

## **Positive and nodal solutions bifurcating from the infinity for a semilinear equation: solutions with compact support**

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*Dedicado a João-Paulo Dias, al que me une tantas  
cosas, en ocasión de su setenta aniversario*

J.I.D.

*Para João-Paulo Dias, con quien escribí mis primeros  
artículos, por más de cuarenta años de amistad*

J.H.

**Abstract.** The countable branches of nodal solutions bifurcating from the infinity for a sublinear semilinear equation are described with two different approaches. In the one-dimensional case we use plane phase methods of ordinary differential equations. The general  $N$ -dimensional problem can be studied by using topological methods and we sketch here some previous results by the second author in collaboration with João-Paulo Dias. One of the main motivations of the present paper was the ambiguity of the mathematical treatment of the Schrödinger equation for the infinite well potential. We study the classes of flat solutions (i.e. with zero normal derivative at the boundary) and solutions with compact support of the semilinear problem which allow to offer a kind of “alternative approach” to the infinite well potential for the Schrödinger equation.

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### 1. Introduction

This paper deals with the study of the countable branches of nodal solutions bifurcating from the infinity for the eigenvalue type problem

$$\begin{cases} -\Delta u + V_0|u|^{m-1}u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where  $\Omega$  is an open bounded set in  $\mathbb{R}^N$ ,  $V_0 > 0$ ,  $m \in (0, 1)$  and  $\lambda$  is a real parameter. We shall show here that the one-dimensional case

$$\begin{cases} -u'' + V_0|u|^{m-1}u = \lambda u & \text{in } ]-R, R[, \\ u(\pm R) = 0, \end{cases} \tag{2}$$

can be studied by using plane phase methods of ordinary differential equations. This kind of arguments were used by the authors in [9] (extended in [12] to the case of  $m \in (-1, 1)$ ) to a variation of the problem which is recalled in Section 2 of this paper. They have the advantage of providing a complete description of the solution set for (2), something that cannot be expected for (1), except in the radial case  $\Omega = B_R(0)$ .

We deal with the one-dimensional problem (2) in Section 2, where we start by proving the existence of a branch of positive solutions for a bounded interval of the parameter,  $\lambda \in (\lambda_1, \lambda_1^*(m))$ . We recall that  $\lambda_1 = \left(\frac{\pi}{2R}\right)^2$  is the first eigenvalue of the linearized problem

$$\begin{cases} -u'' = \lambda u & \text{in } (-R, R), \\ u(\pm R) = 0. \end{cases} \tag{3}$$

Here  $\lambda_1^*(m)$  is a certain value of the parameter whose exact definition depends crucially of the main assumption  $m \in ]0, 1[$ :

$$\lambda_1^*(m) = \frac{1}{2R^2} \left( \int_0^{(2/(1+m))^{1/(1-m)}} \frac{dr}{(F(\mu) - F(r))^{1/2}} \right)^2 \tag{4}$$

with  $F(r) = \frac{r^2}{2} - \frac{r^{m+1}}{m+1}$ . We show that the (unique) positive solution for  $\lambda = \lambda_1^*(m)$  has a peculiar behaviour near the boundary since

$$u'(\pm R) = 0 \tag{5}$$

(in contrast with the fact that

$$u'(R) < 0 \quad \text{and} \quad u'(-R) > 0$$

if  $\lambda \in (\lambda_1, \lambda_1^*(m))$ ). This is the reason why we call any solution which additionally satisfies (5) as “flat solution”. We point out that this type of special solutions was called previously by other authors as “free boundary nonnegative solutions” (see, e.g. [15]), nevertheless in our opinion the use of the expression “free boundary” may be misleading: such terminology is more adequate in a context where the equation (2) is set in the whole real line and not in a bounded interval.

