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Robustness of dichotomies and trichotomies for difference equations

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Abstract. We give a short proof of the robustness of the notion of a nonuniform exponential dichotomy for a sequence of linear operators acting on a Banach space. This means that any sufficiently small linear perturbation of a nonuniform exponential dichotomy exhibits the same type of exponential behavior. The method of proof is based on the notion of admissibility introduced by Perron in the special case of a uniform exponential behavior. In strong contrast to former proofs, we do not need to construct explicitly projections on the stable and unstable directions. As an application, we also give a short proof of the robustness of the notion of a nonuniform exponential trichotomy.

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1. Introduction

Our main aim is to give relatively short proofs of the robustness of the notions of a nonuniform exponential dichotomy and of a nonuniform exponential trichotomy for a nonautonomous dynamics with discrete time generated by a sequence of linear operators acting on a Banach space. This means that any sufficiently small linear perturbation of a nonuniform exponential dichotomy (respectively, trichotomy) has the same type of exponential behavior. We emphasize that the sequence of linear operators need not be bounded, which is a considerable improvement of former work in [4].

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We formulate briefly the results in the particular case of a (classical) uniform exponential behavior. We say that a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators acting on a Banach space X admits a (*uniform*) *exponential dichotomy* if there exist projections P_m for $m \in \mathbb{Z}$ such that

$$\mathscr{A}(m,n)P_n = P_m\mathscr{A}(m,n)$$

for $m, n \in \mathbb{Z}$, where $\mathscr{A}(m,n) = A_{m-1}\dots A_n$, and there exist $a, D > 0$ such that

$$\|\mathscr{A}(m,n)P_n\| \le De^{-a(m-n)} \quad \text{and} \quad \|\mathscr{A}(m,n)^{-1}Q_m\| \le De^{-a(m-n)}$$

for $m \ge n$, where $Q_m = \mathrm{Id} - P_m$. The following result establishes the robustness of (uniform) exponential dichotomies and is a particular case of Theorem 3 below.

Theorem 1. *If a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators admits a (uniform) exponential dichotomy and $\sup_{n\in\mathbb{Z}}\|B_n\|$ is sufficiently small, then the sequence $(A_m + B_m)_{m\in\mathbb{Z}}$ also admits a (uniform) exponential dichotomy.*

Moreover, we say that a sequence of invertible linear operators $(A_m)_{m\in\mathbb{Z}}$ admits a (*uniform*) *exponential trichotomy* if there exist projections P_m, Q_m, R_m for $m \in \mathbb{Z}$ such that

$$P_m + Q_m + R_m = \mathrm{Id},$$
$$P_mQ_m = 0, \qquad P_mR_m = 0, \qquad Q_mR_m = 0,$$
$$\mathscr{A}(m,n)P_n = P_m\mathscr{A}(m,n), \qquad \mathscr{A}(m,n)Q_n = Q_m\mathscr{A}(m,n)$$

for $m, n \in \mathbb{Z}$ and there exist $a, D > 0$ and $b \in [0, a)$ such that

$$\|\mathscr{A}(m,n)P_n\| \le De^{-a(m-n)}, \qquad \|\mathscr{A}(m,n)^{-1}Q_m\| \le De^{-a(m-n)}$$

and

$$\|\mathscr{A}(m,n)R_n\| \le De^{b(m-n)}, \qquad \|\mathscr{A}(m,n)^{-1}R_m\| \le De^{b(m-n)}$$

for $m \ge n$. The following result establishes the robustness of (uniform) exponential trichotomies and is a particular case of Theorem 4 below.

Theorem 2. *If a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators admits a (uniform) exponential trichotomy and $\sup_{n\in\mathbb{Z}}\|B_n\|$ is sufficiently small, then the sequence $(A_m + B_m)_{m\in\mathbb{Z}}$ also admits a (uniform) exponential trichotomy.*

Due to the central role played by the exponential behavior in a large part of the theory of dynamical systems, it is important to understand whether this behavior persists under sufficiently small perturbations and so it is not surprising that the study of robustness has a long history. The problem was discussed by Massera and Schäffer [8], Coppel [6] and in the case of Banach spaces by Dalec'kiĭ and Kreĭn [7], with different approaches and successive generalizations. For more recent works we refer the reader to [5], [9], [11], [12] and the references therein. Moreover, we refer to [2], [3] for the study of robustness in the general setting of a nonuniform exponential behavior, respectively for dichotomies and trichotomies.

We emphasize that the approach in the last two works is different from the present one. Namely, while in [2], [3] we construct explicitly projections on the stable and unstable directions of the perturbation, as fixed points of appropriate operators, here the projections are obtained immediately after knowing that suitable stable and unstable spaces having respectively bounded forward and backward orbits form a direct sum. Overall, this is a considerable shortening of the former approach. Our method of proof is based on the notion of admissibility considered by Perron in [10] in the special case of a uniform exponential behavior. In particular, the stable and unstable subspaces are obtained fairly explicitly depending only on the boundedness respectively of forward and backward orbits. We recall that the notion of admissibility refers to the existence of bounded solutions for any bounded nonlinear perturbation of the original cocycle. This allows us to construct an invertible operator from the set of bounded perturbations to the set of bounded solutions and thus to conclude that under sufficiently small perturbations a similar operator exists for the perturbed cocycle.

A principal motivation for weakening the notion of a uniform exponential behavior is given by ergodic theory. Namely, consider a flow $(\phi_t)_{t \in \mathbb{R}}$ defined by an autonomous equation $x' = f(x)$ in $\mathbb{R}^n$ and assume that it preserves a finite measure μ. This means that

$$\mu\big(\phi_t(A)\big) = \mu(A)$$

for any measurable set $A \subset \mathbb{R}^n$ and any $t \in \mathbb{R}$. One can show that the trajectory of μ-almost every point x with nonzero Lyapunov exponents has a linear variational equation

$$v' = A_x(t)v, \qquad \text{with } A_x(t) = d_{\phi_t x} f,$$

that admits a nonuniform exponential dichotomy. We refer the reader to [1] for a detailed discussion of the ubiquity of nonuniform exponential behavior.

2. Robustness of dichotomies

In this section we establish the robustness of the notion of a nonuniform exponential dichotomy. This means that any sufficiently small linear perturbation of a nonuniform exponential dichotomy has the same exponential behavior as the original dichotomy. The method of proof is based on the notion of admissibility introduced by Perron in the special case of a uniform exponential behavior. This referred originally to the characterization of a uniform exponential dichotomy in terms of the existence of unique bounded solutions for any time-dependent bounded perturbation of the linear dynamics.

Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators acting on a Banach space X, we define

$$\mathscr{A}(m,n)=\begin{cases} A_{m-1}\dots A_n & \text{if } m>n,\\ \mathrm{Id} & \text{if } m=n,\\ A_m^{-1}\dots A_{n-1}^{-1} & \text{if } m<n \end{cases}$$

for each $m,n\in\mathbb{Z}$. We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *nonuniform exponential dichotomy* if there exist projections P_m for $m\in\mathbb{Z}$ such that

$$\mathscr{A}(m,n)P_n=P_m\mathscr{A}(m,n)$$

for $m,n\in\mathbb{Z}$ and there exist constants $a,D>0$ and $\varepsilon\geq 0$ with $\varepsilon<2a$ such that

$$\|\mathscr{A}(m,n)P_n\|\leq De^{-a(m-n)+\varepsilon|n|} \tag{1}$$

and

$$\|\mathscr{A}(n,m)Q_m\|\leq De^{-a(m-n)+\varepsilon|m|} \tag{2}$$

for $m\geq n$, where $Q_m=\mathrm{Id}-P_m$. The condition $\varepsilon<2a$ ensures that the exponential rates of the stable and unstable subspaces $\operatorname{Im}P_n$ and $\operatorname{Im}Q_n$ are separated.

Now we consider a nonautonomous linear perturbation of a nonuniform exponential dichotomy. Namely, given another sequence $(B_m)_{m\in\mathbb{Z}}$ of linear operators such that A_m+B_m is invertible for each m, let

$$\mathscr{F}(m,n)=\begin{cases} (A_{m-1}+B_{m-1})\dots(A_n+B_n) & \text{if } m>n,\\ \mathrm{Id} & \text{if } m=n,\\ (A_m+B_m)^{-1}\dots(A_{n-1}+B_{n-1})^{-1} & \text{if } m<n \end{cases}$$

for each $m,n\in\mathbb{Z}$.

The following result establishes the robustness of the nonuniform exponential dichotomies.

Theorem 3. *If a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators admits a nonuniform exponential dichotomy, and ε and $\sup_{n\in\mathbb{Z}}(\|B_n\|e^{\varepsilon|n|})$ are sufficiently small, then the sequence $(A_m+B_m)_{m\in\mathbb{Z}}$ also admits a nonuniform exponential dichotomy.*

Proof. We separate the proof into steps.

Step 1. *Construction of auxiliary norms.* We first introduce appropriate Lyapunov norms that allow us to control the nonuniform behavior of the exponential dichotomy. Write

$$\mathscr{B}(m,n) = \mathscr{A}(m,n)P_n \quad \text{and} \quad \mathscr{C}(m,n) = \mathscr{A}(m,n)Q_n.$$

For each $v \in X$ and $n \in \mathbb{Z}$, we consider the norm

$$\|v\|_n^* = \|v\|_n^s + \|v\|_n^u,$$

where

$$\|v\|_n^s = \sup\{\|\mathscr{B}(k,n)v\|e^{a(k-n)} : k \geq n\}$$

and

$$\|v\|_n^u = \sup\{\|\mathscr{C}(k,n)v\|e^{-a(k-n)} : k \leq n\}.$$

It follows readily from (1) and (2) that the suprema are finite.

Lemma 1. *For each $v \in X$ and $n \in \mathbb{Z}$ we have*

$$\|v\| \leq \|v\|_n^* \leq 2De^{\varepsilon|n|}\|v\|. \tag{3}$$

Proof of the lemma. Clearly,

$$\|v\|_n^s + \|v\|_n^u \geq \|P_n v\| + \|Q_n v\| \geq \|v\|.$$

On the other hand, by (1) and (2), we have

$$\|v\|_n^s \leq De^{\varepsilon|n|}\|v\| \quad \text{and} \quad \|v\|_n^u \leq De^{\varepsilon|n|}\|v\|.$$

This yields the second inequality in (3). □

Step 2. *Admissibility property.* Now we establish an appropriate version of the admissibility property that is expressed in terms of the Lyapunov norms. It tells us that there exists a unique bounded solution for any time-dependent bounded perturbation of the linear dynamics.

Consider the vector space

$$\mathscr{L} = \left\{f = \big(f(n)\big)_{n\in\mathbb{Z}} : \sup_{n\in\mathbb{Z}} \|f(n)\|_n^* < \infty\right\}$$

endowed with the norm

$$\|f\|^* = \sup_{n\in\mathbb{Z}} \|f(n)\|_n^*.$$

One can easily verify that $\mathscr{L}$ is a Banach space. Given $f \in \mathscr{L}$, we define a sequence x_f by

$$x_f(n) = \sum_{p=-\infty}^{n} \mathscr{B}(n,p)f(p) - \sum_{p=n+1}^{\infty} \mathscr{C}(n,p)f(p)$$

for $n \in \mathbb{Z}$.

Lemma 2. *For each $f \in \mathscr{L}$:*

1. *x_f is a well-defined sequence in $\mathscr{L}$ and*

$$\|x_f\|^* \le \frac{1+e^{-a}}{1-e^{-a}}\|f\|^*.$$

2. *we have*

$$x_f(n+1) = A_n x_f(n) + f(n+1), \qquad n \in \mathbb{Z}. \tag{4}$$

Proof of the lemma. We first note that

$$\|\mathscr{B}(n,p)\|^* := \sup_{x\neq 0} \frac{\|\mathscr{B}(n,p)x\|_n^*}{\|x\|_p^*} \le e^{a(p-n)}$$

for $n \ge p$. Indeed,

$$\begin{aligned}\|\mathscr{B}(n,p)x\|_n^* &= \sup\{\|\mathscr{B}(k,n)\mathscr{B}(n,p)x\|e^{a(k-n)} : k \ge n\}\\ &= \sup\{\|\mathscr{B}(k,p)x\|e^{a(k-n)} : k \ge n\}\\ &\le e^{a(p-n)}\{\|\mathscr{B}(k,p)x\|e^{a(k-p)} : k \ge p\}\\ &= e^{a(p-n)}\|x\|_p^*.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{p=-\infty}^{n} \|\mathscr{B}(n,p)f(p)\|_n^* &\le \sum_{p=-\infty}^{n} \|\mathscr{B}(n,p)\|^*\|f(p)\|_p^*\\ &\le \|f\|^* \sum_{p=-\infty}^{n} e^{a(p-n)} = \frac{\|f\|^*}{1-e^{-a}}.\end{aligned} \tag{5}$$

Furthermore, for $n \le p$ we have

$$\|\mathscr{C}(n,p)\|^* := \sup_{x \neq 0} \frac{\|\mathscr{C}(n,p)x\|_n^*}{\|x\|_p^*} \le e^{-a(p-n)}$$

since

$$\begin{aligned}
\|\mathscr{C}(n,p)x\|_n^* &= \sup\{\|\mathscr{C}(k,n)\mathscr{C}(n,p)x\|e^{-a(k-n)} : k \le n\} \\
&= \sup\{\|\mathscr{C}(k,p)x\|e^{-a(k-n)} : k \le n\} \\
&\le e^{-a(p-n)}\{\|\mathscr{C}(k,p)x\|e^{-a(k-p)} : k \le p\} \\
&= e^{-a(p-n)}\|x\|_p^*.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{p=n+1}^{\infty} \|\mathscr{C}(n,p)f(p)\|_n^* &\le \sum_{p=n+1}^{\infty} \|\mathscr{D}(n,p)\|^* \|f(p)\|_p^* \\
&\le \|f\|^* \sum_{p=n+1}^{\infty} e^{-a(p-n)} = \frac{\|f\|^*}{e^a - 1}.
\end{aligned}$$

