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On the exactly synchronizable state to a coupled system of wave equations

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Dedicated to Professor João Paulo de Carvalho Dias on the occasion of his 70th birthday

Abstract. In this paper, we discuss the determination of the exactly synchronizable state to a coupled system of wave equations. In a special case, the exactly synchronizable state can be uniquely determined whatever the boundary controls would be chosen. In the general case, the determination of the exactly synchronizable state depends on the boundary controls which realize the exact synchronization, however, we can give an estimate to the difference between the exactly synchronizable state and the solution to a problem independent of boundary controls.

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§1. Introduction

The phenomenon of synchronization was first observed by Huygens in 1665 [4]. The theoretical research on synchronization phenomena dates back to Fujisaka and Yamada's study of synchronization for coupled equations in 1983 [2], and since then the previous studies focused only on systems described by ODEs. The exact synchronization in the PDEs case was first studied for a coupled system of wave equations both for the higher-dimensional case in the framework of weak solutions by Li-Rao [5], [6], and for the 1-D case in the framework of classical solutions in Li-Rao-Hu [3], [7].

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary Γ. Let $\Gamma = \Gamma_1 \cup \Gamma_0$ be a partition of Γ such that $\bar{\Gamma}_1 \cap \bar{\Gamma}_0 = \emptyset$. Assume that the usual geometric

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control condition (see Bordos-Lebau-Rauch [1]) is satisfied, for instance, we assume that there exists $x_0 \in \mathbb{R}^n$ such that, setting $m = x - x_0$, we have

$$(m, \nu) > 0, \qquad \forall x \in \Gamma_1; \qquad (m, \nu) \leq 0, \qquad \forall x \in \Gamma_0, \tag{1.1}$$

where ν is the unit outward normal vector and $(\cdot\,,\cdot)$ denotes the inner product in $\mathbb{R}^n$ (cf. Lions [9]).

Let

$$U = (u^{(1)}, \dots, u^{(N)})^T. \tag{1.2}$$

Consider the following coupled system of wave equations with Dirichlet boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \Gamma_0, \\ U = DH & \text{on } \Gamma_1, \\ t = 0\colon \quad U = U_0, \quad U' = U_1, \end{cases} \tag{1.3}$$

where A and D are matrices of order N with constant elements, and

$$H = (h^{(1)}, \dots, h^{(N)})^T \tag{1.4}$$

is the boundary control.

Differently from the statement given in Li-Rao [5], a control matrix D is added to the boundary condition on Γ_1. This way is more flexible since one can adjust the number of boundary controls by changing the rank of D correspondingly. Moreover, the introduction of D enables us to precisely express the form of boundary controls which realize the exact synchronization.

As in Li-Rao [5], we recall the following

Definition 1.1. Problem (1.3) is exactly synchronizable at the moment $T > 0$, if for any given initial data $(U_0, U_1) \in \big(L^2(\Omega) \times H^{-1}(\Omega)\big)^N$, there exist suitable boundary controls $H \in \big(L^2\big(0, +\infty; L^2(\Gamma_1)\big)\big)^N$ with compact support in $[0, T]$, such that the corresponding solution $U = U(t, x)$ satisfies the final condition

$$t \geq T\colon \qquad u^{(1)} \equiv \cdots \equiv u^{(N)} := u, \tag{1.5}$$

and u is called the corresponding exactly synchronizable state.

It is proved in Li-Rao [5] that if problem (1.3) is exactly synchronizable but not exactly null controllable, then the coupling matrix A should satisfy the following condition of compatibility:

$$\sum_{j=1}^{N} a_{ij} := \lambda \qquad (1 \leq i \leq N), \tag{1.6}$$

where λ is a constant independent of i. Moreover, under conditions (1.1) and (1.6), problem (1.3) is certainly exactly synchronizable by means of suitable boundary controls, provided that $T > 0$ is large enough.

Let C be the matrix of synchronization of type $(N-1) \times N$, defined by

$$C = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \cdot & \cdot & \\ & & & 1 & -1 \end{pmatrix}. \tag{1.7}$$

Then condition (1.6), which means that $\text{Ker}(C)$ is an invariant subspace of A, is equivalent to the existence of a matrix $\bar{A}$ of order $(N-1)$, such that

$$CA = \bar{A}C \tag{1.8}$$

(cf. [8]). On the other hand, it was proved in Li-Rao [6] that if the rank of D is less than N, then problem (1.3) is not exactly null controllable. Thus, in order to exclude the exact null controllability and to realize the exact synchronization essentially by means of $(N-1)$ boundary controls, in what follows we always assume that the rank of D is equal to $(N-1)$. The corresponding result of synchronization will be precisely given in Section 2.

If problem (1.3) is exactly synchronizable at the moment T, then the exactly synchronizable state u satisfies

$$t \geq T: \quad \begin{cases} u'' - \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.9}$$

where λ is given by (1.6). However, the value of (u, u') at $t = T$ should depend on the original initial data (U_0, U_1) as well as the boundary control H which realizes the exact synchronization.

