

## A bound on the group velocity for Bloch wave packets

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Dedicated to our friend and colleague Joao Paulo de Carvalho Dias  
celebrating years of comradery

**Abstract.** We give a direct proof that the group velocities of Bloch wave packet solutions of periodic second order wave equations cannot exceed the maximal speed of propagation of the periodic wave equation.

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### 1. Main result

This note complements our studies [2], [3] of Bloch wave packets. Bloch wave packets are short wavelength asymptotic solutions of periodic wave equations and their perturbations. They propagate with a group velocity that is given by the gradient of the Bloch dispersion relation. The dispersion in turn is defined by the Bloch eigenvalues of a periodic elliptic partial differential operator. Using the rigorously established asymptotic solutions one can conclude that these group velocities cannot exceed the maximal speeds of the periodic wave equation. In this note we give a direct proof that the group velocity is no larger than that maximal speed. This is reminiscent of the pair of papers of Lax [9] and Weinberger [11]. The first derives a result about matrix eigenvalues from properties of hyperbolic partial differential equations. The second gives a direct algebraic proof.

With the notations from [3],  $\mathbb{T}^N$  denotes the unit torus and for periodic symmetric matrix valued  $A_0(y)$  and scalar valued  $\rho_0(y)$  both in  $L^\infty(\mathbb{T}^N)$  the central object is the wave equation

$$\rho_0(y)u_{tt} - \operatorname{div}_y(A_0(y) \operatorname{grad}_y u) = 0. \quad (1.1)$$

For any fixed  $y \in \mathbb{T}^N$  the maximal speed of propagation for (1.1) at  $y$  is given by

$$c(y) := \max_{1 \leq j \leq N} \sqrt{\mu_j(y)}$$

where  $\mu_j(y) > 0$  are the roots of the polynomial

$$p(y, \mu) := \det(A_0(y) - \mu \rho_0(y)I).$$

The maximal speed of propagation is

$$c_{\max} := \max_{y \in \mathbb{T}^N} c(y). \quad (1.2)$$

Denote the Bloch parameter  $\theta \in [0, 1]^N$ . The Bloch spectral cell problem for  $\psi_n$  periodic in  $y$  is

$$-(\operatorname{div}_y + 2i\pi\theta)(A_0(y)(\operatorname{grad}_y + 2i\pi\theta)\psi_n) = \lambda_n(\theta)\rho_0(y)\psi_n \quad \text{in } \mathbb{T}^N. \quad (1.3)$$

Assuming that  $A_0(y)$  is symmetric and uniformly coercive and that  $\rho_0(y)$  is uniformly bounded away from zero, it is well known that (1.3) admits a countable infinite family of positive real eigenvalues  $\lambda_n(\theta)$  (repeated according to their multiplicity) and associated eigenfunctions  $\psi_n(\theta, y)$  that, as functions of  $y$ , belong to  $H^1(\mathbb{T}^N)$  [5], [6], [10], [12]. The eigenvalues, labeled in increasing order, are uniformly Lipschitzian as functions of  $\theta$ . They are not more regular because of possible crossings. Simple eigenvalues are analytic functions of  $\theta$  [8]. Simple eigenvalues are generic [1]. Normalize the eigenfunctions by

$$\int \rho_0(y) |\psi_n(y, \theta)|^2 dy = 1. \quad (1.4)$$

**Assumption.** Fix  $\theta_0 \in [0, 1]^N$ ,  $n \in \mathbb{N}$  and assume that  $\lambda_n(\theta_0)$  is a **simple** eigenvalue.

Define the associated nonnegative frequency  $\omega_n(\theta_0)$  as the solution of the dispersion relation

$$4\pi^2 \omega_n^2(\theta) = \lambda_n(\theta), \quad \omega_n \geq 0. \quad (1.5)$$

Corresponding plane wave solutions of (1.1) are

$$e^{2\pi i(\omega_n(\theta)t + y \cdot \theta)} \psi_n(y, \theta).$$

Superposition in  $\theta$  yields wave packets with group velocity defined by (see [3])

$$\mathcal{V} := -\nabla_\theta \omega_n(\theta_0) = \frac{-\nabla_\theta \lambda_n(\theta_0)}{4\pi \sqrt{\lambda_n(\theta_0)}}. \quad (1.6)$$