Together with (5) this implies that x_f is well defined and that

$$\begin{aligned}
\|x_f(n)\|_n^* &\le \sum_{p=-\infty}^{n} \|\mathscr{B}(n,p)f(p)\|_n^* + \sum_{p=n+1}^{\infty} \|\mathscr{C}(n,p)f(p)\|_n^* \\
&\le \frac{1+e^{-a}}{1-e^{-a}}\|f\|^*.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
x_f(n+1) &= \sum_{p=-\infty}^{n+1} \mathscr{B}(n+1,p)f(p) - \sum_{p=n+2}^{\infty} \mathscr{C}(n+1,p)f(p) \\
&= \sum_{p=-\infty}^{n} A_n\mathscr{B}(n,p)f(p) + P_{n+1}f(n+1) \\
&\quad - \sum_{p=n+1}^{\infty} A_n\mathscr{C}(n,p)f(p) + Q_{n+1}f(n+1) \\
&= A_n x_f(n) + f(n+1)
\end{aligned}$$

for $n \in \mathbb{Z}$ and so identity (4) holds. □

The first property in Lemma 2 allows one to define a bounded linear operator $M : \mathscr{L} \to \mathscr{L}$ by $M(f) = x_f$. Clearly,

$$\|M\| \leq \frac{1 + e^{-a}}{1 - e^{-a}}.$$

Lemma 3. *The operator M is invertible.*

Proof of the lemma. If $x_f = 0$, then it follows from (4) that

$$f(n+1) = x_f(n+1) - A_n x_f(n) = 0$$

for $n \in \mathbb{Z}$ and so $f = 0$. This shows that M is one-to-one.

Now take $g \in \mathscr{L}$ and consider the sequence $f : \mathbb{Z} \to X$ defined by

$$f(n) = g(n) - A_{n-1} g(n-1).$$

For each $n \in \mathbb{Z}$ we have

$$\begin{aligned}
\|f(n)\|_n^* &\leq \|g(n)\|_n^* + \|A_{n-1} g(n-1)\|_n^* \\
&\leq \|g(n)\|_n^* + \sup\{\|\mathscr{B}(k,n) A_{n-1} g(n-1)\| e^{a(k-n)} : k \geq n\} \\
&\quad + \sup\{\|\mathscr{C}(k,n) A_{n-1} g(n-1)\| e^{-a(k-n)} : k \leq n\} \\
&\leq \|g(n)\|_n^* + e^{-a} \sup\{\|\mathscr{B}(k,n-1) g(n-1)\| e^{a(k-n+1)} : k \geq n\} \\
&\quad + e^{a} \sup\{\|\mathscr{C}(k,n-1) g(n-1)\| e^{-a(k-n+1)} : k \leq n\} \\
&\leq \|g(n)\|_n^* + e^{-a} \sup\{\|\mathscr{B}(k,n-1) g(n-1)\| e^{a(k-n+1)} : k \geq n\} \\
&\quad + e^{a} \|Q_{n-1} g(n-1)\| \\
&\quad + e^{a} \sup\{\|\mathscr{C}(k,n-1) g(n-1)\| e^{-a(k-n+1)} : k \leq n-1\} \\
&\leq \|g(n)\|_n^* + e^{a} \|Q_{n-1} g(n-1)\|_{n-1}^* \\
&\quad + e^{-a} \sup\{\|\mathscr{B}(k,n-1) g(n-1)\| e^{a(k-n+1)} : k \geq n-1\} \\
&\quad + e^{a} \sup\{\|\mathscr{C}(k,n-1) g(n-1)\| e^{-a(k-n+1)} : k \leq n-1\} \\
&\leq \|g(n)\|_n^* + e^{a} \|g(n-1)\|_{n-1}^* + e^{a} \|g(n-1)\|_{n-1}^*.
\end{aligned} \tag{6}$$

Therefore,

$$\|f\|^* \leq (1 + 2e^{a}) \|g\|^* < +\infty \tag{7}$$

and hence $f \in \mathscr{L}$. Moreover, by construction we have $g = M(f)$ and so the operator M is onto. $\square$

It follows from the proof of Lemma 3 that the inverse $M^{-1} : \mathscr{L} \to \mathscr{L}$ is given by

$$(M^{-1}g)(n) = g(n) - A_{n-1}g(n-1), \qquad n \in \mathbb{Z}. \tag{8}$$

Moreover, by (7) the operator M^{-1} is bounded.

Step 3. *Admissibility for the perturbation.* The next step is to establish a corresponding admissibility property for the perturbed linear dynamics. It is obtained by showing that the corresponding candidate for an invertible bounded linear operator is a small perturbation of the invertible operator for the original dynamics.

We define a linear operator L on $\mathscr{L}$ by

$$L(g)(n) = g(n) - (A_{n-1} + B_{n-1})g(n-1), \qquad n \in \mathbb{Z}.$$

Lemma 4. *If*

$$b := \sup_{n\in\mathbb{Z}} (\|B_n\| e^{\varepsilon|n+1|}) < \frac{1}{2D\|M\|},$$

then $L : \mathscr{L} \to \mathscr{L}$ is an invertible bounded linear operator.

Proof of the lemma. We first show that $L(g) \in \mathscr{L}$ for each $g \in \mathscr{L}$. By (6) and (8), for each $n \in \mathbb{Z}$ we have

$$\begin{aligned} \|L(g)(n)\|_n^* &\le \|(M^{-1}g)(n)\|_n^* + \|B_{n-1}g(n-1)\|_n^* \\ &\le \|g(n)\|_n^* + 2e^a\|g(n-1)\|_{n-1}^* + 2De^{\varepsilon|n|}\|B_{n-1}g(n-1)\| \\ &\le \|g(n)\|_n^* + (2e^a + 2Db)\|g(n-1)\|_{n-1}^*. \end{aligned}$$

Taking the supremum over $n \in \mathbb{Z}$ yields that

$$\|L(g)\|^* \le (1 + 2e^a + 2Db)\|g\|^* < +\infty$$

and so $L(g) \in \mathscr{L}$. This also shows that L is bounded. Moreover, since

$$(M^{-1} - L)(g)(n) = B_{n-1}g(n-1),$$

we have

$$\|(M^{-1} - L)(g)(n)\|_n^* \le \|B_{n-1}g(n-1)\|_n^* \le 2Db\|g(n-1)\|_{n-1}^*$$

and so

$$\|\mathrm{Id} - LM\| \le \|M^{-1} - L\| \cdot \|M\| \le 2Db\|M\| < 1. \tag{9}$$

Hence LM is invertible and since $L = (LM)M^{-1}$, we conclude that L is also invertible. □

Step 4. *Construction of invariant subspaces.* Now we start obtaining the structural elements of the exponential dichotomy for the perturbed dynamics. We first construct candidates for the stable and unstable subspaces. Their definition is very simple minded: we consider the subspaces formed by those vectors having respectively a bounded forward orbit and a bounded backward orbit.