It was pointed out in Li-Rao [5] that the attainable set of all possible values of (u, u') at $t = T$ is the whole space $L^2(\Omega) \times H^{-1}(\Omega)$ when the initial data (U_0, U_1) vary in the space $\left(L^2(\Omega) \times H^{-1}(\Omega)\right)^N$. In this paper, we will try to determine the exactly synchronizable state u for each given initial data (U_0, U_1).

The condition of compatibility (1.6) means that $e = (1, 1, \ldots, 1)^T$ is a right eigenvector of A, corresponding to the eigenvalue λ. Let E^T be a left eigenvector of A, corresponding to the same eigenvalue λ. Assume that $(E, e) = 1$ and D is chosen such that $E^T D = 0$. Then the exactly synchronizable state u is uniquely determined by $u = \phi$ for $t \geq T$, where ϕ is the solution to the following problem with homogeneous Dirichlet boundary condition:

$$\begin{cases} \phi'' - \Delta\phi + \lambda\phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma, \\ t = 0: \quad \phi = (E, U_0), \quad \phi' = (E, U_1). \end{cases} \tag{1.10}$$

Inversely, if the exactly synchronizable state u is independent of boundary control H, then we have necessarily $(E, e) = 1$. In that case, the exactly synchronizable state u is uniquely determined by the new initial data (E, U_0) and (E, U_1) which are the weighted averages of the original initial data U_0 and U_1 (see Theorems 3.1 and 3.2).

In the case that $(E, e) = 0$ for all left eigenvectors E^T of A, corresponding to the eigenvalue λ, we can find a Jordan chain $E_1, E_2, \dots, E_r$ $(r > 1)$ such that

$$A^T E_k = \lambda E_k + E_{k-1}, \qquad 1 \le k \le r \text{ with } E_0 = 0. \tag{1.11}$$

Then the exactly synchronizable state u can be determined by $u = \psi_r$ for $t \ge T$, where $(\psi_1, \dots, \psi_r)$ is the solution to the following system of problems $(1 \le k \le r)$ with $\psi_0 = 0$ (see Theorem 3.2):

$$\begin{cases} \psi_k'' - \Delta\psi_k + \lambda\psi_k + \psi_{k-1} = 0 & \text{in } \Omega, \\ \psi_k = 0 & \text{on } \Gamma_0, \\ \psi_k = E_k^T D H & \text{on } \Gamma_1, \\ t = 0: \quad \psi_k = (E_k, U_0), \quad \psi_k' = (E_k, U_1). \end{cases} \tag{1.12}$$

In this case, the exactly synchronizable state u depends not only on the new initial data (E_k, U_0) and (E_k, U_1) for $1 \le k \le r$, but also on the boundary control H, then it can not be uniquely determined in general. Nevertheless, when $C(U_0, U_1)$ is small, if $E_r^T D = 0$, then we have the following estimate:

$$\begin{aligned} t \ge T: \quad & \|(u, u')(t) - (\tilde{u}, \tilde{u}')(t)\|_{H_0^1(\Omega) \times L^2(\Omega)} \\ & \le c \|C(U_0, U_1)\|_{(L^2(\Omega) \times H^{-1}(\Omega))^{N-1}}, \end{aligned} \tag{1.13}$$

where $\tilde{u}$ is the solution to a wave equation with homogeneous Dirichlet boundary condition (see Theorem 4.1).

The paper is organized as follows. In Section 2, we first give some more general results on the exact synchronization, which complete the results obtained in Li-Rao [5]. In Section 3, the determination of the exactly synchronizable state is discussed. In Section 4, we give an estimate on the exactly synchronizable state in the general case. Finally, in Section 5, some remarks are given for the exact null controllability and synchronization by 2-groups.

§2. Complements for the exact synchronization

In this section we will give some more information on the exact synchronization discussed in Li-Rao [5].

Theorem 2.1. *Let*

$$\mathscr{D}_{N-1} = \{D \in \mathbb{M}^N(\mathbb{R}) : \mathrm{rank}(D) = \mathrm{rank}(CD) = N-1\}. \tag{2.1}$$

Assume that the geometric control condition (1.1) *and the condition of compatibility* (1.8) *are satisfied. Then for any given control matrix* $D \in \mathscr{D}_{N-1}$, *problem* (1.3) *is exactly synchronizable.*

Proof. Setting

$$W = CU, \qquad \overline{H} = CDH, \qquad W_0 = CU_0, \qquad W_1 = CU_1 \tag{2.2}$$

and noting (1.8), from problem (1.3) we get the following reduced problem:

$$\begin{cases} W'' - \Delta W + \bar{A}W = 0 & \text{in } \Omega, \\ W = 0 & \text{on } \Gamma_0, \\ W = \overline{H} & \text{on } \Gamma_1, \\ t = 0: \quad W = W_0, \quad W' = W_1. \end{cases} \tag{2.3}$$

It was proved in Li-Rao [5] that there exists $T_0 > 0$, such that for all $T > T_0$, the reduced problem (2.3) is exactly null controllable at the moment T for all $(W_0, W_1) \in (L^2(\Omega))^{N-1} \times (H^{-1}(\Omega))^{N-1}$ by means of some boundary controls $\overline{H} \in (L^2(0, T; L^2(\Gamma_1)))^{N-1}$.