**Theorem 1.1.** *The group velocity defined by (1.6) and the maximal speed (1.2) satisfy*

$$|\mathcal{V}| \leq c_{\max}.$$

*Proof.* Introduce the operator

$$\mathbb{A}(\theta)\psi := -(\operatorname{div}_y + 2i\pi\theta)(A_0(y)(\operatorname{grad}_y + 2i\pi\theta)\psi) - \lambda_n(\theta)\rho_0(y)\psi.$$

Equation (1.3) asserts that  $\mathbb{A}(\theta)\psi_n = 0$ . At the point  $\theta_0$ , differentiate (1.3) with respect to  $\theta$  in the direction  $\xi$  to find

$$(\xi \cdot \nabla_\theta \mathbb{A}(\theta))\psi_n(\theta) + \mathbb{A}(\theta)(\xi \cdot \nabla_\theta \psi_n(\theta)) = 0.$$

Therefore

$$\begin{aligned} \mathbb{A}(\theta)\xi \cdot \nabla_\theta \psi_n &= 2i\pi\xi \cdot A_0(y)(\nabla_y + 2i\pi\theta)\psi_n \\ &\quad + 2i\pi(\operatorname{div}_y + 2i\pi\theta)(A_0(y)\xi\psi_n) + \xi \cdot \nabla_\theta(\lambda_n\rho_0(y)\psi_n). \end{aligned} \quad (1.7)$$

Multiply (1.7) by  $\overline{\psi_n}$  and integrate  $dy$ . Equation (1.4) shows that

$$\xi \cdot \nabla_\theta \int \overline{\psi_n}(y, \theta)\rho(y)\psi_n(y, \theta) dy = 0. \quad (1.8)$$

Therefore

$$\begin{aligned} \xi \cdot \nabla_\theta \lambda_n(\theta) &= 2i\pi \int_{\mathbb{T}^N} (\psi_n A_0(y)\xi \cdot \overline{(\nabla_y + 2i\pi\theta)\psi_n} - \overline{\psi_n}\xi \cdot A_0(y)(\nabla_y + 2i\pi\theta)\psi_n) dy \\ &= 4i\pi \operatorname{Re} \int_{\mathbb{T}^N} (\psi_n A_0(y)\xi \cdot \overline{(\nabla_y + 2i\pi\theta)\psi_n}) dy. \end{aligned}$$

Write the integrand as the product of three terms,

$$(\rho_0^{1/2}\psi_n)(\rho_0^{-1/2}A_0(y)^{1/2}\xi) \cdot (A_0(y)^{1/2}\overline{(\nabla_y + 2i\pi\theta)\psi_n}).$$

Constrain to unit vectors  $\xi$  and estimate for each  $y$  using the Euclidean norm on vectors and the induced matrix norm,

$$\|\rho_0^{-1/2}A_0(y)^{1/2}\xi\| \leq \|\rho_0^{-1/2}A_0(y)^{1/2}\| \|\xi\| = \|(\rho_0^{-1/2}A_0(y)^{1/2})^2\|^{1/2} = c(y) \leq c_{\max}.$$

The Cauchy-Schwartz inequality estimates the integral expression for  $\xi \cdot \nabla_\theta \lambda_n$  by

$$|\xi \cdot \nabla_\theta \lambda_n(\theta)| \leq 4\pi c_{\max} \|\rho_0^{1/2} \psi_n\|_{L^2(\mathbb{T}^N)} \|A_0^{1/2} (\nabla_y + 2i\pi\theta) \psi_n\|_{L^2(\mathbb{T}^N)}.$$

Using the normalization (1.4) and then the eigenvalue equation yields

$$\begin{aligned} |\xi \cdot \nabla_\theta \lambda_n(\theta)| &\leq 4\pi c_{\max} \|A_0^{1/2} (\nabla_y + 2i\pi\theta) \psi_n\|_{L^2(\mathbb{T}^N)} \\ &= 4\pi c_{\max} \sqrt{\lambda_n(\theta)}. \end{aligned} \quad (1.9)$$

Since  $\xi$  is an arbitrary unit vector, this estimate together with (1.6) implies the Theorem.  $\square$

**Remark 1.2.** i. Estimate (1.9) yields a bound for the propagation speed as a function of direction by the corresponding fastest speeds of the original system (see [4] for an analogous result).

ii. The simplicity of  $\lambda_n$  is only used to yield smoothness of  $\lambda_n$ . The proof applies without modification away from eigenvalue crossings or for directional derivatives when crossing occurs.

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