The associated solution  $u_{\lambda_1^*(m), V_0}$  (when extended by zero to the real line  $\mathbb{R}$ ) gives rise to a continuum of nonnegative solutions  $u_{\lambda, V_0}$  for any  $\lambda > \lambda_1^*(m)$  through a double rescaling (in amplitude and in the argument of application). This type of solutions have compact support in the sense that

$$\text{support } (u_{\lambda, V_0}) \Subset \Omega,$$

and, in fact, the boundary of the support must be understood as a free boundary of the problem. In a second result we show a qualitatively similar result for the branches of nodal solutions changing a finite number of times of sign and emanating from the infinity from the simple eigenvalues  $\lambda_n$ , for  $n > 1$ , of the linear problem (3). The global bifurcation diagram is qualitatively described in the following Figure 3 below. Some of these results were already announced in [10].

The general formulation, problem (1), can be studied by using either variational or topological methods. Concerning the latter, existence of a continuum of nonnegative solutions was proved in [13] by using a differentiability result for the solution operator in [5] [6] (see also [2]) and general results for asymptotic bifurcation [22], [1] and [4]. Much later, Porretta [18] proved existence of nonnegative solutions for any  $\lambda > \lambda_1$  (with  $\lambda_1$  the first eigenvalue of the linear problem (3)) by using variational methods. In [11] the authors use Nehari manifolds to find non-negative solutions and more information on both positive, flat and compact support solutions is obtained by using, in particular, a Pohozaev identity for starshaped  $\Omega$  (see [14]). We sketch the asymptotic bifurcation approach in Section 3. Some of these ideas were presented in [10] and will be developed in [11].

One of the main motivations of the present paper was the series of lectures by the first author ([7]) on the ambiguity of the mathematical treatment offered in most of the textbooks on Quantum Mechanics for the study of bound states of the Schrödinger equation

$$-\Delta u + V(x)u = \lambda u \quad \text{in } \mathbb{R}^N \tag{6}$$

for the infinite well potential  $V(x)$ . For instance, for the one-dimensional case such a potential is given by

$$V(x) = \begin{cases} V_0 & \text{if } x \in (-R, R), \\ +\infty & \text{if } x \notin (-R, R). \end{cases} \tag{7}$$

It was pointed out by this author (see also the detailed exposition made in [8]), it seems that for the first time in the literature, that in fact what is usually presented as “the corresponding solution of this Schrödinger equation” is not strictly true since the solution generates two Dirac deltas over the boundary points  $x = \pm R$ . Nevertheless, it is possible to offer an alternative to this type of “localizing” process by considering other kinds of different potentials. Here by a solution  $u$  of the above problem we must understand any function  $u = \lim_{q \rightarrow \infty} u_q$  with  $u_q$  solution of the problem associated to the truncated potential  $V_q$  (see Remark 2.0 of [8]). If we consider, again, the one-dimensional case, as a consequence of the present paper, the *linear* eigenvalue problem

$$\begin{cases} -u'' + V_0|u_{\lambda^*}(x)|^{m-1}u = \lambda u & \text{in } (-R, R), \\ u(\pm R) = 0, \end{cases} \quad (8)$$

has a first (principal) eigenvalue  $\lambda^*$  with a positive eigenfunction  $u_{\lambda^*} > 0$  such that,  $u'_{\lambda^*}(\pm R) = 0$ . Now, this function can be extended by zero outside of  $(-R, R)$ , without generating any Dirac delta on the boundaries  $x = \pm R$ . Thus, such an extension is a correct bound solution of the Schrödinger equation for the potential

$$V(x) = \begin{cases} V_0|u_{\lambda^*}(x)|^{m-1} & \text{if } x \in (-R, R), \\ +\infty & \text{if } x \notin (-R, R). \end{cases}$$

It is in this sense that all this can be interpreted as an “alternative” approach to the infinite well potential for the Schrödinger equation. A more general statement is presented in [8].

In Section 1 we deal with the one-dimensional case by using phase plane methods and give a description of the branch of both positive and compact support nonnegative solutions. In Section 2 we sketch the application of asymptotic bifurcation in the case of a general domain  $\Omega$ .