On the other hand, since CD is of rank$(N-1)$, the linear system of H

$$CDH = \overline{H} \tag{2.4}$$

has always solutions, this yields the equivalence between the exact synchronization of (1.3) and the exact null controllability of the reduced problem (2.3). The proof is complete.

Theorem 2.2. *Let us denote by* $\mathscr{H}$ *the set of all the boundary controls* H *which realize the exact synchronization of problem* (1.3) *at the moment* T. *Assume that the geometric control condition* (1.1) *and the condition of compatibility* (1.8) *are satisfied. Then for* $\varepsilon > 0$ *small enough, the values of* $H \in \mathscr{H}$ *on* $]T-\varepsilon, T[\times \Gamma_1$ *can be arbitrarily chosen.*

Proof. First recall that there exists a positive constant $T_0 > 0$ independent of initial data such that for all $T > T_0$ the reduced problem (2.3) is exactly null controllable at the moment T.

Next let $\varepsilon > 0$ be such that $T - \varepsilon > T_0$ and

$$\hat{H}_\varepsilon \in L^2\big(T - \varepsilon, T; L^2(\Gamma_1)\big)^{N-1} \tag{2.5}$$

be arbitrarily given. We solve the reduced backward problem (2.3) to get a solution $\hat{W}_\varepsilon = \hat{W}_\varepsilon(t, x)$ on the time interval $[T - \varepsilon, T]$ with the boundary function $\overline{H} = \hat{H}_\varepsilon$ and the final data

$$t = T: \qquad \hat{W}_\varepsilon = \hat{W}_\varepsilon' = 0. \tag{2.6}$$

Since $T - \varepsilon > T_0$, the reduced problem (2.3) is still exactly controllable on the interval $[0, T - \varepsilon]$, then we can find a boundary control

$$\tilde{H}_\varepsilon \in L^2\big(0, T - \varepsilon; L^2(\Gamma_1)\big)^{N-1}, \tag{2.7}$$

such that the corresponding solution $\tilde{W}_\varepsilon$ satisfies the initial condition:

$$t = 0: \qquad \tilde{W}_\varepsilon = W_0, \qquad \tilde{W}_\varepsilon' = W_1 \tag{2.8}$$

and the final condition:

$$t = T - \varepsilon: \qquad \tilde{W}_\varepsilon = \hat{W}_\varepsilon, \qquad \tilde{W}_\varepsilon' = \hat{W}_\varepsilon'. \tag{2.9}$$

Thus, setting

$$\overline{H} = \begin{cases} \hat{H}_\varepsilon & t \in]T - \varepsilon, T[, \\ \tilde{H}_\varepsilon & t \in]0, T - \varepsilon[, \end{cases} \qquad W = \begin{cases} \hat{W}_\varepsilon & t \in]T - \varepsilon, T[, \\ \tilde{W}_\varepsilon & t \in]0, T - \varepsilon[, \end{cases} \tag{2.10}$$

we check easily that W is a weak solution of the reduced problem (2.3) and the boundary control $\overline{H}$ realizes the exact null controllability. By this way, we can construct a family of boundary controls $\overline{H}$ with arbitrarily given values on $]T - \varepsilon, T[\times \Gamma_1$.

Finally, once the controls $\overline{H}$ are found, we determine the corresponding controls H by solving the linear system (2.4). It is easy to see that the values of H on $]T - \varepsilon, T[\times \Gamma_1$ can be arbitrarily given. The proof is then complete.

Remark 2.1. Though the control H can not be uniquely determined, but the effective control DH is uniquely determined by the boundary control $\overline{H}$ for the exact null controllability of the reduced problem (2.3). More precisely, we have the following result.

Proposition 2.1. *For any given matrix $D \in \mathscr{D}_{N-1}$, the solutions H of linear system* (2.4) *satisfy*

$$DH = \hat{D}(C\hat{D})^{-1}\overline{H}, \tag{2.11}$$

where $\hat{D}$ is a full column-rank sub-matrix of D.

Proof. Since D is of rank$(N-1)$, there exists an invertible matrix Q such that

$$DQ = (\hat{D}, 0), \tag{2.12}$$

where $\hat{D}$ is a full column-rank sub-matrix of D. Accordingly, setting

$$Q^{-1}H = \begin{pmatrix} \hat{H} \\ h \end{pmatrix}, \tag{2.13}$$

the linear system (2.4) becomes

$$C\hat{D}\hat{H} = \overline{H}. \tag{2.14}$$

Since CD is of rank$(N-1)$, so is $C\hat{D}$, then it follows that

$$\hat{H} = (C\hat{D})^{-1}\overline{H}. \tag{2.15}$$

Using (2.12), (2.13) and (2.15), we have

$$DH = (\hat{D}, 0)Q^{-1}H = \hat{D}\hat{H} = \hat{D}(C\hat{D})^{-1}\overline{H}.$$

The proof is complete.

