\phi^{-1}(v)}$, $v(0) = s$;

(2) $f : (-1,1) \to [-1,1] : s \to v_s(1)$;

(3) *if* $f(c) > -1$ *then* $b = c$ *and* $c = \frac{a+b}{2}$, *else* $a = c$ *and* $c = \frac{a+b}{2}$;

then

$$\alpha(0) \simeq c.$$

The precision can be improved by taking a (resp b) closer to -1 (resp. 1), and ε closer to 0. Let us notice also that $f(s)$ will be 1 (resp. -1) if the function v_s is not global, i.e., reaches 1 (resp. -1) before that t reaches 1.

Remark A.2. Another way to obtain the graph of $\alpha(0)$ following p, is to observe the value at 0 of the solution of Cauchy problem with the function $e^{\phi^{-1}(v)}$ extended by 0 when $v = -1$, and with initial condition $v(r_1) = -1$ (or $v(r_2) = -1$ following p).

The routine used for Figure 3 is obtained by considering Problem $(\tilde{\mathscr{P}}_{b,\lambda})$ (with $b = 1$) which, from the constant function 0 (solution of $(\tilde{\mathscr{P}}_{b,\lambda})$ for $\lambda = 0$ and of $\int_0^1 (h_{p,q} + e^u) = 0$), leads to the function v_p.

Routine A.3.

$$v_0 := 0, \qquad \forall i \in \{1, \ldots, n\} : \qquad \lambda_i := \frac{i}{n},$$

for i *equals* 1 *to* n:

(1) $v_{i,s}$ *is the solution of the Cauchy's Problem* $v' = \lambda_i\big(h_{p,q}(t) + e^{\phi^{-1}(v)}\big)$, $v(0) = s$;

(2) $f_i : \mathbb{R} \to \mathbb{R} : s \to v_{i,s}(1)$;

(3) $u_0 \leftarrow$ *the closest zero of* u_0 *of function* f_i;

then

$$u_p \simeq u_{n,a_n}.$$

This is a consequence of the proof of Theorem 1 done in Section 3. Indeed, the solutions of problem $(\tilde{\mathscr{P}}_{b,\lambda})$ with $b = 1$ are solutions of $\big(\phi(u)\big)' = \lambda f(t,u)$ because of the boundary conditions and the fact that reals $\tilde{K}_1$, $\tilde{K}_2$ given by (16), (17) are equal to K_1, K_2 given by (13), (14).

The numerical errors of this routine can appear for two reasons: first because we used a numerical method in order to find root of f_i, what gives just an approximation, and secondly because we used a numerical method to compute the Poincaré function $P(s) = V(1,s)$, what gives again just an approximation. We can not say anything about the second error except that we used the software MatLab and the function *ode-45* of this software. For the first kind of errors, we can say some words. When the routine run following λ, the errors made for λ will not increase when λ tend to 1, because at each step, the function *fzero* of Matlab searching zero corrects this error. So we are not in front of a warning of explosion of the error with the running of the routine, but instead the converse. All the error due to *fzero* will be due to the last step, when $\lambda = 1$. But even at this step, we think that the error is not bad. Indeed, the function *fzero* search a root of $id - P$ and is better than the simple algorithm of Newton which needs, to run, that $(id - P)'(s_0) \neq 0$ where s_0 is the searched root. As

$$(id - P)'(s_0) = 1 - \frac{d}{ds} V(1,s)|_{s_0} \neq 0 \iff \varepsilon(1, s_0) \neq 1 \tag{24}$$

where $\varepsilon(t,s) := \frac{d}{ds} V(t,s)$, we know that condition 24 is equivalent to

$$\int_0^1 \frac{\partial h}{\partial v}\big(\sigma, V(\sigma, s_0)\big)\, \mathrm{d}\sigma \neq 0 \tag{25}$$

where h is the nonlinearity $h(t) + e^{\phi^{-1}(v)}$ of our equation, because the general theory about ordinary differential equations implies that

$$\varepsilon' = \frac{\partial h}{\partial v}\big(t, V(t,s)\big)\varepsilon, \quad \varepsilon(0,s) = 1 \iff \varepsilon(t,s) = \exp\Big(\int_0^t \frac{\partial h}{\partial v}\big(\sigma, V(\sigma, s)\big)\, \mathrm{d}\sigma\Big).$$

As Condition (25) is true because of

$$0 \neq \deg_{\mathrm{LS}}[id - T, B(0,K)] = \int_0^1 \frac{\partial h}{\partial v}\big(\sigma, V(\sigma, s_0)\big)\, \mathrm{d}\sigma,$$

(where T and K are explained in Section 3, where the inequality is due to the proof of Theorem 1, and where the equality can be found for example in [13]) we can conclude as claimed that the error made by the function *fzero* is not so bad.

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V. Bouchez, Institut de Recherche en Mathématique et physique, Université catholique de Louvain, Chemin du cyclotron 2, B-1348 Louvain-la-Neuve, Belgium

E-mail: vincent.bouchez@uclouvain.be

J. Mawhin, Institut de Recherche en Mathématique et physique, Université catholique de Louvain, Chemin du cyclotron 2, B-1348 Louvain-la-Neuve, Belgium

E-mail: jean.mawhin@uclouvain.be