Portugal. Math. (N.S.)
Vol. 72, Fasc. 2-3, 2015, 125–144
DOI 10.4171/PM/1961

Portugaliae Mathematica

On the Cauchy problem for the evolution $p(x)$-Laplace equation

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(Communicated by Hugo Beirão da Veiga and José Francisco Rodrigues)

Dedicated to Professor João Paulo de Carvalho Dias on the occasion of his 70th birthday

Abstract. We consider the Cauchy problem for the equation

$$u_t - \operatorname{div}\big(a(x,t)|\nabla u|^{p(x)-2}\nabla u\big) = f(x,t) \qquad \text{in } S_T = \mathbb{R}^n \times (0,T)$$

with measurable but possibly discontinuous variable exponent $p(x) : \mathbb{R}^n \mapsto [p^-, p^+] \subset (1,\infty)$. It is shown that for every $u(x,0) \in L^2(\mathbb{R}^n)$ and $f \in L^2(S_T)$ the problem has at least one weak solution $u \in C^0\big([0,T]; L^2_{loc}(\mathbb{R}^n)\big) \cap L^2(S_T)$, $|\nabla u|^{p(x)} \in L^1(S_T)$. We derive sufficient conditions for global boundedness of weak solutions and show that the bounded weak solution is unique.

Mathematics Subject Classification (primary; secondary): 35K92; 35K15

Keywords: Nonlinear parabolic equation, variable nonlinearity, $p(x)$-Laplace, Cauchy problem

1. Statement of the problem and results

Let us consider the Cauchy problem for the $p(x)$-Laplace equation

$$\begin{cases} u_t - \operatorname{div} \mathscr{A}(z, \nabla u) = f(z) & \text{in } S_T = \mathbb{R}^n \times (0,T], \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^n), \end{cases} \tag{1}$$

*The first author was partially supported by the Research Projects Grant No. 15-11-20019, Russian Science Foundation, Russia, and MTM2011-26119, MICINN, Spain.

**The second author acknowledges the support of the Research Project MTM2010-18427, MICINN, Spain.

‡Both authors were partially supported by the Research Grants CAPES-PVE-88887.059583/2014-00 and 88881.0303888/2013-01 (Brazil)

where $\mathscr{A}(z, \nabla u) = a(z)|\nabla u|^{p(x)-2}\nabla u$ and $z = (x, t)$ denotes the point of S_T. It is assumed that

$$\begin{cases} a(z) \text{ is a measurable function such that} \\ 0 < a_- \le a(z) \le a_+ < \infty \text{ for a.e. } z \in S_T \end{cases} \tag{2}$$

with some constants $a_\pm$,

$$\begin{cases} p : \mathbb{R}^n \mapsto \mathbb{R} \text{ is a measurable function with} \\ p^- = \operatorname{ess\,inf}_{\mathbb{R}^n} p(x) > 1, \quad p^+ = \operatorname{ess\,sup}_{\mathbb{R}^n} p(x) < \infty. \end{cases} \tag{3}$$

The continuity of $p(x)$ is not required. By $C_0^\infty(\mathbb{R}^n)$ we denote the space of infinitely differentiable functions with compact support and define the space W as the closure of $C^\infty\big([0, T]; C_0^\infty(\mathbb{R}^n)\big)$ with respect to the norm

$$\|u\|_W = \|u\|_{2, S_T} + \|\nabla u\|_{p(\cdot), S_T},$$

where $\|\cdot\|_{p(\cdot), S_T}$ denotes the Luxemburg norm

$$\|h\|_{p(\cdot), S_T} = \inf\Big\{\lambda > 0 : \int_{S_T} |h/\lambda|^{p(x)}\, dx\, dt < \infty\Big\}$$

on the space of functions

$$L^{p(\cdot)}(S_T) = \Big\{u(z) \text{ is measurable in } S_T : \int_{S_T} |u(z)|^{p(x)}\, dx\, dt < \infty\Big\}.$$

The dual space to W is denoted by W'. This is the space of linear functionals over W,

$$\Phi \in W' \iff \begin{cases} \exists \phi_0 \in L^2(S_T),\ \phi_i \in L^{p'(x)}(S_T), \quad i = 1, \dots, n, \\ \forall v \in W \quad (v, \Phi)_{2, S_T} = (v, \phi_0)_{2, S_T} + \sum_{i=1}^n (D_i v, \phi_i)_{2, S_T}, \end{cases}$$

endowed with the usual norm

$$\|\Phi\|_{W'} = \sup\{(\Phi, u)_{2, S_T} : \|u\|_W = 1\}.$$

Definition 1.1. A function $u : S_T \mapsto \mathbb{R}$ is called weak solution of problem (1) if

(1) $u \in W$, $u_t \in W'$,

(2) for every $\phi \in C^1\big([0, T]; C_0^1(\mathbb{R}^n)\big)$

$$\int_{S_T} \big(u_t \phi + \mathscr{A}(z, \nabla u) \cdot \nabla \phi - f\phi\big)\, dx\, dt = 0, \tag{4}$$

(3) $u \in C^0_w([0,T]; L^2(\mathbb{R}^n))$, in particular, for every $\phi(x) \in C^1_0(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \phi(x)(u - u_0)\, dx \to 0 \qquad \text{as } t \to 0.$$

Theorem 1.2. *Let conditions* (2), (3) *be fulfilled. Then for every* $u_0 \in L^2(\mathbb{R}^n)$ *and* $f \in L^2\ (S_T)$ *problem* (1) *has at least weak solution in the sense of Definition* 1.1. *This solution satisfies the energy estimate*

$$\frac{1}{2}\,\text{ess}\sup_{(0,T)} \|u(t)\|^2_{2,\mathbb{R}^n} + \int_{S_T} |\nabla u|^{p(x)}\, dz \le \frac{1}{2}\|u_0\|^2_{2,\mathbb{R}^n} + \int_0^T \|f(\cdot,s)\|_{2,\mathbb{R}^n}\, ds. \quad (5)$$

Uniqueness of weak solutions is established under additional restrictions on the data of problem (1).

Theorem 1.3. *Let the conditions of Theorem* 1.2 *be fulfilled.*

(1) *If* $\sup_{S_T}|f| \le C_f$ *and* $\sup_{\mathbb{R}^n}|u_0| \le M$ *with finite positive constants* C_f, M, *then the solution of problem* (1) *satisfies the estimate*

$$|u| \le (1+M)\mathrm{e}^{(C_f/(1+M))T} \qquad \text{a.e. in } S_T. \quad (6)$$

(2) *If* $p(x) \ge 2$ *a.e. in* $\mathbb{R}^n$, *then problem* (1) *cannot have more than one bounded solution.*

(3) *If* $u_0 \in L^2(\mathbb{R}^n) \cap W_0^{1,p(x,0)}(\mathbb{R}^n)$ *with compact support,* $f \in L^2(S_T)$ *and* $|a_t| \le a_T = const$, *then* $u_t \in L^2(S_T)$, $|\nabla u|^{p(x)} \in L^\infty(0,T; L^1(\mathbb{R}^n))$ *and*

$$\|u_t\|^2_{2,S_T} + \text{ess}\sup_{(0,T)} \int_{\mathbb{R}^n} |\nabla u|^{p(x)}\, dx \le C$$

with a finite constant C.