For each $(n, x) \in \mathbb{Z} \times X$, we define sequences $s_{n,x}, u_{n,x} : \mathbb{Z} \to X$ by

$$s_{n,x}(k) = \begin{cases} \mathscr{F}(k, n)x, & k \geq n, \\ 0, & k < n \end{cases}$$

and

$$u_{n,x}(k) = \begin{cases} 0, & k > n, \\ \mathscr{F}(k, n)x, & k \leq n. \end{cases}$$

Moreover, for each $n \in \mathbb{Z}$, let

$$E_n = \{x \in X : s_{n,x} \in \mathscr{L}\}$$

and

$$F_n = \{x \in X : u_{n,x} \in \mathscr{L}\}.$$

One can easily verify that E_n and F_n are vector spaces. Now we establish their invariance under the dynamics.

Lemma 5. *For each $n \in \mathbb{Z}$ we have*

$$(A_n + B_n)E_n = E_{n+1} \quad \text{and} \quad (A_n + B_n)F_n = F_{n+1}. \tag{10}$$

Proof of the lemma. Since

$$s_{n+1,(A_n+B_n)x}(k) = \begin{cases} \mathscr{F}(k, n)x, & k \geq n+1, \\ 0, & k < n+1, \end{cases}$$

we have

$$\|s_{n,x}\|^* = \max\{\|x\|_n^*, \|s_{n+1,(A_n+B_n)x}\|^*\}.$$

Therefore, $s_{n,x} \in \mathscr{L}$ if and only if $s_{n+1,(A_n+B_n)x} \in \mathscr{L}$, that is, $x \in E_n$ if and only if $(A_n + B_n)x \in E_{n+1}$. This yields the first identity in (10).

For the second identity, we first note that

$$u_{n+1,(A_n+B_n)x}(k) = \begin{cases} 0, & k > n+1, \\ \mathscr{F}(k,n)x, & k \le n+1 \end{cases}$$

and hence,

$$\|u_{n+1,(A_n+B_n)x}\|^* = \max\{\|(A_n+B_n)x\|_{n+1}^*, \|u_{n,x}\|^*\}.$$

Therefore, $u_{n,x} \in \mathscr{L}$ if and only if $u_{n+1,(A_n+B_n)x} \in \mathscr{L}$, that is, $x \in F_n$ if and only if $(A_n+B_n)x \in F_{n+1}$, which yields the second identity in (10). □

Moreover, the spaces E_n and F_n form a direct sum.

Lemma 6. *For each $n \in \mathbb{Z}$ we have $X = E_n \oplus F_n$.*

Proof of the lemma. Take $x \in E_n \cap F_n$ and consider the sequence $g : \mathbb{Z} \to X$ defined by

$$g(k) = \begin{cases} s_{n,x}(k), & k \ge n, \\ u_{n,x}(k), & k \le n. \end{cases}$$

We note that $s_{n,x}(n) = u_{n,x}(n) = x$. Since

$$\|g\|^* \le \|s_{n,x}\|^* + \|u_{n,x}\|^* < +\infty,$$

we have $g \in \mathscr{L}$. Moreover, g is a solution of the equation

$$x_{m+1} = (A_m + B_m)x_m + f_{m+1}, \qquad m \in \mathbb{Z}$$

with $f = 0$, that is, $L(g) = 0$. It follows from Lemma 4 that $g = 0$ and hence $x = 0$. This shows that $E_n \cap F_n = \{0\}$.

Now take $n \in \mathbb{Z}$ and $x \in X$. We consider the sequence $\delta_{n,x} : \mathbb{Z} \to X$ defined by

$$\delta_{n,x}(m) = \begin{cases} x, & m = n, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Clearly, $\delta_{n,x} \in \mathscr{L}$ and so there exists a unique $g \in \mathscr{L}$ such that $L(g) = \delta_{n,x}$. We note that $g(m) = \mathscr{F}(m,n)g(n)$ for all $m \ge n$. Since $g \in \mathscr{L}$, this shows that $s_{n,g(n)} \in \mathscr{L}$ and hence $g(n) \in E_n$. Now we observe that

$$u_{n,x-g(n)}(k) = \begin{cases} 0, & k > n, \\ x - g(n), & k = n, \\ -g(k), & k < n. \end{cases}$$

Again since $g \in \mathscr{L}$, this shows that $x - g(n) \in F_n$. Therefore,

$$x = g(n) + \big(x - g(n)\big) \in E_n + F_n,$$

which completes the proof of the lemma. □

Now let P_n and Q_n be the projections associated to the decomposition $X = E_n \oplus F_n$. Given $n \in \mathbb{Z}$ and $x \in X$, we have

$$(A_n + B_n)x = (A_n + B_n)P_n x + (A_n + B_n)Q_n x \tag{12}$$

and

$$(A_n + B_n)x = P_{n+1}(A_n + B_n)x + Q_{n+1}(A_n + B_n)x \tag{13}$$

On the other hand, by Lemma 5,

$$(A_n + B_n)P_n x \in E_{n+1} \quad \text{and} \quad (A_n + B_n)Q_n x \in F_{n+1}.$$

Hence, it follows from (12) and (13) that

$$(A_n + B_n)P_n = P_{n+1}(A_n + B_n) \tag{14}$$

and

$$(A_n + B_n)Q_n = Q_{n+1}(A_n + B_n) \tag{15}$$

for $n \in \mathbb{Z}$ (notice that identities (14) and (15) are in fact equivalent).

Step 5. *Estimates on the stable direction.* Now we obtain an exponential bound along the stable direction. We first show that the dynamics in uniformly bounded on the initial time (recall that the space E_n is defined in terms of the boundedness of a sequence starting at time zero).

Lemma 7. *For each $n \in \mathbb{Z}$, $x \in E_n$ and $m \geq n$ we have*

$$\|\mathscr{F}(m,n)x\|_m^* \leq K\|x\|_n^*,$$

where $K = \|M\|/(1 - 2Db\|M\|)$.

Proof of the lemma. Take $n \in \mathbb{Z}$ and $x \in E_n$. Since $x \in E_n$, we have $\delta_{n,x} \in \mathscr{L}$ (see (11)) and one can easily verify that $s_{n,x} = L^{-1}(\delta_{n,x})$. On the other hand,

$$L = [(LM - \mathrm{Id}) + \mathrm{Id}]M^{-1}$$

and thus,

$$L^{-1} = M\sum_{k=0}^{\infty}(-1)^k(LM - \mathrm{Id})^k.$$

It follows from (9) that

$$\|L^{-1}\| \le \|M\|\sum_{k=0}^{\infty}\|LM - \mathrm{Id}\|^k \le \frac{\|M\|}{1-2Db\|M\|} = K. \tag{16}$$

Therefore, for $m \ge n$ we have

$$\begin{aligned}\|\mathscr{F}(m,n)x\|_m^* \le \|s_{n,x}\|^* &= \|L^{-1}(\delta_{n,x})\|^* \\ &\le \|L^{-1}\|\cdot\|\delta_{n,x}\|^* \le K\|x\|_n^*,\end{aligned}$$

which yields the desired inequality. □

The following result yields an exponential bound along the stable direction.