§3. Determination of the synchronizable part

Noting (1.6) or (1.8), $(1, 1, \dots, 1)^T$ is a right eigenvector of A, corresponding to the real eigenvalue λ given in (1.6). Let $e_1, e_2, \dots, e_r$ (resp. $E_1, E_2, \dots, E_r$) be a Jordan chain of length $r \geq 1$ of A (resp. A^T) corresponding to the eigenvalue λ, such that

$$\begin{cases} Ae_l = \lambda e_l + e_{l+1}, & 1 \leq l \leq r, \\ A^T E_k = \lambda E_k + E_{k-1}, & 1 \leq k \leq r, \\ (E_k, e_l) = \delta_{kl}, & 1 \leq k, l \leq r, \\ e_r = (1, 1 \dots, 1)^T, \quad e_{r+1} = 0, \quad E_0 = 0. \end{cases} \tag{3.1}$$

Consider the projection P on the bi-orthogonal systems $e_1, e_2, \dots, e_r$ and $E_1, E_2, \dots, E_r$ as follows:

$$P = \sum_{k=1}^{r} e_k \otimes E_k, \tag{3.2}$$

where

$$(e \otimes E)U = (E, U)e = E^T U e, \qquad \forall U \in \mathbb{R}^N. \tag{3.3}$$

P can be represented by a matrix of order N. We can decompose

$$\mathbb{R}^N = \mathrm{Im}(P) \oplus \mathrm{Ker}(P). \tag{3.4}$$

Moreover, we have

$$\mathrm{Im}(P) = \mathrm{Span}\{e_1, e_2, \dots, e_r\}, \qquad \mathrm{Ker}(P) = (\mathrm{Span}\{E_1, E_2, \dots, E_r\})^{\perp} \tag{3.5}$$

and

$$PA = AP. \tag{3.6}$$

Now let $U = U(t, x)$ be the solution to problem (1.3). We define

$$U_c := (I - P)U, \qquad U_s := PU. \tag{3.7}$$

If problem (1.3) is exactly synchronizable, we have

$$t \geq T: \qquad U = ue_r, \tag{3.8}$$

where $u = u(t, x)$ is the exactly synchronizable state and $e_r = (1, \dots, 1)^T$. Then, noting (3.5), we have

$$t \geq T: \qquad U_c = u(I - P)e_r = 0, \qquad U_s = uPe_r = ue_r. \tag{3.9}$$

Thus U_c and U_s will be called the controllable part and the synchronoizable part of U, respectively.

Lemma 3.1. *The controllable part U_c is the solution to the following system:*

$$\begin{cases} U_c'' - \Delta U_c + AU_c = 0 & \text{in } \Omega, \\ U_c = 0 & \text{on } \Gamma_0, \\ U_c = (I - P)DH & \text{on } \Gamma_1, \\ t = 0: \quad U_c = (I - P)U_0, \quad U_c' = (I - P)U_1, \end{cases} \tag{3.10}$$

while, the synchronoizable part U_s is the solution to the following system:

$$\begin{cases} U_s'' - \Delta U_s + AU_s = 0 & \text{in } \Omega, \\ U_s = 0 & \text{on } \Gamma_0, \\ U_s = PDH & \text{on } \Gamma_1, \\ t = 0: \quad U_s = PU_0, \quad U_s' = PU_1. \end{cases} \tag{3.11}$$

Proof. Noting (3.6) and applying the projection P on problem (1.3), we get immediately (3.10) and (3.11).

Remark 3.1. In fact, the boundary control H realizes the exact null controllability for U_c with initial data $\big((I-P)U_0, (I-P)U_1\big) \in \mathrm{Ker}(P) \times \mathrm{Ker}(P)$ on one hand, and the exact synchronization for U_s with the initial data $(PU_0, PU_1) \in \mathrm{Im}(P) \times \mathrm{Im}(P)$ on the other hand.

Lemma 3.2. *There exists a control matrix $D \in \mathscr{D}_{N-1}$ such that $E_r^T D = 0$.*

Proof. Let D be a matrix of order N such that

$$\mathrm{Im}(D) = (\mathrm{Span}\{E_r\})^{\perp}. \tag{3.12}$$

Clearly, we have $\mathrm{rank}(D) = N-1$ and $E_r^T D = 0$.

We next show that $\mathrm{rank}(CD) = N-1$. Let $x \in \mathrm{Ker}(CD)$. We have

$$Dx \in \mathrm{Ker}(C) = \mathrm{Span}\{e_r\}, \tag{3.13}$$

then there exists a real number α such that

$$Dx = \alpha e_r. \tag{3.14}$$

Since $E_r^T D = 0$ and $(E_r, e_r) = 1$, taking the inner product of (3.14) with E_r, we get $\alpha = 0$, then $x \in \mathrm{Ker}(D)$. Thus, it is easy to see that $\mathrm{Ker}(CD) = \mathrm{Ker}(D)$. It follows that

$$\mathrm{rank}(CD) = N - \dim \mathrm{Ker}(CD) = N - \dim \mathrm{Ker}(D) = N-1. \tag{3.15}$$

The proof is complete.