In the recent decade, nonlinear PDEs with variable nonlinearity have been studied very intensively. Most of results concerning existence and uniqueness of solutions of parabolic PDEs of the type (1) were established under certain regularity restrictions on the variable exponent $p(x)$ (or $p(x,t)$), which allow one to approximate the elements of the space W by infinitely differentiable functions—see, e.g., [5], [8]. The question of solvability of the homogeneous Dirichlet problem for equation (1) with measurable but not necessarily continuous exponent $p(x,t)$ was discussed by several authors. It is shown in [3], [4] that problem (1) with measurable and bounded exponent $p(x,t)$ admits a weak solution, but this solution need not satisfy the energy identity of the type (26) below. The solution possesses better properties if the exponent p is independent of t. The proofs given in [4] rely on the theory of monotone operators and a singular perturbation of the operator $\mathscr{A}(z, \nabla u)$. The sub-differential approach is used in [1], [2] to prove solvability of the homogeneous Dirichlet problem for equation (1) with discontinuous exponents $p(x)$. This method is applicable if the coefficient a is independent of t.

To the best of our knowledge, by now there are no results on the solvability of the Cauchy problem for parabolic equations with variable nonlinearity.

2. Lebesgue spaces with variable exponents

A solution of the Cauchy problem (1) will be constructed as the limit of a sequence of solutions of the Cauchy-Dirichlet problems posed in expanding cylinders. The members of this sequence are elements of the Lebesgue and Sobolev spaces with variable exponents defined in the present section.

2.1. Basic properties. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz-continuous boundary $\partial\Omega$ and $Q_T = \Omega \times (0, T]$. The space $W(Q_T)$ is defined as the closure of $C^\infty\big([0,T]; C_0^\infty(\Omega)\big)$ with respect to the norm

$$\|u\|_{W(Q_T)} = \|u\|_{2,Q_T} + \|\nabla u\|_{p(\cdot),Q_T}, \tag{7}$$

where

$$\|v\|_{p(\cdot),Q_T} = \inf\Big\{\lambda > 0 : \int_{Q_T} |v/\lambda|^{p(x)}\,dx < \infty\Big\}. \tag{8}$$

We will repeatedly use the following known properties of the spaces $L^{p(x)}$. Let the exponent $p(x)$ satisfy conditions (3). Then

$$\begin{gathered}\forall f \in L^{p(\cdot)}(Q_T), \qquad g \in L^{p(\cdot)'}(Q_T), \qquad p' = \frac{p}{p-1}, \\ \Big|\int_{Q_T} fg\,dz\Big| \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|f\|_{p(\cdot),Q_T}\|g\|_{p'(\cdot),Q_T}\end{gathered} \tag{9}$$

and

$$\min\{\|f\|_{p(\cdot),Q_T}^{p^-}, \|f\|_{p(\cdot),Q_T}^{p^+}\} \le \int_{Q_T} |f|^{p(x)}\,dz \le \max\{\|f\|_{p(\cdot),Q_T}^{p^-}, \|f\|_{p(\cdot),Q_T}^{p^+}\}. \tag{10}$$

Inequality (10) means that $\|f_k - f\|_{p(\cdot), Q_T} \to 0$ if and only if $\int_{Q_T} |f_n - f|^{p(x)}\, dz \to 0$. The proofs of properties (9) and (10) can be found in [9], [7].

2.2. Steklov's means. Denote by f_h the Steklov mean of the function $f \in L^{p(x)}(Q_T)$:

$$f_h = \frac{1}{h}\int_t^{t+h} f(x,\tau)\, d\tau.$$

Proposition 2.1. *The operator $S_h : f \to f_h$ maps $L^{p(x)}(Q_T)$ into $L^{p(x)}(Q_{T-h})$.*

Proof. By virtue of Hölder's inequality we have that for an arbitrary given $\varepsilon \in (0, T)$, every $h \in (0, \varepsilon)$ and a.e. $x \in \Omega$

$$\begin{aligned} |f_h|^{p(x)} &= \frac{1}{h^{p(x)}}\left|\int_0^h f(x, t+\tau)\, d\tau\right|^{p(x)} \\ &\le \frac{1}{h^{p(x)}}\left(\int_0^h |f(x,t+\tau)|^{p(x)}\, d\tau\right)\left(\int_0^h d\tau\right)^{p(x)/p'(x)} \\ &= \frac{1}{h}\int_0^h |f(x,t+\tau)|^{p(x)}\, d\tau = (|f|^{p(x)})_h. \end{aligned} \tag{11}$$

In particular,

$$\frac{1}{h^{p(x)}}\left|\int_0^h \big(f(x,t+\tau) - f(x,t)\big)\, d\tau\right|^{p(x)} \le \frac{1}{h}\int_0^h |f(x,t+\tau) - f(x,t)|^{p(x)}\, d\tau. \tag{12}$$

Applying (12) we immediately obtain

$$\begin{aligned} \int_{Q_{T-\varepsilon}} |f_h|^{p(x)}\, dx\, dt &= \int_{Q_{T-\varepsilon}} \frac{1}{h^{p(x)}}\left|\int_0^h f(x,t+\tau)\, d\tau\right|^{p(x)} dx\, dt \\ &\le \frac{1}{h}\int_{Q_{T-\varepsilon}}\left(\int_0^h |f(x,t+\tau)|^{p(x)}\, d\tau\right) dx\, dt \\ &\le \frac{1}{h}\int_0^h \left(\int_{Q_T} |f|^{p(x)}\, dx\, dt\right) d\tau = \int_{Q_T} |f|^{p(x)}\, dx\, dt. \end{aligned}$$ □

Proposition 2.2. *The translation operator is continuous: for every $\varepsilon > 0$ and $f \in L^{p(x)}(Q_T)$*

$$\|f(x,t+h) - f(x,t)\|_{p(x), Q_{T-\varepsilon}} \to 0 \quad \text{as } h \to 0.$$

Proof. Take $f \in L^{p(x)}(Q_T)$ and an arbitrary $\varepsilon > 0$. Since $C^0(\overline{Q}_T)$ is dense in $L^{p(x)}(Q_T)$, there exists $\phi \in C^0(\overline{Q}_T)$ such that $\|f - \phi\|_{p(x), Q_T} < \varepsilon$. For $\varepsilon \le 1$ this means that

$$\int_{Q_T} |f - \phi|^{p(x)}\, dz < \varepsilon^{p^-}.$$

Since ϕ is uniformly continuous in $\overline{Q}_T$, there exists $\delta > 0$ such that

$$\forall x \in \Omega, \qquad t, \tau \in [0, T] \qquad |t - \tau| < \delta \implies |\phi(x,t) - \phi(x,\tau)| < \frac{\varepsilon}{(1 + 2|\Omega|T)^{1/p^-}}.$$