Lemma 8. *There exist constants $C, \lambda > 0$ such that*

$$\|\mathscr{F}(m,n)x\|_m^* \le Ce^{-\lambda(m-n)}\|x\|_n^*$$

for $n \in \mathbb{Z}$, $x \in E_n$ and $m \ge n$.

Proof of the lemma. Given $n \in \mathbb{Z}$, $p \in \mathbb{N}$ and $x \in E_n$, we define a sequence $f : \mathbb{Z} \to X$ by

$$f(m) = \begin{cases}\mathscr{F}(m,n)x, & n \le m < n+p,\\ 0, & \text{otherwise}.\end{cases}$$

By Lemma 7, for each $m \ge n$ we have

$$\|f(m)\|_m^* \le \|\mathscr{F}(m,n)x\|_m^* \le K\|x\|_n^* \tag{17}$$

and thus $\|f\|^* \le K\|x\|_n^*$. On the other hand,

$$L^{-1}(f)(m) = \begin{cases}0, & m < n,\\ (m-n+1)\mathscr{F}(m,n)x, & n \le m < n+p,\\ p\mathscr{F}(m,n)x, & m \ge n+p\end{cases}$$

(since $x \in E_n$). By Lemma 5, we have $\mathscr{F}(l,n)x \in E_l$ for $l \geq m$. Hence, using Lemma 7, we obtain

$$\begin{aligned}
\frac{p(p+1)}{2}\|\mathscr{F}(n+p,n)x\|_{n+p}^* &= \Big\| \sum_{l=n}^{n+p-1} (l-n+1)\mathscr{F}(n+p,n)x \Big\|_{n+p}^* \\
&\leq \sum_{l=n}^{n+p-1} (l-n+1)\|\mathscr{F}(n+p,n)x\|_{n+p}^* \\
&= \sum_{l=n}^{n+p-1} (l-n+1)\|\mathscr{F}(n+p,l)\mathscr{F}(l,n)x\|_{n+p}^* \\
&\leq K \sum_{l=n}^{n+p-1} (l-n+1)\|\mathscr{F}(l,n)x\|_l^* \\
&= K \sum_{l=n}^{n+p-1} \|L^{-1}(f)(l)\|_l^* \leq Kp\|L^{-1}(f)\|^*.
\end{aligned}$$

It follows from (16) and (17) that

$$\frac{p(p+1)}{2}\|\mathscr{F}(n+p,n)x\|_{n+p}^* \leq K^2p\|f\|^* \leq K^3p\|x\|_n^*.$$

Therefore,

$$\|\mathscr{F}(n+p,n)\|^* = \sup_{x \neq 0} \frac{\|\mathscr{F}(n+p,n)x\|_{n+p}^*}{\|x\|_n^*} \leq \frac{2K^3}{p+1}.$$

Now we take $p_0 \in \mathbb{N}$ sufficiently large so that

$$s := \frac{2K^3}{p_0+1} < 1. \tag{18}$$

Given $m, n \in \mathbb{N}$ with $m \geq n$, let $r = [(m-n)/p_0]$ where $[\cdot]$ denotes the integer part. Then

$$\mathscr{F}(m,n) = \mathscr{F}(m, m+p_0r)\mathscr{F}(n+p_0r, n),$$

and by Lemma 7 we have

$$\begin{aligned}
\|\mathscr{F}(m,n)\|^* &\leq K\|\mathscr{F}(n+p_0r,n)\|^* \leq Ks^r \leq Ks^{(m-n)/p_0-1} \\
&\leq (K/s)e^{(m-n)(1/p_0)\log s} = Ce^{-\lambda(m-n)},
\end{aligned}$$

where

$$C = K/s \quad \text{and} \quad \lambda = -(1/p_0)\log s > 0.$$

This completes the proof of the lemma. $\square$

Step 6. *Estimates on the unstable direction.* We also obtain an exponential bound along the unstable direction. The approach is analogous: we first show that the dynamics in uniformly bounded on the initial time.

Lemma 9. *For each $n \in \mathbb{Z}$, $x \in F_n$ and $m \geq n$ we have*

$$\|\mathscr{F}(m,n)x\|_m^* \geq \frac{\|x\|_n^*}{K}.$$

Proof of the lemma. Take $n \in \mathbb{Z}$ and $x \in F_n$. We have $u_{m-1,y} = L^{-1}(\delta_{m,z})$ for each $m \geq n$, where

$$y = \mathscr{F}(m-1,n)x \quad \text{and} \quad z = -\mathscr{F}(m,n)x$$

(since $x \in F_n$). Moreover, since $u_{m-1,y}(n) = x$ we have $\|u_{m-1,y}\|^* \geq \|x\|_n^*$. Therefore, by (16),

$$\|x\|_n^* \leq \|u_{m-1,y}\|^* = \|L^{-1}(\delta_{m-1,y})\|^* \leq K\|\mathscr{F}(m,n)x\|_m^*$$

for $m \geq n$. This completes the proof of the lemma. $\square$

The following result yields an exponential bound along the unstable direction.

Lemma 10. *For every $n \in \mathbb{Z}$, $x \in F_n$ and $m \leq n$ we have*

$$\|\mathscr{F}(m,n)x\|_m^* \leq Ce^{-\lambda(n-m)}\|x\|_n^*.$$

Proof of the lemma. Given $n \in \mathbb{Z}$, $p \in \mathbb{N}$ and $x \in F_n$, we define a sequence $f : \mathbb{Z} \to X$ by

$$f(m) = \begin{cases} \mathscr{F}(m,n)x, & n-p < m \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Lemma 9 that

$$\|f(m)\|_m^* \leq \|\mathscr{F}(m,n)x\|_m^* \leq K\|x\|_n^*$$

for $m \le n$ (since $\mathscr{F}(m,n)x \in F_m$) and thus,

$$\|f\|^* \le K\|x\|_n^*. \tag{19}$$

On the other hand,

$$\begin{aligned}\frac{p(p+1)}{2}\|\mathscr{F}(n-p,n)x\|_{n-p}^* &= \Big\| \sum_{l=n-p}^{n-1} (l-n)\mathscr{F}(n-p,n)x \Big\|_{n-p}^* \\ &\le \sum_{l=n-p}^{n-1} (l-n)\|\mathscr{F}(n-p,n)x\|_{n-p}^* \\ &= \sum_{l=n-p}^{n-1} (l-n)\|\mathscr{F}(n-p,l)\mathscr{F}(l,n)x\|_{n-p}^*.\end{aligned}$$