Theorem 3.1. *Let the projection P be defined by* (3.2). *Assume that problem* (1.3) *is exactly synchronizable. When $r = 1$, we can take a control matrix $D \in \mathscr{D}_{N-1}$ such that $PD = 0$, then the synchronizable part U_s is independent of boundary controls*

H. Inversely, if the synchronizable part U_s is independent of boundary controls H, then we should have

$$r = 1 \quad \text{and} \quad PD = 0. \tag{3.16}$$

In particular, if $PU_0 = PU_1 = 0$, then problem (1.3) *is exactly null controllable for the initial data (U_0, U_1) of this kind.*

Proof. When $r = 1$, by Lemma 3.2, we can take a control matrix $D \in \mathscr{D}_{N-1}$ such that $E_1^T D = 0$. From (3.5), we have $\mathrm{Ker}(P) = (\mathrm{Span}\{E_1\})^\perp = \mathrm{Im}(D)$, then $PD = 0$. Therefore (3.11) becomes a problem with homogeneous Dirichlet boundary condition, then the solution U_s is independent of boundary controls H.

Inversely, let H_1 and H_2 be two boundary controls which realize simultaneously the exact synchronization of (1.3). If the corresponding solution U_s to (3.11) is independent of the boundary controls H_1 and H_2, then, noting Lemma 3.1, we have

$$PD(H_1 - H_2) = 0 \quad \text{on }]0, T[\times \Gamma_1. \tag{3.17}$$

By Theorem 2.2, the values of $(H_1 - H_2)$ on $]T - \varepsilon, T[\times \Gamma_1$ can be arbitrarily chosen, this yields that $PD = 0$. It follows that

$$\mathrm{Im}(D) \subseteq \mathrm{Ker}(P). \tag{3.18}$$

Noting (3.5), we have $\dim \mathrm{Ker}(P) = N - r$ and $\dim \mathrm{Im}(D) = N - 1$, then, we have necessarily $r = 1$. The proof is complete.

Corollary 3.1. *Assume that* $\mathrm{Ker}(C)$ *and* $\mathrm{Im}(C^T)$ *are simultaneously invariant subspaces of A. Then there exists a control matrix $D \in \mathscr{D}_{N-1}$ such that problem* (1.3) *is exactly synchronizable and the synchronizable part U_s is independent of boundary controls H.*

Proof. Recall that $\mathrm{Ker}(C) = \mathrm{Span}\{e\}$ with $e = (1, 1, \ldots, 1)^T$. Since $\mathrm{Ker}(C)$ is an invariant subspace of A, the condition of compatibility (1.8) holds. Then by Theorem 2.1, problem (1.3) is exactly synchronizable for any given control matrix $D \in \mathscr{D}_{N-1}$. On the other hand, $\mathrm{Im}(C^T)$ being an invariant subspace of A, $\left(\mathrm{Im}(C^T)\right)^\perp = \mathrm{Ker}(C)$ is an invariant subspace of A^T. Thus, e^T is a left eigenvector of A corresponding to the same eigenvalue λ given by (1.6), then taking $E = e/N$, we have $(E, e) = 1$ and then $r = 1$. Thus by Theorem 3.1 we can chose a control matrix $D \in \mathscr{D}_{N-1}$ such that the synchronizable part U_s of problem (1.3) is independent of boundary controls H. The proof is complete.

Remark 3.2. The condition $r = 1$ means that λ is a single eigenvalue of A, or equivalently, there exists a left eigenvector E^T of A, such that

$$(E, e) = 1. \tag{3.19}$$

Clearly, if A is symmetric or A^T satisfies also the condition of compatibility (1.6), then e is also an eigenvector of A^T. Consequently, we can take $E = e/N$ such that $(E, e) = 1$. However, this condition is not always satisfied for any given matrix A. For example, let

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.20}$$

We have

$$\lambda = 1, \qquad e = (1, 1)^T, \qquad E = (1, -1)^T, \tag{3.21}$$

then

$$(E, e) = 0. \tag{3.22}$$

In general, if (3.22) holds, then

$$E \in (\mathrm{Span}\{e\})^{\perp} = \big(\mathrm{Ker}(C)\big)^{\perp} = \mathrm{Im}(C^T). \tag{3.23}$$

This means that (E, U) is just a combination of CU, therefore it does not provide any new information for the synchronizable part U_s.

Remark 3.3. Let $(\hat{c}_1, \ldots, \hat{c}_{N-1})$ be a basis of $(\mathrm{Span}\{E\})^{\perp}$. Then

$$A(e, \hat{c}_1, \ldots, \hat{c}_{N-1}) = (e, \hat{c}_1, \ldots, \hat{c}_{N-1}) \begin{pmatrix} \lambda & 0 \\ 0 & A_{22} \end{pmatrix}, \tag{3.24}$$

where A_{22} is a matrix of order $(N-1)$. Therefore, A is diagonalizable by blocks under the basis $(e, \hat{c}_1, \ldots, \hat{c}_{N-1})$.

We next discuss the general case $r \geq 1$. Let us denote

$$\psi_k = (E_k, U), \qquad 1 \leq k \leq r \tag{3.25}$$

and write

$$U_s = \sum_{k=1}^{r} (E_k, U) e_k = \sum_{k=1}^{r} \psi_k e_k. \tag{3.26}$$

Then, $(\psi_1, \dots, \psi_r)$ are the coordinates of U_s on the bi-orthogonal basis $e_1, e_2, \dots, e_r$ and $E_1, E_2, \dots, E_r$.