Then

$$\begin{aligned}\int_{Q_{T-h}} |f(x,t+h) - f(x,t)|^{p(x)}\, dx\, dt &\le \int_{Q_{T-h}} |f(x,t+h) - \phi(x,t+h)|^{p(x)}\, dx\, dt \\ &\quad + \int_{Q_{T-h}} |\phi(x,t+h) - \phi(x,t)|^{p(x)}\, dx\, dt \\ &\quad + \int_{Q_{T-h}} |f(x,t) - \phi(x,t)|^{p(x)}\, dx\, dt \\ &\le 2\varepsilon^{p^-} + \frac{\varepsilon^{p^-}|\Omega|T}{1 + 2|\Omega|T} \le 3\varepsilon^{p^-}. \qquad \square\end{aligned}$$

Remark 2.3. Proposition 2.2 is false if p depends on t. It is known that in the case $p = p(x,t)$ the translation operator is unbounded unless $p = const$—see ([7], Proposition 3.6.1)

Proposition 2.4. *If $f \in W(Q_T)$, then $\|f_h - f\|_{W(Q_{T-h})} \to 0$ as $h \to 0$.*

Proof. It is sufficient to check that for every $f \in L^{p(x)}(Q_T)$ $\|f_h - f\|_{p(x), Q_{T-\varepsilon}} \to 0$ as $h \to 0$. By (12)

$$\begin{aligned}\int_{Q_{T-\varepsilon}} |f_h - f|^{p(x)}\, dx\, dt &= \int_{Q_{T-\varepsilon}} \left|\frac{1}{h}\int_0^h \big(f(x,t+\tau) - f(x,t)\big)\, d\tau\right|^{p(x)} dx\, dt \\ &\le \frac{1}{h}\int_{Q_{T-\varepsilon}} \left(\int_0^h |f(x,t+\tau) - f(x,t)|^{p(x)}\, d\tau\right) dx\, dt \\ &= \frac{1}{h}\int_0^h \left(\int_{Q_{T-\varepsilon}} |f(x,t+\tau) - f(x,t)|^{p(x)}\, dx\, dt\right) d\tau \\ &\equiv \frac{1}{h}\int_0^h F(\tau)\, d\tau.\end{aligned}$$

By Proposition 2.2 $F(\tau) \to 0$ as $\tau \to 0+$. For every $\varepsilon > 0$ there is $h(\varepsilon)$ such that $F(\tau) < \varepsilon$ if $\tau < h(\varepsilon)$. It follows that

$$\forall h < h(\varepsilon) \qquad \frac{1}{h}\int_0^h F(\tau)\,d\tau < \varepsilon,$$

whence the assertion. □

Proposition 2.5. *If $u \in W(Q_T)$ and $u_t \in W'(Q_T)$, then $(u_t)_h = (u_h)_t$ and $(u_h)_t \rightharpoonup u_t$ in $W'(Q_{T-\varepsilon})$ for every $\varepsilon > 0$.*

Proposition 2.6. *If $u, v \in W(Q_T)$ and $u_t, v_t \in W'(Q_T)$, then for a.e. $t_1, t_2 \in (0, T)$*

$$\int_{t_1}^{t_2} u_t v\,dx\,dt + \int_{t_1}^{t_2} u v_t\,dx\,dt = \int_\Omega uv\,dx\Big|_{t=t_1}^{t=t_2}.$$

We omit the proofs of Propositions 2.5, 2.6 which are imitations of the proofs of Lemmas 4.2, 4.3 in [6].

Proposition 2.7. *If $u \in W(Q_T)$ and $u_t \in W'(Q_T)$, then $u \in C^0\big([0, T-\varepsilon]; L^2(\Omega)\big)$ for every $\varepsilon \in (0, T)$.*

Proof. Since $u_h \in L^2(Q_T)$ and $(u_h)_t \in L^2(Q_{T-\varepsilon})$, it follows from ([10], Ch. 1, Lemma 1.2) that $u_h \in C^0\big([0, T-\varepsilon]; L^2(\Omega)\big)$ after possible redefining on a set of zero measure in $(0, T-\varepsilon)$. Thus, for every $h_1, h_2 \in (0, T-\varepsilon)$

$$\|u_{h_1} - u_{h_2}\|^2_{2,\Omega}(t) = 2\int_{Q_t} (u_{h_1} - u_{h_2})(u_{h_1} - u_{h_2})_t\,dz + \|u_{0h_1} - u_{0h_2}\|^2_{2,\Omega}.$$

By Proposition 2.4 $u_h \to u$ in $W(Q_{T-\varepsilon})$ as $h \to 0$, i.e., $\{u_h\}$ is a Cauchy sequence in $W(Q_{T-\varepsilon})$. By Proposition 2.5 $(u_h)_t \rightharpoonup u_t$ in $W'(Q_{T-\varepsilon})$ and, thus, $(u_h)_t$ are bounded in the norm of $W'(Q_{T-\varepsilon})$. Notice also that $(u_0)_h \equiv u_0$ and $\|u_{0h_2} - u_{0h_1}\|_{2,\Omega} = 0$. For every $t \in (0, T-\varepsilon)$

$$\begin{aligned}\|u_{h_1} - u_{h_2}\|^2_{2,\Omega}(t) &= 2\int_{Q_t} (u_{h_1} - u_{h_2})(u_{h_1} - u_{h_2})_t\,dz \\ &\le 2\|(u_{h_1} - u_{h_2})_t\|_{W'}\|u_{h_1} - u_{h_2}\|_W \to 0 \qquad \text{as } h_1, h_2 \to 0.\end{aligned}$$

This means that $\{u_h\}$ is a Cauchy sequence in $C^0\big([0, T-\varepsilon]; L^2(\Omega)\big)$ and there is a function $\tilde{u} \in C^0\big([0, T-\varepsilon]; L^2(\Omega)\big)$ such that $u_h \to \tilde{u}$. Since $u_h \to u \in W(Q_{T-\varepsilon})$, it necessary that $u = \tilde{u}$ after possible redefining on a set of zero measure in $(0, T-\varepsilon)$. □

Corollary 2.8. *It is easy to see that all the above propositions, except Proposition* 2.2, *remain true if the cylinder Q_T is substituted by the layer $S_T = \mathbb{R}^n \times (0, T]$. The proof of Proposition* 2.2 *can be modified as follows. If $f \in L^{p(x)}(S_T)$, for every $\varepsilon > 0$ there is $R > 0$ such that*

$$\int_0^T \int_{\mathbb{R}^n \setminus B_R(0)} |f(x, t+h) - f(x, t)|^{p(x)} \, dz < \varepsilon.$$

Set $Q_{T-h,R} = B_R(0) \times (0, T-h)$. Using the representation

$$\begin{aligned}\int_{S_{T-h}} |f(x, t+h) - f(x, t)|^{p(x)} \, dz &= \int_{S_{T-h} \setminus Q_{T-h,R}} |f(x, t+h) - f(x, t)|^{p(x)} \, dz \\ &\quad + \int_{Q_{T-h,R}} |f(x, t+h) - f(x, t)|^{p(x)} \, dz \\ &< \varepsilon + \int_{Q_{T-h,R}} |f(x, t+h) - f(x, t)|^{p(x)} \, dz\end{aligned}$$

we complete the proof applying Proposition 2.2.