Since

$$L^{-1}(f)(m) = \begin{cases} 0, & m \ge n, \\ (m-n)\mathscr{F}(m,n)x, & n-p \le m < n, \\ -p\mathscr{F}(m,n)x, & m < n-p \end{cases}$$

(because $x \in F_n$), we obtain

$$\begin{aligned}\frac{p(p+1)}{2}\|\mathscr{F}(n-p,n)x\|_{n-p}^* &\le K \sum_{l=n-p}^{n-1} (l-n)\|\mathscr{F}(l,n)x\|_l^* \\ &= K \sum_{l=n-p}^{n-1} \|L^{-1}(f)(l)\|_l^* \le Kp\|L^{-1}(f)\|^*.\end{aligned}$$

Moreover, by (16) we have $\|L^{-1}(f)\|^* \le K\|f\|^*$, and hence, by (19),

$$\frac{p(p+1)}{2}\|\mathscr{F}(n-p,n)x\|_{n-p}^* \le K^2 p\|f\|^* \le K^3 p\|x\|_n^*.$$

Therefore,

$$\|\mathscr{F}(n-p,n)\|^* \le \frac{2K^3}{p+1}.$$

Given $m,n \in \mathbb{N}$ with $m \le n$, let $r = [(n-m)/p_0]$ with p_0 as in (18). Then

$$\mathscr{F}(m,n) = \mathscr{F}(m,m-p_0r)\mathscr{F}(n-p_0r,n)$$

and hence,

$$\begin{aligned}\|\mathscr{F}(m,n)\|^* &\leq K\|\mathscr{F}(n-p_0r,n)\|^* \\ &\leq Ks^r \leq Ks^{(n-m)/p_0-1} \\ &= (K/s)e^{(n-m)(1/p_0)\log s} = Ce^{-\lambda(n-m)}.\end{aligned}$$

This completes the proof of the lemma. □

Step 7. Existence of an exponential dichotomy. Finally, we show that the perturbed linear dynamics admits a nonuniform exponential dichotomy having E_n and F_n respectively has the stable and unstable subspaces. This amounts to obtaining exponential bounds along the stable and unstable directions in terms of the original norm (we recall that the bounds were obtained in terms of the Lyapunov norms) and to estimate the norms of the projections P_n and Q_n.

For $x \in E_n$ it follows from Lemma 8 that

$$\|\mathscr{F}(m,n)x\|_m^* \leq Ce^{-\lambda(m-n)}\|x\|_n^*, \qquad m \geq n.$$

By (3) we obtain

$$\begin{aligned}\|\mathscr{F}(m,n)x\| \leq \|\mathscr{F}(m,n)x\|_m^* &\leq Ce^{-\lambda(m-n)}\|x\|_n^* \\ &\leq 2CDe^{-\lambda(m-n)+\varepsilon|n|}\|x\|\end{aligned} \tag{20}$$

for $m \geq n$. Similarly, for $x \in F_n$ it follows from Lemma 10 that

$$\|\mathscr{F}(m,n)x\|_m^* \leq Ce^{-\lambda(m-n)}\|x\|_n^*, \qquad m \leq n$$

and proceeding as in (20) we get

$$\begin{aligned}\|\mathscr{F}(m,n)x\| \leq \|\mathscr{F}(m,n)x\|_m^* &\leq Ce^{-\lambda(n-m)}\|x\|_n^* \\ &\leq 2CDe^{-\lambda(n-m)+\varepsilon|n|}\|x\|\end{aligned} \tag{21}$$

for $m \leq n$. Finally, we estimate the norms of the projections P_n and Q_n. Using the notation in the proof of Lemma 6, given $x \in X$ we have $P_nx = g(n)$, where $g = L^{-1}(f)$ with $f = \delta_{n,x}$. Therefore,

$$\|P_n\| = \sup_{x\neq 0}\frac{\|g(n)\|}{\|x\|} = \sup_{x\neq 0}\frac{\|L^{-1}(f)(n)\|}{\|x\|}.$$

Using (3) we obtain

$$\begin{aligned}\|L^{-1}(f)(n)\| &\le \|L^{-1}(f)(n)\|_n^* \le \|L^{-1}(f)\|^* \\ &\le \|L^{-1}\| \cdot \|f\|^* = \|L^{-1}\| \cdot \|x\|_n^* \\ &\le 2De^{\varepsilon|n|}\|L^{-1}\| \cdot \|x\|\end{aligned}$$

and hence,

$$\|P_n\| \le 2De^{\varepsilon|n|}\|L^{-1}\|. \tag{22}$$

Therefore, we also obtain

$$\begin{aligned}\|Q_n\| &\le \|\mathrm{Id} - P_n\| \le 1 + \|P_n\| \\ &\le \max\{1, 2D\|L^{-1}\|\}e^{\varepsilon|n|}. \end{aligned}\tag{23}$$

It follows from (20) and (21) together with (22) and (23) that

$$\|\mathscr{F}(m,n)P_n\| \le 4CD^2\|L^{-1}\|e^{-\lambda(m-n)+2\varepsilon|n|}$$

for $m \ge n$ and

$$\|\mathscr{F}(m,n)Q_n\| \le 2\max\{1, 2D\|L^{-1}\|\}CDe^{-\lambda(n-m)+2\varepsilon|n|}$$

for $m \le n$. Therefore, provided that ε is sufficiently small the sequence $(A_m + B_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy. □

3. Robustness of trichotomies

In this section we obtain a corresponding robustness result for the notion of a nonuniform exponential trichotomy. This means that any sufficiently small linear perturbation of a nonuniform exponential trichotomy has the same exponential behavior as the original trichotomy. The result is obtained by applying Theorem 3 to appropriate shifts of the perturbed dynamics: essentially, a nonuniform exponential trichotomy is obtained from intersecting two nonuniform exponential dichotomies that are obtained from shifting the original dynamics to the right and to the left (see (27)).

We say that a sequence of invertible linear operators $(A_m)_{m\in\mathbb{Z}}$ admits a *nonuniform exponential trichotomy* if there exist projections P_m, Q_m, R_m for $m \in \mathbb{Z}$ such that

$$P_m + Q_m + R_m = \mathrm{Id},$$
$$P_mQ_m = 0, \quad P_mR_m = 0, \quad Q_mR_m = 0$$

and

$$\mathscr{A}(m,n)P_n = P_m\mathscr{A}(m,n),$$
$$\mathscr{A}(m,n)Q_n = Q_m\mathscr{A}(m,n),$$
$$\mathscr{A}(m,n)R_n = R_m\mathscr{A}(m,n)$$

for $m,n \in \mathbb{Z}$ and there exist constants $a,D > 0$ and $b,\varepsilon \geq 0$ with $\varepsilon < a-b$ such that

$$\|\mathscr{B}(m,n)\| \leq De^{-a(m-n)+\varepsilon|n|},$$
$$\|\mathscr{C}(n,m)\| \leq De^{-a(m-n)+\varepsilon|m|},$$
$$\|\mathscr{D}(m,n)\| \leq De^{b(m-n)+\varepsilon|n|}$$

and

$$\|\mathscr{D}(n,m)\| \leq De^{b(m-n)+\varepsilon|m|}$$

for $m \geq n$, where

$$\mathscr{B}(m,n) = \mathscr{A}(m,n)P_n, \qquad \mathscr{C}(m,n) = \mathscr{A}(m,n)Q_n, \qquad \mathscr{D}(m,n) = \mathscr{A}(m,n)R_n.$$

The following result establishes the robustness of the nonuniform exponential trichotomies.