Theorem 3.2. *Let $e_1, e_2, \dots, e_r$ (resp. $E_1, E_2, \dots, E_r$) be a Jordan chain of A (resp. A^T) corresponding to the eigenvalue λ and $e_r = (1, \dots, 1)^T$. Then the synchronizable part $U_s = (\psi_1, \dots, \psi_r)$ can be determined by the solution of the following system $(1 \le k \le r)$:*

$$\begin{cases} \psi_k'' - \Delta\psi_k + \lambda\psi_k + \psi_{k-1} = 0 & \text{in } \Omega, \\ \psi_k = 0 & \text{on } \Gamma_0, \\ \psi_k = h_k & \text{on } \Gamma_1, \\ t = 0\colon \quad \psi_k = (E_k, U_0), \quad \psi_k' = (E_k, U_1), \end{cases} \tag{3.27}$$

where

$$\psi_0 = 0 \quad \text{and} \quad h_k = E_k^T DH. \tag{3.28}$$

Moreover, the exactly synchronizable state is given by $u = \psi_r$ for $t \ge T$.

Proof. First, for $1 \le k \le r$, we have

$$(E_k, U) = (E_k, U_s) = \psi_k, \tag{3.29}$$

$$E_k^T PDH = \sum_{l=1}^{r} (E_l, DH)(E_k, e_l) = (E_k, DH) = E_k^T DH, \tag{3.30}$$

$$E_k^T PU_0 = (E_k, U_0), \qquad E_k^T PU_1 = (E_k, U_1). \tag{3.31}$$

Taking the inner product of (3.11) with E_k, we get (3.27)–(3.28).

On the other hand, noting (3.8), we have

$$t \ge T\colon \quad \psi_k(t) = \big(E_k, U(t)\big) = \big(E_k, u(t)e_r\big) = u(t)\delta_{kr}, \quad 1 \le k \le r. \tag{3.32}$$

Thus, the exactly synchronizable state u is given by

$$t \ge T\colon \quad u = u(t, x) = \psi_r(t, x). \tag{3.33}$$

The proof is complete.

In the special case $r = 1$, by Theorems 3.1 and 3.2, we have

Corollary 3.2. *When $r = 1$, we can take $D \in \mathscr{D}_{N-1}$ such that $E_1^T D = 0$. Then the exactly synchronizable state u is determined by $u = \phi$ for $t \ge T$, where ϕ is the solution of the following problem with homogeneous Dirichlet boundary condition:*

$$\begin{cases} \phi'' - \Delta\phi + \lambda\phi = 0 & in\ \Omega, \\ \phi = 0 & on\ \Gamma, \\ t = 0: \quad \phi = (E_1, U_0), \quad \phi' = (E_1, U_1). \end{cases} \tag{3.34}$$

Inversely, if the synchronizable part $U_s = (\psi_1, \dots, \psi_r)$ is independent of the boundary controls H, then we have necessarily

$$r = 1 \quad \text{and} \quad E_1^T D = 0. \tag{3.35}$$

Consequently, the exactly synchronizable state u is given by $u = \phi$ for $t \geq T$, where ϕ is the solution of (3.34). In particular, if

$$(E_1, U_0) = (E_1, U_1) = 0, \tag{3.36}$$

then problem (1.3) is exactly null controllable for such initial data (U_0, U_1).

§4. Approximation of the exactly synchronizable state

The relation (3.32) shows that only the last component ψ_r is synchronized, while, the others are steered to zero. However, in order to get ψ_r, we have to solve the whole system (3.27)–(3.28) for $(\psi_1, \dots, \psi_r)$. Therefore, except in the case $r = 1$, the exactly synchronizable state u depends on the boundary controls which realize the exact synchronization, and then, generically speaking, one can not uniquely determine the exactly synchronizable state u. However, we have the following result.

Theorem 4.1. *Assume that* (1.3) *is exactly synchronizable by means of some control matrix* $D \in \mathscr{D}_{N-1}$. *Let* $\tilde{u}$ *be the solution of the following homogeneous problem:*

$$\begin{cases} \tilde{u}'' - \Delta\tilde{u} + \lambda\tilde{u} = 0 & in\ \Omega, \\ \tilde{u} = 0 & on\ \Gamma, \\ t = 0: \quad \tilde{u} = (E_r, U_0), \quad \tilde{u}' = (E_r, U_1). \end{cases} \tag{4.1}$$

Assume furthermore that

$$E_r^T D = 0. \tag{4.2}$$

Then there exists a positive constant $c > 0$ *such that the exactly synchrizable state* u *satisfies the following estimate:*

$$\begin{aligned} t \geq T: \quad & \|(u, u')(t) - (\tilde{u}, \tilde{u}')(t)\|_{H_0^1(\Omega)\times L^2(\Omega)} \\ & \leq c\|C(U_0, U_1)\|_{(L^2(\Omega)\times H^{-1}(\Omega))^{N-1}}. \end{aligned} \tag{4.3}$$