3. Regularization. Problems in bounded cylinders

Let us denote $B_k = \{x \in \mathbb{R}^n : |x| < k\}$, $Q_{T,k} = B_k \times (0, T]$, $(k \in \mathbb{N})$, and consider the sequence of regularized problems

$$\begin{cases} u_t - \operatorname{div} \mathscr{A}(z, \nabla u) = f \quad \text{in } Q_{T,k}, \\ u = 0 \text{ on } \partial B_k \times (0, T], \quad u(x, 0) = u_0(x) \in L^2(B_k), \end{cases} \tag{13}$$

where for the initial data we take the restriction of u_0 to B_k, f is the restriction of f to $Q_{T,k}$. The natural energy space for problem (13) is defined by (7)–(8). The solution is understood is the sense of Definition 1.1 with obvious changes: u is a weak solution of the regularized problem (13) if $u \in C^0\big([0, T]; L^2(B_k)\big) \cap W(Q_{T,k})$, $u_t \in W'(Q_{T,k})$, for every test-function $\phi \in C^1\big([0, T]; C_0^1(B_k)\big)$

$$\int_{Q_{T,k}} \big(u_t \phi + \mathscr{A}(z, \nabla u) \cdot \nabla \phi - f \phi\big) \, dz = 0, \tag{14}$$

and $(\phi, u - u_0)_{2, B_k} \to 0$ as $t \to 0$ for every $\phi(x) \in C_0^1(B_k)$.

Theorem 3.1. *Let condition* (3) *be fulfilled. For every*

$$u_0 \in L^2(\mathbb{R}^n), \qquad f \in L^2(S_T), \qquad k \in \mathbb{N},$$

problem (13) *has a unique weak solution $u \equiv u_k$.*

3.1. Galerkin's approximations. Let $\{\psi_i\}$ be the orthonormal basis of $L^2(B_k)$ composed of the eigenfunctions of the operator

$$(w, \psi_i)_{H_0^s(B_k)} = \lambda_i (w, \psi_i)_{2, B_k} \qquad \forall w \in H_0^s(B_k).$$

The integer s is chosen so big that the embedding $H_0^s(\Omega) \subset W_0^{1,p^+}(\Omega)$ is compact: $\dfrac{s-1}{n} \geq \dfrac{1}{2} - \dfrac{1}{p^+}$. Accept the notation

$$\mathscr{P}_N = \operatorname{span}\{\psi_1, \ldots, \psi_N\} \subset L^{p^+}\big(0, T; H_0^s(B_k)\big) \subset W(Q_{T,k}).$$

Lemma 3.2. *The space $L^{p^+}\big(0, T; H_0^s(B_k)\big)$ is dense in $W(Q_{T,k})$.*

Proof. Let $u \in W(Q_{T,k})$. By the definition there is a sequence $\{u_s\}$ such that $u_s \in C^\infty\big(0, T; C_0^\infty(B_k)\big)$ and $u_s \to u$ in $W(Q_{T,k})$. Since $C^\infty\big(0, T; C_0^\infty(B_k)\big) \subset L^{p^+}\big(0, T; H_0^s(B_k)\big)$, the assertion follows. □

A solution of problem (13) will be obtained as the limit of the sequence

$$u^{(N)} = \sum_{i=1}^{N} \psi_i(x) d_{i,N}(t) \in \mathscr{P}_N \tag{15}$$

with the coefficients $d_{i,N}(t)$ to be defined. Substituting $u^{(N)}$ into equation (13), multiplying by ψ_i and integrating over B_k we obtain the system of ODEs for the coefficients $\mathbf{d}_N = \{d_{1,N}(t), \ldots, d_{N,N}(t)\}$:

$$\begin{cases} \mathbf{d}_N'(t) = \mathscr{F}\big(t, \mathbf{d}_N(t)\big), & t > 0 \\ d_{i,N}(0) = (u_0, \psi_i)_{2, B_k}, & i = 1, \ldots, N, \end{cases} \tag{16}$$

$$\mathscr{F}_i\big(t, \mathbf{d}_N(t)\big) = -\int_{B_k} \mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla \psi_i \, dx + \int_{B_k} f \psi_i \, dx.$$

Since $\mathscr{F}(t, \mathbf{d}_N)$ is continuous with respect to $d_{j,N}$ and t, it follows from Peano's theorem that system (16) has a solution on an interval $[0, T_N)$.

3.2. Uniform a priori estimates.

Lemma 3.3. *For every N the function $u^{(N)}$ satisfies the estimate*

$$\operatorname*{ess\,sup}_{(0,T)} \|u^{(N)}(t)\|_{2, B_k}^2 + \int_{Q_{T,k}} |\nabla u^{(N)}|^{p(x)} \, dz \leq C \tag{17}$$

with a constant C depending on a_-, T, $\|u_0\|_{2,\mathbb{R}^n}$ and $\|f\|_{2,S_T}$, but independent of k and N.

Proof. Set $y(t) = \|u^{(N)}(t)\|^2_{2,B_k}$. Multiplying the ith equation of system (16) by $d_{i,N}(t)$ and summing up the results we obtain the inequality

$$\frac{1}{2}y'(t) + a_- \int_{B_k} |\nabla u^{(N)}|^p \, dx \le \int_{B_k} |f|\,|u^{(N)}|\, dx \le \sqrt{y(t)}\|f\|_{2,B_k}, \tag{18}$$

whence

$$\sqrt{y(t)} \le \sqrt{y(0)} + \int_0^t \|f(\cdot,s)\|_{2,B_k}\, ds.$$

Substituting this inequality into (18) and integrating over the integral $(0,t)$ we conclude that

$$\begin{aligned}&\|u^{(N)}(t)\|^2_{2,B_k} + 2a_- \int_{Q_{T,k}} |\nabla u^{(N)}|^p \, dz \\ &\quad \le \|u_0\|^2_{2,B_k} + 2\Big(\|u_0\|_{2,B_k} + \int_0^T \|f(\cdot,s)\|_{2,B_k}\, ds\Big)\int_0^T \|f(\cdot,s)\|_{2,B_k}\, ds.\end{aligned}$$

This inequality yields the assertion because

$$\Big(\int_0^T \|f(\cdot,s)\|_{2,B_k}\, ds\Big)^2 \le T\|f\|^2_{2,Q_{T,k}} \le T\|f\|^2_{2,S_T}$$

and

$$2\|u_0\|_{2,B_k}\int_0^T \|f(\cdot,s)\|_{2,B_k}\, ds \le \|u_0\|^2_{2,\mathbb{R}^n} + \Big(\int_0^T \|f(\cdot,s)\|_{2,B_k}\, ds\Big)^2. \qquad \square$$

Corollary 3.4. *Estimates* (17) *are independent of N, which allows one to continue each of $u^{(N)}$ to the maximal existence interval* $[0,T]$.