Theorem 4. *If a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators admits a nonuniform exponential trichotomy, and ε and $\sup_{n\in\mathbb{Z}}(\|B_n\|e^{\varepsilon|n|})$ are sufficiently small, then the sequence $(A_m + B_m)_{m\in\mathbb{Z}}$ also admits a nonuniform exponential trichotomy.*

Proof. Let $\kappa = (a+b)/2$. Since the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential trichotomy, the sequence $(e^{\kappa}A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy with projections

$$P_{1,m} = P_m \quad \text{and} \quad Q_{1,m} = Q_m + R_m$$

for $m \in \mathbb{Z}$. Provided that $c := \sup_{n\in\mathbb{Z}}(\|B_n\|e^{\varepsilon|n|})$ is sufficiently small, it follows from Theorem 3 that the sequence $\big(e^{\kappa}(A_m + B_m)\big)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy, say with projections $\hat{P}_{1,m}$ and $\hat{Q}_{1,m}$. In particular, the subspaces

$$\hat{E}_{1,m} = \hat{P}_{1,m}(X) \quad \text{and} \quad \hat{F}_{1,m} = \hat{Q}_{1,m}(X)$$

satisfy

$$\hat{E}_{1,m} \oplus \hat{F}_{1,m} = X. \tag{24}$$

Similarly, the sequence $(e^{-\kappa} A_m)_{m \in Z}$ admits a nonuniform exponential dichotomy with projections

$$P_{2,m} = P_m + R_m \quad \text{and} \quad Q_{2,m} = Q_m$$

for $m \in \mathbb{Z}$. Provided that c is sufficiently small, it follows from Theorem 3 that the sequence $\left(e^{-\kappa}(A_m + B_m)\right)_{m \in \mathbb{Z}}$ admits a nonuniform exponential dichotomy, say with projections $\hat{P}_{2,m}$ and $\hat{Q}_{2,m}$. In particular, the subspaces

$$\hat{E}_{2,m} = \hat{P}_{2,m}(X) \quad \text{and} \quad \hat{F}_{2,m} = \hat{Q}_{2,m}(X)$$

satisfy

$$\hat{E}_{2,m} \oplus \hat{F}_{2,m} = X. \tag{25}$$

We also consider the maps

$$\mathscr{F}_\kappa(m,n) = e^{\kappa(m-n)} \mathscr{F}(m,n) \quad \text{and} \quad \mathscr{F}_{-\kappa}(m,n) = e^{-\kappa(m-n)} \mathscr{F}(m,n). \tag{26}$$

Lemma 11. *For every $n \in \mathbb{Z}$ we have*

$$\hat{E}_{1,n} \subset \hat{E}_{2,n} \quad \textit{and} \quad \hat{F}_{2,n} \subset \hat{F}_{1,n}.$$

Proof of the lemma. Let

$$\mu(x) = \limsup_{m \to +\infty} \frac{1}{m} \log\|\mathscr{F}_\kappa(m,n)x\|.$$

If there exists $x \in \hat{E}_{1,n} \setminus \hat{E}_{2,n}$, then we write $x = y + z$ with $y \in \hat{E}_{2,n}$ and $z \in \hat{F}_{2,n}$. Since $x \in \hat{E}_{1,n}$, by (20) we have

$$\|\mathscr{F}_\kappa(m,n)x\| \le 2CDe^{-\lambda_1(m-n)+\varepsilon|n|}\|x\|$$

for some $\lambda_1 > 0$ and hence $\mu(x) \le -\lambda_1$. Moreover, we have $z \ne 0$ (otherwise $x = y \in \hat{E}_{2,n}$ which is false by hypothesis). Hence,

$$\begin{aligned} \mu(x) &= \max\{\mu(y), \mu(z)\} = \mu(z) \\ &= \limsup_{m \to +\infty} \frac{1}{m} \log\|\mathscr{F}_\kappa(m,n)z\|. \end{aligned}$$

Since $z \in \hat{F}_{2,n}$, for $m \geq n$ we have

$$\begin{aligned}\|\mathscr{F}_{\kappa}(m,n)z\| &= e^{2\kappa(m-n)}\|\mathscr{F}_{-\kappa}(m,n)z\| \\ &\geq \frac{1}{2CD}\|z\|e^{(2\kappa+\lambda_2)(m-n)-\varepsilon|m|}\end{aligned}$$

for some $\lambda_2 > 0$ and hence,

$$\mu(x) \geq \lambda_2 + 2\kappa - \varepsilon.$$

But this contradicts to the inequality $\mu(x) \leq -\lambda_1$ since

$$\varepsilon < a - b < a + b = 2\kappa.$$

Therefore, $\hat{E}_{1,n} \subset \hat{E}_{2,n}$. One can show in a similar manner that $\hat{F}_{2,n} \subset \hat{F}_{1,n}$ for each $n \in \mathbb{Z}$. □

Lemma 12. *For every $n \in \mathbb{Z}$ we have*

$$(\hat{E}_{2,n} \cap \hat{F}_{1,n}) \oplus \hat{E}_{1,n} \oplus \hat{F}_{2,n} = X. \tag{27}$$

Proof of the lemma. It follows from (24) that

$$(\hat{E}_{2,n} \cap \hat{E}_{1,n}) \oplus (\hat{E}_{2,n} \cap \hat{F}_{1,n}) = \hat{E}_{2,n}.$$

But in view of Lemma 11 we have $\hat{E}_{2,n} \cap \hat{E}_{1,n} = \hat{E}_{1,n}$ and hence,

$$\hat{E}_{1,n} \oplus (\hat{E}_{2,n} \cap \hat{F}_{1,n}) = \hat{E}_{2,n}.$$

The desired statement follows now immediately from (25). □

Lemma 13. *For each $m \in \mathbb{Z}$ we have*

$$\hat{P}_{1,m}\hat{Q}_{2,m} = \hat{Q}_{2,m}\hat{P}_{1,m} = 0.$$

Proof of the lemma. By Lemma 11, for each $x \in X$ we have

$$\hat{Q}_{2,m}x \in \hat{F}_{2,m} \subset \hat{F}_{1,m}$$

and hence,

$$\hat{P}_{1,m}\hat{Q}_{2,m}x \in \hat{P}_{1,m}\hat{F}_{1,m} = \hat{P}_{1,m}\,\mathrm{Im}\,\hat{Q}_{1,m} = \{0\}.$$