Proof. By Lemma 3.2, we can take a control matrix $D \in \mathscr{D}_{N-1}$ such that (4.2) is satisfied. Then considering the r-th equation of (3.27), we get the following problem with homogeneous Dirichlet boundary condition:

$$\begin{cases} \psi_r'' - \Delta\psi_r + \lambda\psi_r = -\psi_{r-1} & \text{in } \Omega, \\ \psi_r = 0 & \text{on } \Gamma, \\ t = 0: \quad \psi_r = (E_r, U_0), \quad \psi_r' = (E_r, U_1). \end{cases} \tag{4.4}$$

From (3.32) we have

$$t \geq T: \qquad \psi_r(t) = u(t), \qquad \psi_{r-1}(t) \equiv 0. \tag{4.5}$$

Noting that problems (4.1) and (4.4) have the same initial data and the same homogeneous Dirichlet boundary condition, by the well-posedness, there exists a positive constant $c_1 > 0$ such that

$$t \geq T: \quad \|(u, u')(t) - (\tilde{u}, \tilde{u}')(t)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq c_1 \int_0^T \|\psi_{r-1}(s)\|^2_{L^2(\Omega)}\, ds. \tag{4.6}$$

But the condition $(E_{r-1}, e_r) = 0$ implies that

$$E_{r-1} \in (\mathrm{Span}\{e_r\})^{\perp} = \big(\mathrm{Ker}(C)\big)^{\perp} = \mathrm{Im}(C^T),$$

then E_{r-1} is a combination of the rows of C. Therefore, there exists a positive constant $c_2 > 0$ such that

$$\|\psi_{r-1}(s)\|^2_{L^2(\Omega)} = \|(E_{r-1}, U)(s)\|^2_{L^2(\Omega)} \leq c_2 \|CU(s)\|^2_{(L^2(\Omega))^{N-1}}. \tag{4.7}$$

But $CU = W$, due to the exact null controllability of the reduced problem (2.3), there exists a positive constant $c_3 > 0$ such that

$$\int_0^T \|CU(s)\|^2_{(L^2(\Omega))^{N-1}}\, ds \leq c_3 \|C(U_0, U_1)\|^2_{(L^2(\Omega)\times H^{-1}(\Omega))^{N-1}}. \tag{4.8}$$

Inserting (4.7)–(4.8) into (4.6), we get (4.3). The proof is complete.

Corollary 4.1. *Let $S \in \mathscr{S}_N$ be a permutation of $\{1, 2, \dots, N\}$. Under the same conditions as in Theorem* 4.1, *we have*

$$\|(u, u')(t) - (\tilde{u}, \tilde{u}')(t)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq c \min_{S \in \mathscr{S}_N} \|CS(U_0, U_1)\|_{(L^2(\Omega)\times H^{-1}(\Omega))^{N-1}} \tag{4.9}$$

for all $t \geq T$.

Proof. In fact, the synchronization condition $CU = 0$ is equivalent to $CSU = 0$ for any given $S \in \mathscr{S}_N$. Then (4.9) is a direct consequence of (4.3).

Remark 4.1. When $(E, e) = 0$, the exactly synchronizable state u of problem (1.3) can not be determined independently of boundary controls H. Nevertheless, by Theorem 4.1, when $C(U_0, U_1)$ is suitably small, then u is closed to the solution of (4.1) whose initial data are given by the weighted average of the original initial data (U_0, U_1) with the weight E_r (a root vector of A^T).

§5. Remarks on the exact null controllability and synchronization by 2-groups

Now let us rearrange the components of U in 2-groups:

$$(u^{(1)}, \ldots, u^{(m)}), \qquad (u^{(m+1)}, \ldots, u^{(N)}). \tag{5.1}$$

For any given initial data $(U_0, U_1) \in \left(L^2(\Omega)\right)^N \times \left(H^{-1}(\Omega)\right)^N$, if there exist suitable boundary controls $H \in \left(L^2(0, +\infty; L^2(\Gamma_1))\right)^N$ with compact support in $[0, T]$, such that the first group is exactly null controllable and the second one is exactly synchronizable:

$$t \geq T: \quad u^{(1)} \equiv \cdots \equiv u^{(m)} \equiv 0, \qquad u^{(m+1)} \equiv \cdots \equiv u^{(N)} := u, \tag{5.2}$$

then, we say that problem (1.3) is exactly null controllable and synchronizable by 2-groups at the moment $T > 0$, and $u = u(t, x)$ is called the partially synchronizable state.

It was proved in Li-Rao [5] that when the rank of D is less than N and if problem (1.3) is exactly null controllable and synchronizable by 2-groups, then we have the following conditions of compatibility:

$$\begin{cases} \displaystyle\sum_{p=m+1}^{N} a_{kp} = 0, & k = 1, \ldots, m, \\ \displaystyle\sum_{p=m+1}^{N} a_{kp} := \tilde{\lambda}, & k = m+1, \ldots, N, \end{cases} \tag{5.3}$$

where $\tilde{\lambda}$ is a constant independent of $k = m+1, \ldots, N$. Inversely, assume that (5.3) hold, then there exists a control matrix D of rank$(N-1)$ such that problem (1.3) is exactly null controllable and synchronizable by 2-groups.