Lemma 3.5. *There is an independent of k and N constant C such that*

$$\|(u^{(N)})_t\|_{L^{(p^+)'}(0,T;H^{-s}(B_k))} \le C. \tag{19}$$

The constant C depends only on $a_\pm$, T, $\|u_0\|_{2,\mathbb{R}^n}$ and $\|f\|_{2,S_T}$.

Proof. Given $\phi \in L^{p^+}\big(0,T;H^s_0(\Omega)\big)$, we denote $\phi^{(N)} = \sum_{i=1}^N \phi_k(t)\psi_k(x) \in \mathscr{P}_N$. Since $\{\psi_k\}$ are orthogonal in $L^2(\Omega)$, the definition of $u^{(N)}$ yields

$$
\begin{aligned}
-\int_{Q_{T,k}} u_t^{(N)}\phi\,dz &= -\int_{Q_{T,k}} u_t^{(N)}\phi^{(N)}\,dz \\
&= \sum_{i=1}^{n}\int_{Q_{T,k}} \mathscr{A}(z,\nabla u^{(N)}\cdot\nabla\phi^{(N)}\,dz + \int_{Q_{T,k}} f\phi^{(N)}\,dz,
\end{aligned}
$$

whence

$$
\begin{aligned}
&\Big|\int_{Q_{T,k}} u_t^{(N)}\phi\,dz\Big| \\
&\quad\le a_+\|\,|\nabla u^{(N)}|^{p-1}\|_{p'(\cdot),Q_{T,k}}\|\nabla\phi^{(N)}\|_{p(\cdot),Q_{T,k}} + \|f\|_{2,Q_{T,k}}\|\phi^{(N)}\|_{2,Q_{T,k}} \\
&\quad\le C(1+\|\nabla u^{(N)}\|_{p(\cdot),Q_{T,k}} + \|f\|_{2,Q_{T,k}})(\|\phi^{(N)}\|_{2,Q_{T,k}} + \|\nabla\phi^{(N)}\|_{p(\cdot),Q_{T,k}}).
\end{aligned}
$$

By virtue of (10)

$$
\begin{aligned}
&\|\nabla\phi^{(N)}\|_{p(\cdot),Q_{T,k}} \\
&\quad\le \max\Big\{\Big(\int_{Q_{T,k}}|\nabla\phi^{(N)}|^{p(x)}\,dz\Big)^{1/p^+},\Big(\int_{Q_{T,k}}|\nabla\phi^{(N)}|^{p(x)}\,dz\Big)^{1/p^-}\Big\} \\
&\quad\le C\bigg(1+\max\Big\{\Big(\int_{Q_{T,k}}|\nabla\phi^{(N)}|^{p^+}\,dz\Big)^{1/p^+},\Big(\int_{Q_{T,k}}|\nabla\phi^{(N)}|^{p^+}\,dz\Big)^{1/p^-}\Big\}\bigg) \\
&\quad\le C(1+\max\{\|\phi^{(N)}\|^{p^+/p^-}_{L^{p^+}(0,T;H_0^s(B_k))},\|\phi^{(N)}\|_{L^{p^+}(0,T;H_0^s(B_k))}\}).
\end{aligned}
$$

Since

$$
\begin{aligned}
\|\phi^{(N)}\|^{p^+}_{L^{p^+}(0,T;W_0^{1,p^+}(B_k))} &\le C\int_0^T \|\phi^{(N)}\|^{p^+}_{H_0^s(B_k)}(t)\,dt \\
&\le C\int_0^T \|\phi\|^{p^+}_{H_0^s(B_k)}(t)\,dt = C\|\phi\|^{p^+}_{L^{p^+}(0,T;H_0^s(B_k))},
\end{aligned}
$$

applying estimate (17) we find that for every

$$
\phi\in L^{p^+}\big(0,T;H_0^s(B_k)\big)\qquad\text{with } \|\phi\|_{L^{p^+}(0,T;H_0^s(B_k))}\le 1
$$

the function $u_t^{(N)}$ satisfies the inequality

$$
\Big|\int_{Q_{T,k}} u_t^{(N)}\phi\,dz\Big|\le C
$$

with an independent on $u^{(N)}$ and ϕ constant C. $\square$

Lemma 3.6. *The sequence* $\{u^{(N)}\}$ *is precompact in* $L^{\mu}(Q_{T,k})$ *with some* $\mu > 1$.

Proof. Estimates (17), (19) and the inclusion $V \subset W_0^{1,p^-}(B_k)$ yield the independent of k, N estimates

$$\|u^{(N)}\|_{L^{p^-}(0,T;W_0^{1,p^-}(B_k))} \leq C. \tag{20}$$

The inclusion $W_0^{1,p^-}(B_k) \subset L^r(B_k)$ with $r < np^-/(n-p^-)$ is compact and $L^r(B_k) \subset H^{-s}(B_k)$. The assertion follows now from Lemma 3.5 and the compactness results in [11]. □

3.3. Passage to the limit. Due to estimates (17) and Lemmas 3.5, 3.6 we may find functions $\chi_k \in L^{p'(\cdot)}(Q_{T,k})$, $u_k \in L^{\mu}(Q_{T,k}) \cap W(Q_{T,k})$, $U \in L^{\infty}\big(0,T;L^2(B_k)\big)$ such that

$$\begin{aligned} &u^{(N)} \to u_k \text{ in } L^{\mu}(Q_{T,k}) \text{ with some } \mu > 1 \text{ and a.e. in } Q_{T,k},\\ &u^{(N)} \to U \ast\text{-weakly in } L^{\infty}\big(0,T;L^2(B_k)\big),\\ &\mathscr{A}(z,\nabla u^{(N)}) \rightharpoonup \chi_k \text{ in } L^{p'}(Q_{T,k}),\\ &\big(u^{(N)}\big)_t \rightharpoonup u_{k,t} \text{ in } L^{(p^+)'}\big(0,T;H^{-s}(B_k)\big). \end{aligned} \tag{21}$$

Lemma 3.7. *For every* $k \in \mathbb{N}$ $u_{k,t} \in W(Q_{T,k})'$ *and*

$$\|u_{k,t}\|_{W_k'} \leq C$$

with an independent of k *constant* C.