Similarly, again by Lemma 11, for each $x \in X$ we have

$$\hat{P}_{1,m}x \in \hat{E}_{1,m} \subset \hat{E}_{2,m}$$

and hence,

$$\hat{Q}_{2,m}\hat{P}_{1,m}x \in \hat{Q}_{2,m}\hat{E}_{2,m} = \hat{Q}_{2,m}\,\mathrm{Im}\,\hat{P}_{2,m} = \{0\}.$$

This completes the proof of the lemma. □

We proceed with the proof of the theorem. Let

$$\hat{P}_m = \hat{P}_{1,m}, \qquad \hat{Q}_m = \hat{Q}_{2,m} \qquad \text{and} \qquad \hat{R}_m = \mathrm{Id} - \hat{P}_{1,m} - \hat{Q}_{2,m}.$$

We also consider the subspaces

$$\hat{E}_m = \hat{P}_m(X), \qquad \hat{F}_m = \hat{Q}_m(X) \qquad \text{and} \qquad \hat{G}_m = \hat{R}_m(X).$$

In view of (14) and (15) we have respectively

$$\mathscr{F}_\kappa(m,n)\hat{P}_n = \hat{P}_m\mathscr{F}_\kappa(m,n) \qquad \text{and} \qquad \mathscr{F}_{-\kappa}(m,n)\hat{Q}_n = \hat{Q}_m\mathscr{F}_{-\kappa}(m,n),$$

which by (26) yields that

$$\mathscr{F}(m,n)\hat{P}_n = \hat{P}_m\mathscr{F}(m,n) \qquad \text{and} \qquad \mathscr{F}(m,n)\hat{Q}_n = \hat{Q}_m\mathscr{F}(m,n).$$

This readily implies that

$$\mathscr{F}(m,n)\hat{R}_n = \hat{R}_m\mathscr{F}(n,m).$$

Furthermore, the operators $\hat{P}_m$ and $\hat{Q}_m$ are projections and by Lemma 13 we have

$$\begin{aligned}
\hat{R}_m^2 &= (\mathrm{Id} - \hat{P}_{1,m} - \hat{Q}_{2,m})^2 \\
&= \mathrm{Id} - 2\hat{P}_{1,m} - 2\hat{Q}_{2,m} + \hat{P}_{1,m}^2 + \hat{Q}_{2,m}^2 + \hat{P}_{1,m}\hat{Q}_{2,m} + \hat{Q}_{2,m}\hat{P}_{1,m} \\
&= \mathrm{Id} - \hat{P}_{1,m} - \hat{Q}_{2,m} = \hat{R}_m.
\end{aligned}$$

By (23) we have

$$\begin{aligned}
\|\hat{R}_m\| &= \|\mathrm{Id} - \hat{P}_{1,m} - \hat{Q}_{2,m}\| \\
&\le 1 + 2\max\{1, 2D\|L^{-1}\|\}e^{\varepsilon|m|} \\
&\le (1 + 2\max\{1, 2D\|L^{-1}\|\})e^{\varepsilon|m|}.
\end{aligned} \tag{28}$$

By (20), since $\hat{P}_m = \hat{P}_{1,m}$, for every $m \geq n$ we have

$$\begin{aligned}\|\mathscr{F}(m,n)\hat{P}_n\| &= \|e^{-\kappa(m-n)}\mathscr{F}_\kappa(m,n) \,|\, \operatorname{Im}\hat{P}_n\| \cdot \|\hat{P}_n\| \\ &= K_1 e^{-\kappa(m-n)} e^{-\lambda_1(m-n)+2\varepsilon|n|} \\ &= K_1 e^{-(\lambda_1+\kappa)(m-n)+2\varepsilon|n|}\end{aligned}$$

for some constant $K_1 > 0$. Similarly, since $\hat{Q}_m = \hat{Q}_{2,m}$, for every $m \geq n$ we have

$$\begin{aligned}\|\mathscr{F}(m,n)^{-1}\hat{Q}_m\| &= \|e^{-\kappa(m-n)}\mathscr{F}_{-\kappa}(m,n)^{-1} \,|\, \operatorname{Im}\hat{Q}_m\| \cdot \|\hat{Q}_m\| \\ &\leq K_2 e^{-\kappa(m-n)} e^{-\lambda_2(m-n)+2\varepsilon|m|} \\ &\leq K_2 e^{-(\lambda_2+\kappa)(m-n)+2\varepsilon|m|}\end{aligned}$$

for some constant $K_2 > 0$. Furthermore, for every $m \geq n$ we have

$$\begin{aligned}\|\mathscr{F}(m,n)\hat{R}_n\| &\leq \|\mathscr{F}(m,n) \,|\, \hat{G}_n\| \cdot \|\hat{R}_n\| \\ &= \|\mathscr{F}(m,n) \,|\, (\hat{E}_{2,n} \cap \hat{F}_{1,n})\| \cdot \|\hat{R}_n\| \\ &\leq \|\mathscr{F}(m,n) \,|\, \hat{E}_{2,n}\| \cdot \|\hat{R}_n\| \\ &= e^{\kappa(m-n)}\|\mathscr{F}_{-\kappa}(m,n) \,|\, \hat{E}_{2,n}\| \cdot \|\hat{R}_n\| \qquad (29)\end{aligned}$$

and analogously, for every $m \geq n$,

$$\begin{aligned}\|\mathscr{F}(m,n)^{-1}\hat{R}_m\| &\leq \|\mathscr{F}(m,n)^{-1} \,|\, \hat{F}_{1,m}\| \cdot \|\hat{R}_m\| \\ &= e^{\kappa(m-n)}\|\mathscr{F}_\kappa(m,n)^{-1} \,|\, \hat{F}_{1,m}\| \cdot \|\hat{R}_m\|. \qquad (30)\end{aligned}$$

By (28), it follows from (29) that for every $m \geq n$,

$$\|\mathscr{F}(m,n)\hat{R}_n\| \leq (1 + 2\max\{1, 2D\|L^{-1}\|\})K_2 e^{(\kappa-\lambda_2)(m-n)+2\varepsilon|n|}$$

and it follows from (30) that for every $m \geq n$,

$$\|\mathscr{F}(m,n)^{-1}\hat{R}_n\| \leq (1 + 2\max\{1, 2D\|L^{-1}\|\})K_1 e^{(\kappa-\lambda_1)(m-n)+2\varepsilon|n|}.$$

Taking

$$a' = \min\{\lambda_1, \lambda_2\} + \kappa \quad \text{and} \quad b' = -\min\{\lambda_1, \lambda_2\} + \kappa,$$

we obtain

$$a' - b' = 2\min\{\lambda_1, \lambda_2\} > \varepsilon$$

provided that ε is sufficiently small and so the sequence $(A_m + B_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential trichotomy. This completes the proof of the theorem. $\square$

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