Let

$$\tilde{e} = (\overbrace{0, 0, \ldots, 0}^{m}, \overbrace{1, 1, \ldots, 1}^{N-m})^T. \tag{5.4}$$

The conditions of compatibility (5.3) imply that $\tilde{e}$ is an eigenvector of A, corresponding to the eigenvalue $\tilde{\lambda}$ given in (5.3). Moreover, there exists a Jordan chain $\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_r$ (resp. $\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_r$) of A (resp. A^T), corresponding to the eigenvalue $\tilde{\lambda}$, such that

$$\begin{cases} A\tilde{e}_l = \tilde{\lambda}\tilde{e}_l + \tilde{e}_{l+1}, & 1 \le l \le r, \\ A^T\tilde{E}_k = \tilde{\lambda}\tilde{E}_k + \tilde{E}_{k-1}, & 1 \le k \le r, \\ (\tilde{E}_k, \tilde{e}_l) = \delta_{kl}, & 1 \le k, l \le r, \\ \tilde{e}_r = (0, \dots, 0, 1, \dots, 1)^T, \quad \tilde{e}_{r+1} = 0, \quad \tilde{E}_0 = 0. \end{cases} \tag{5.5}$$

Noting that the partially synchronizable state u satisfies

$$t \ge T: \qquad U = u\tilde{e}, \tag{5.6}$$

similar results as in Theorems 3.1, 3.2 and 4.1 can be easily established as follows.

Theorem 5.1. *Assume that* (1.3) *is exactly null controllable and synchronizable by 2-groups. Then the partially synchronizable state u can be determined by $u = \tilde{\psi}_r$ for $t \ge T$, where $(\tilde{\psi}_1, \dots, \tilde{\psi}_r)$ is the solution to the following system of problems $(1 \le k \le r)$:*

$$\begin{cases} \tilde{\psi}_k'' - \Delta\tilde{\psi}_k + \tilde{\lambda}\tilde{\psi}_k + \tilde{\psi}_{k-1} = 0 & \text{in } \Omega, \\ \tilde{\psi}_k = 0 & \text{on } \Gamma_0, \\ \tilde{\psi}_k = \tilde{h}_k & \text{on } \Gamma_1, \\ t = 0: \quad \tilde{\psi}_k = (\tilde{E}_k, U_0), \quad \tilde{\psi}_k' = (\tilde{E}_k, U_1), \end{cases} \tag{5.7}$$

where

$$\tilde{\psi}_0 = 0 \qquad \text{and} \qquad \tilde{h}_k = \tilde{E}_k^T DH. \tag{5.8}$$

Moreover, when $r = 1$, we can chose a control matrix D such that

$$\tilde{E}_1^T D = 0. \tag{5.9}$$

Then the partially synchronizable state u is uniquely given by $u = \tilde{\phi}$ for $t \ge T$, where $\tilde{\phi}$ is the solution to the following problem with homogeneous Dirichlet boundary condition:

$$\begin{cases} \tilde{\phi}'' - \Delta\tilde{\phi} + \tilde{\lambda}\tilde{\phi} = 0 & \text{in } \Omega, \\ \tilde{\phi} = 0 & \text{on } \Gamma, \\ t = 0: \quad \tilde{\phi} = (\tilde{E}_1, U_0), \quad \tilde{\phi}' = (\tilde{E}_1, U_1). \end{cases} \tag{5.10}$$

Inversely, if the solution $(\tilde{\psi}_1, \ldots, \tilde{\psi}_r)$ of (5.7)–(5.8) is independent of boundary controls H, then we have necessarily

$$r = 1 \quad \text{and} \quad \tilde{E}_1^T D = 0. \tag{5.11}$$

Consequently, the partially synchronizable state u is determined by $u = \tilde{\phi}$ for $t \geq T$, where $\tilde{\phi}$ is the solution of (5.10). In particular, if

$$(\tilde{E}_1, U_0) = (\tilde{E}_1, U_1) = 0, \tag{5.12}$$

then problem (1.3) is exactly null controllable for such initial data (U_0, U_1).

Theorem 5.2. *Assume that* (1.3) *is exactly null controllable and synchronizable by* 2*-groups. Assume that*

$$\tilde{E}_r^T D = 0 \tag{5.13}$$

and let $\tilde{u}$ *be the solution of the following problem with homogeneous Dirichlet boundary condition:*

$$\begin{cases} \tilde{u}'' - \Delta\tilde{u} + \tilde{\lambda}\tilde{u} = 0 & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \Gamma, \\ t = 0: \quad \tilde{u} = (\tilde{E}_r, U_0), \quad \tilde{u}' = (\tilde{E}_r, U_1). \end{cases} \tag{5.14}$$

Then there exists a positive constant $c > 0$ *such that the partially synchrizable state* u *satisfies the following estimate:*

$$\begin{aligned} t \geq T: \quad & \|(u, u')(t) - (\tilde{u}, \tilde{u}')(t)\|_{H_0^1(\Omega)\times L^2(\Omega)} \\ & \leq c\|C(U_0, U_1)\|_{(L^2(\Omega)\times H^{-1}(\Omega))^{N-1}}. \end{aligned} \tag{5.15}$$

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