Proof. Take some $N \in \mathbb{N}$, fix $j \leq N$ and test (14) with a function $\phi \in \mathscr{P}_j$:

$$\int_{Q_{T,k}} \big(u_t^{(N)}\phi + \mathscr{A}(z,\nabla u^{(N)}) \cdot \nabla\phi\big)\,dz = \int_{Q_{T,k}} f\phi\,dz. \tag{22}$$

Passing to the limit as $N \to \infty$ we have that

$$\forall\phi \in \mathscr{P}_j \qquad \int_{Q_{T,k}} (u_{k,t}\phi + \chi_k \cdot \nabla\phi)\,dz = \int_{Q_{T,k}} f\phi\,dz. \tag{23}$$

Since $L^{p^+}\big(0,T;H_0^s(B_k)\big)$ is dense in $W(Q_{T,k})$, letting $j \to \infty$ we conclude that (23) holds for an arbitrary $\phi \in W(Q_{T,k})$. It follows then that for every $\phi \in W(Q_{T,k})$

$$\left|\int_{Q_{T,k}} u_{k,t}\phi\,dz\right| \leq C\|\phi\|_{W(Q_{T,k})}$$

with the constant C from (17). □

Lemma 3.8. *For every k and a.e. $t \in (0, T)$*

$$\frac{1}{2}\int_{B_k} u_k^2\,dx\Big|_{\tau=0}^{\tau=t} + \int_0^t\int_{B_k} \chi_k \cdot \nabla u_k\,dz = \int_{Q_{T,k}} fu_k\,dz. \tag{24}$$

Proof. It is shown in the proof of Lemma 3.7 that

$$\int_{Q_{T,k}} (u_{k,t}u_k + \chi_k \cdot \nabla u_k - fu_k)\,dz = 0.$$

The assertion follows now from Proposition 2.6 with $u = v = u_k$. □

Lemma 3.9. $u_k \in C_w\big(0, T; L^2(\Omega)\big)$ *and* $\big(\eta(x), u_k - u_0\big)_{2, B_k} \to 0$ *as* $t \to 0$ *for every* $\eta \in C_0^\infty(B_k)$.

Proof. Let us take for the test-function $\phi = \eta(x)\theta(t)$ with $\theta(t) \in C^1[0, T]$ such that $\theta(0) = \theta(T) = 0$ and $\eta(x) \in C_0^1(B_k)$. Denote $F(t) = \int_{B_k} u_k\eta\,dx$. By (23)

$$\int_0^T \theta'(t)F(t)\,dt = \int_0^T \theta(t)\Psi(t)\,dt, \qquad \Psi = \int_{B_k} (\chi_k \cdot \nabla\eta - f\eta)\,dx \in L^1(0, T).$$

It follows that $F(t) \in W^{1,1}(0, T)$, whence $F(t)$ is absolutely continuous on $(0, T)$ and the limits $F(0)$ and $F(T)$ are well-defined. □

Lemma 3.10. $\chi_k = \mathscr{A}(z, \nabla u_k)$ *a.e. in* $Q_{T,k}$.

Proof. We apply the standard monotonicity argument. Recall that $\big(\mathscr{A}(z, \mathbf{s}) - \mathscr{A}(z, \mathbf{r})\big) \cdot (\mathbf{s} - \mathbf{r}) \geq 0$ for a.e. $z \in Q_{T,k}$ and all $\mathbf{s}, \mathbf{r} \in \mathbb{R}^n$. By construction

$$\frac{1}{2}\int_{B_k} (u^{(N)})^2\,dx\Big|_{t=0}^{t=T} + \int_{Q_{T,k}} \mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla u^{(N)}\,dz = \int_{Q_{T,k}} fu_k\,dz.$$

For every $\phi \in \mathscr{P}_N$

$$\begin{aligned}
\mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla u^{(N)} &= \mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla(u^{(N)} - \phi) + \mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla\phi \\
&= \big(\mathscr{A}(z, \nabla u^{(N)}) - \mathscr{A}(z, \nabla\phi) \cdot \nabla(u^{(N)} - \phi) \\
&\quad + \mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla\phi + \mathscr{A}(z, \nabla\phi) \cdot \nabla(u^{(N)} - \phi)
\end{aligned}$$

whence, by monotonicity,

$$\int_{Q_{T,k}} \mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla u^{(N)}\, dz \geq \int_{Q_{T,k}} \mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla \phi\, dz + \int_{Q_{T,k}} \mathscr{A}(z, \nabla \phi) \cdot \nabla (u^{(N)} - \phi)\, dz.$$

Since $u^{(N)} \in \mathscr{P}_N$

$$\begin{aligned} 0 &= \frac{1}{2}\int_{B_k} (u^{(N)})^2\, dx\Big|_{t=0}^{t=T} + \int_{Q_{T,k}} \mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla u^{(N)}\, dz - \int_{Q_{T,k}} f u^{(N)}\, dz \\ &\geq \frac{1}{2}\int_{B_k} (u^{(N)})^2\, dx\Big|_{t=0}^{t=T} + \int_{Q_{T,k}} \mathscr{A}(z, \nabla u^{(N)}) \cdot \nabla \phi\, dz \\ &\quad + \int_{Q_{T,k}} \mathscr{A}(z, \nabla \phi) \cdot \nabla (u^{(N)} - \phi)\, dz - \int_{Q_{T,k}} f u^{(N)}\, dz. \end{aligned}$$

Fix an arbitrary $\phi \in \mathscr{P}_K$, $K \leq N$. Letting $N \to \infty$ and using (21) we obtain

$$\frac{1}{2}\int_{B_k} u_k^2\, dx\Big|_{t=0}^{t=T} + \int_{Q_{T,k}} \chi_k \cdot \nabla \phi\, dz + \int_{Q_{T,k}} \mathscr{A}(z, \nabla \phi) \cdot \nabla (u_k - \phi)\, dz - \int_{Q_{T,k}} f u_k\, dz \leq 0.$$

Writing (23) in the form

$$\frac{1}{2}\int_{B_k} u_k^2\, dx\Big|_{t=0}^{t=T} - \int_{Q_{T,k}} f u_k\, dz = -\int_{Q_{T,k}} \chi_k \cdot \nabla u_k\, dz$$

and substituting the result into the previous inequality we obtain

$$\int_{Q_{T,k}} \big(\chi_k - \mathscr{A}(z, \nabla \phi)\big) \cdot \nabla (u_k - \phi) \geq 0$$

for every $\phi \in \mathscr{P}_K$. Letting $K \to \infty$ we may take for ϕ an arbitrary element from the main functions space. Let us now choose ϕ in the special way: $\phi = u_k + \lambda w$ with $\lambda > 0$ and $w \in W$: letting $\lambda \to 0$ we have

$$\int_{Q_{T,k}} \big(\chi_k - \mathscr{A}(z, \nabla u_k)\big) \cdot \nabla w \geq 0,$$

which is only possible if $\chi_k = \mathscr{A}(z, \nabla u_k)$ a.e. in $Q_{T,k}$.

This completes the proof of existence of a weak solution of problem (13). Uniqueness of the weak solution is an immediate byproduct of monotonicity of the operator $\mathscr{A}$. Assume that there are two different solutions u_i of problem (13) and set $w = u_k^{(1)} - u_k^{(2)}$. Choosing $u_k^{(i)}$ for the test-function in identities (14) for $u_k^{(i)}$ and gathering the results we have that for every $\tau \in (0, T]$

$$\frac{1}{2}\int_{B_k} w(\tau)^2\,dx \le -\int_0^\tau\int_{B_k}\big(\mathscr{A}(z,u_1) - \mathscr{A}(z,u_2)\big)\cdot\nabla(u_1 - u_2)\,dz \le 0,$$

whence $w = 0$ a.e. in $Q_{T,k}$. □

Remark 3.11. Identity (14) holds true for the test-functions $\phi \in W$ with $\phi_t \in W'$.

Lemma 3.12 (Improved regularity). *Let $u_0 \in L^2(\mathbb{R}^n) \cap W_0^{1,p(x,0)}(\mathbb{R}^n)$, $f \in L^2(S_T)$ and $|a_t| \le a_T = const.$ Then*

$$\|u_{k,t}\|^2_{2,Q_{T,k}} + \operatorname*{ess\,sup}_{(0,T)}\int_{B_k}|\nabla u_k|^{p(x)}\,dx \le C \tag{25}$$

with a constant C depending on $\|u_0\|_{2,\mathbb{R}^n}$, $\|\nabla u_0\|_{p(\cdot),\mathbb{R}^n}$ and $\|f\|_{2,S_T}$, a_T, but independent of k.

Proof. To prove the lemma it suffices to show that (25) holds for the Galerkin approximations of the regularized problems $u^{(N)}$. Let us multiply the ith equation of system (16) by $d'_{i,N}(t)$, integrate over B_k and sum up:

$$\begin{aligned}\|u_t^{(N)}(t)\|^2_{2,B_k} &+ \frac{d}{dt}\Big(\int_{B_k}\frac{a(z)}{p(x)}|\nabla u^{(N)}(t)|^{p(x)}\,dx\Big)\\ &= \int_{B_k}\frac{a_t(z)}{p(x)}|\nabla u^{(N)}(t)|^{p(x)}\,dx + \int_{B_k} f u_t^{(N)}\,dx.\end{aligned}$$

Integrating in t and applying Young's inequality we arrive at the estimate

$$\begin{aligned}\|u_t^{(N)}\|^2_{2,Q_{T,k}} + \int_{B_k}\frac{a(z)}{p(x)}|\nabla u^{(N)}(t)|^{p(x)}\,dx &\le \frac{1}{2}\|f\|^2_{2,Q_{T,k}} + \frac{1}{2}\|u_t^{(N)}\|^2_{2,Q_{T,k}}\\ &+ \int_{B_k}\frac{a_+}{p(x)}|\nabla u_0^{(N)}|^{p(x)}\,dx\\ &+ \frac{a_T}{a_-}\int_0^t\int_{B_k}\frac{a(z)}{p(x)}|\nabla u^{(N)}(t)|^{p(x)}\,dz.\end{aligned}$$

Set

$$Y_N(t) = \int_0^t \int_{B_k} \frac{a(z)}{p(x)} |\nabla u^{(N)}(t)|^{p(x)}\, dz.$$

The function $Y_N(t)$ satisfies the linear differential inequality

$$Y_N'(t) \le \frac{a_T}{a_-} Y_N(t) + \frac{1}{2} \|f\|^2_{2, Q_{T,k}} + \int_{B_k} \frac{a_+}{p(x)} |\nabla u_0^{(N)}|^{p(x)}\, dx.$$

Integration of this inequality gives the required uniform estimate:

$$\begin{aligned} Y_N(t) &\le \frac{a_-}{a_T} (e^{(a_T/a_-)T} - 1) \Big(\frac{1}{2} \|f\|^2_{2, Q_{T,k}} + \int_{B_k} \frac{a_+}{p(x)} |\nabla u_0^{(N)}|^{p(x)}\, dx \Big) \\ &\le \frac{a_-}{a_T} (e^{(a_T/a_-)T} - 1) \Big(\frac{1}{2} \|f\|^2_{2, Q_{T,k}} + C \int_{B_k} |\nabla u_0|^{p(x)}\, dx \Big). \end{aligned}$$

□

4. The Cauchy problem

Let $\{u_k\}$ be the sequence of solutions of the regularized problems (13). Define the sequence of functions extended to the whole S_T

$$w_k = \begin{cases} u_k & \text{in } Q_{T,k}, \\ 0 & \text{in } S_T \setminus Q_{T,k}, \end{cases} \qquad f_k = \begin{cases} f & \text{in } Q_{T,k}, \\ 0 & \text{in } S_T \setminus Q_{T,k}. \end{cases}$$

According to the uniform estimates of Lemmas 3.3, 3.7

$$\begin{aligned} &w_k \text{ are bounded in } L^\infty\big(0, T; L^2(\mathbb{R}^n)\big) \text{ and in } W, \\ &\mathscr{A}(z, \nabla w_k) \text{ and } w_{k,t} \text{ are bounded in } W'. \end{aligned}$$

It follows that there exist functions $w \in W$, $\chi \in W'$, $U \in L^2(\mathbb{R}^n)$ such that

$$\begin{aligned} &w_k(T) \to U \text{ in } L^2(\mathbb{R}^n), \qquad w_k \rightharpoonup w \text{ in } W, \\ &w_k \rightharpoonup w \ *\text{-weak in } L^\infty\big(0, T; L^2(\mathbb{R}^n)\big), \\ &\mathscr{A}(z, \nabla w_k) \rightharpoonup \chi \quad \text{and} \quad w_{k,t} \rightharpoonup w_t \text{ in } W'. \end{aligned}$$

Lemma 4.1. *The sequence $\{w_k\}$ contains a subsequence which converges in $C^0\big([0, T]; L^2(\Omega)\big)$ on every compact $\Omega \subset \mathbb{R}^n$.*

Proof. By Proposition 2.7 for every k there is a subsequence $\{w_{m_k}\}$ converging in $C^0([0,T];L^2(B_k))$. Choosing subsequences $\{w_{m_{k+1}}\} \subset \{w_{m_k}\}$, $k = 1, 2, \dots$, we conclude that the diagonal sequence $\{w_{m_m}\}$ converges in $C^0([0,T];L^2(B_k))$ for every k. □

An immediate corollary from Lemma 4.1 is the equality $U = w(T)$. Take an arbitrary $\phi \in C^1(0,T;C_0^1(\mathbb{R}^n))$ and choose $k_0 \in \mathbb{N}$ so big that supp $\phi(\cdot,t) \subset B_k$ for all $k \geq k_0$ and $t \in [0,T]$. Since $C^1([0,T];C_0^1(B_k)) \subset W(Q_{T,k})$ for all $k \geq k_0$, the extended functions w_k satisfy the identity

$$\int_{S_T} \left(w_{k,t}\phi + \mathscr{A}(z,\nabla w_k)\cdot\nabla\phi + f_k\phi\right) dz = 0.$$

Letting $k \to \infty$ we obtain

$$\int_{S_T} (w_t\phi + \chi\cdot\nabla\phi - f\phi)\, dz = 0.$$

Since $C^1(0,T;C_0^1(\mathbb{R}^n))$ is dense in $W(Q_{T,k})$ for every k, the same is true for $\phi = w_k$, which gives the energy identity as $k \to \infty$:

$$\int_{S_T} (w_t w + \chi\cdot\nabla w - fw)\, dz = 0. \tag{26}$$

By virtue of Proposition 2.6 and Corollary 2.8 the energy equality holds:

$$\frac{1}{2}\int_{\mathbb{R}^n} w^2\, dx\Big|_{t=0}^{t=T} + \int_{S_T} (\chi\cdot\nabla w - fw)\, dx\, dt = 0.$$

It is now standard to check that $\chi = \mathscr{A}(z,\nabla w)$ a.e. in S_T.

5. Boundedness and uniqueness of weak solutions

In this section we give the proof of Theorem 1.3, which is split into three assertions.

Lemma 5.1 (The maximum principle). *Let* $\sup_{S_T}|f| \leq C_f$ *and* $\sup_{\Omega}|u_0| \leq M_0$ *with finite positive constants* C_f *and* M_0*. Then the solution of problem* (1) *satisfies the estimate*

$$|u| \leq Me^{(C_f/M)T} \qquad \text{a.e. in } S_T \text{ with } M = 1 + M_0. \tag{27}$$

Proof. It is sufficient to show that estimate (27) holds true for the solutions of the auxiliary problems (13) u_k. Set $v_k := \mathrm{e}^{-\lambda t} u_k$ with a constant λ to be defined and write identity (14) in the form

$$\int_{Q_{T,k}} \mathrm{e}^{\lambda t}\big(v_{k,t}\phi + \mathscr{A}(z, \mathrm{e}^{\lambda t}\nabla v_k)\cdot\nabla\phi\big)\,dz = -\int_{Q_{T,k}} (\lambda \mathrm{e}^{\lambda t} v_k - f)\phi\,dz.$$

Now set

$$\lambda = \frac{C_f}{M}, \qquad v_{k,M} = \max\{v_k - M, 0\}, \qquad \phi = \mathrm{e}^{-\lambda t} v_{k,M}.$$

By virtue of Remark 3.11 ϕ is an admissible test-function. Notice that

$$v_{k,t}v_{k,M} = \frac{1}{2}\partial_t v_{k,M}^2, \qquad \mathscr{A}(z, \mathrm{e}^{\lambda t}\nabla v_k)\cdot\nabla v_{k,M} = \mathscr{A}(z, \mathrm{e}^{\lambda t}\nabla v_{k,M})\cdot\nabla v_{k,M} \ge 0.$$

Then

$$\begin{aligned}\int_{Q_{T,k}} &\big(v_{k,t}v_{k,M} + \mathscr{A}(z, \mathrm{e}^{\lambda t}\nabla v_k)\cdot\nabla v_{k,M}\big)\,dz \\ &= \frac{1}{2}\int_{B_k} v_{k,M}^2(T)\,dx + \int_{Q_{T,k}} \mathscr{A}(z, e^{\lambda t}\nabla v_k)\cdot\nabla v_{k,M}\,dz := J\end{aligned}$$

with

$$\begin{aligned}J &= -\int_{Q_{T,k}} (\lambda \mathrm{e}^{\lambda t} v_k - f)\phi\,dz = -\int_{Q_{T,k}\cap\{v_k > M\}} (\lambda v_k - \mathrm{e}^{-\lambda t} f) v_{k,M}\,dz \\ &\le -\int_{Q_{T,k}\cap(v_k>M)} (\lambda M - C_f) v_{k,M}\,dz \le 0.\end{aligned}$$

It follows that $v_k \le M$ a.e. in $Q_{T,k}$. In the same way we check then that $-v_k \le M$. Thus, $|v_k| \le M$ and $|u_k| \le M \mathrm{e}^{(C_f/M)T}$. □

Lemma 5.2 (Uniqueness of bounded solutions). *If $p(x) \ge 2$ a.e. in $\mathbb{R}^n$, problem* (1) *cannot have more than one bounded solution.*

Proof. Let u_1, u_2 be two weak solutions of problem (1). Assume that there exists a finite constant M such that $|u_i| \le M$ a.e. in S_T, set $w = u_1 - u_2$ and introduce the function

$$\psi(s) = \begin{cases} 1 & \text{if } |s| \le R, \\ s - R + 1 & \text{if } R < |s| < R+1, \\ 0 & \text{if } R+1 \le |s|. \end{cases}$$

It is easy to see that $|\psi'(s)| \leq 1$. Taking $\psi(|x|)u_i$ for the test-functions in the integral identities (4) for u_i and subtracting the results we arrive at the relation

$$\frac{1}{2}\int_{\mathbf{R}^n} w^2(T)\psi(|x|)\,dx + \int_0^T\int_{\mathbb{R}^n} \psi(|x|)\big(\mathscr{A}(z,u_1) - \mathscr{A}(z,u_2)\big)\cdot\nabla w\,dz = I$$

with

$$I = -\int_0^T\int_{\mathbb{R}^n} w\big(\mathscr{A}(z,u_1) - \mathscr{A}(z,u_2)\big)\cdot\nabla\psi(|x|)\,dz.$$

This integral is estimated as follows:

$$\begin{aligned}
|I| &\leq a_+\int_0^T\int_{\mathbb{R}^n\setminus B_R(0)} (|\nabla u_1|^{p-1} + |\nabla u_2|^{p-1})(|u_1| + |u_2|)\,dz \\
&\leq C\Big(\int_0^T\int_{\mathbb{R}^n\setminus B_R(0)} (|\nabla u_1|^p + |\nabla u_2|^p)\,dz + \int_0^T\int_{\mathbb{R}^n\setminus B_R} (|u_1|^p + |u_2|^p)\,dz\Big) \\
&\equiv I_1(R) + I_2(R), \qquad C = C(a_+, p_\pm).
\end{aligned}$$

Due to estimate (5) it is necessary that $I_1(R) \to 0$ as $R \to \infty$. Let us consider the integral $I_2(R)$. By assumption $|u_i| \leq M$ with a finite constant M. Since $p(x) \geq 2$ a.e. in $\mathbb{R}^n$, applying (5) we obtain

$$I_2(R) \leq T\sup_{\mathbb{R}^n} M^{p(x)-2}\,\operatorname*{ess\,sup}_{(0,T)}\int_{\mathbb{R}^n\setminus B_R(0)} (|u_1|^2 + |u_2|^2)\,dx \to 0 \qquad \text{as } R \to \infty. \quad \square$$

The assertion of item 3) of Theorem 1.3 follows from Lemma 3.12 as $k \to \infty$.

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Received July 16, 2014; revised November 20, 2014

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