## 2. The one-dimensional case

In this Section we study the equation (1) by using ODE arguments. This allows to obtain all the solutions to the problem and give a complete description of the set of solutions and study the qualitative properties and its qualitative changes when the parameter  $\lambda$  varies. Some of these results were presented in [10]. We sketch first the results obtained in the study of a similar problem just to illustrate the nature of the solutions and its multiplicity.

We consider the semilinear elliptic equation

$$P(m, q) \begin{cases} -u'' + u^m = \lambda u^q & \text{in } (-1, 1), \\ u(\pm 1) = 0, \end{cases} \quad (9)$$

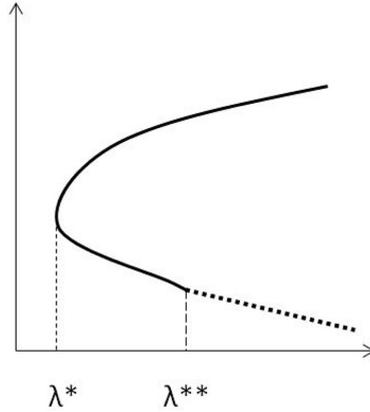


Figure 1. Bifurcation diagram for problem  $P(m, q)$ .

where  $\lambda$  is a real parameter and

$$0 < m < q < 1. \tag{10}$$

This problem was studied in [9] and the extension to  $-1 < m < q < p - 1$  and the  $p$ -Laplacian as well was carried out in [12]. The results are illustrated in the diagram above.

There exists  $0 < \lambda^* < \lambda^{**}$  such that:

- i) First, there is no solution if  $0 < \lambda < \lambda^*$ ;
- ii) For  $\lambda > \lambda^*$  there is an upper branch of positive solutions  $u_\lambda > 0$  such that  $\frac{\partial u_\lambda}{\partial n}(\pm 1) < 0$  which is continued in a lower branch  $v_\lambda > 0$  with  $\frac{\partial v_\lambda}{\partial n}(\pm 1) < 0$  if  $\lambda^* < \lambda < \lambda^{**}$  but such that  $\frac{\partial v_{\lambda^{**}}}{\partial n}(\pm 1) = 0$ .

In this way we obtain a very special flat solution  $v_{\lambda^{**}}$  of (9) such that

$$v_{\lambda^{**}}(\pm 1) = v'_{\lambda^{**}}(\pm 1) = 0.$$

Hence this solution is also a solution of (9) on the whole real line and can be “translated” freely “in both directions”. This possibility together with “stretching” manipulations gives rise to a variety of compact support solutions (i.e., solutions  $u$  such that  $\text{supp}(u) \subset \Omega$ ).

Now we study problem (2) where  $V_0 > 0$  and  $0 < m < 1$ . We shall prove our main result

**Theorem 2.1.** *We define*

$$\gamma(\mu) := \frac{1}{\sqrt{2}} \int_0^\mu \frac{dr}{(F(\mu) - F(r))^{1/2}} \quad (11)$$

with  $F(r) = \frac{r^2}{2} - \frac{r^{m+1}}{m+1}$ . Let  $r_F = (2/(1+m))^{1/(1-m)}$ . Then the mapping  $\gamma : [r_F, +\infty) \rightarrow \mathbb{R}$  has the following properties

- i)  $\gamma \in C[r_F, \infty) \cap C^1(r_F, \infty)$ ;
- ii)  $\gamma'(\mu) \rightarrow -\infty$  as  $\mu \downarrow r_F$ ;
- iii) For any  $\mu > r_F$ ,  $\gamma'(\mu) < 0$ ;
- iv)  $\lim_{\mu \rightarrow +\infty} \gamma(\mu) = \frac{\pi}{2}$ .

Moreover, if we call

$$\lambda_1^*(m) = \frac{1}{2R^2} \left( \int_0^{r_F} \frac{dr}{(F(\mu) - F(r))^{1/2}} \right)^2 \quad (12)$$

then we have:

- a) if  $\lambda \in (0, (\frac{\pi}{2R})^2)$  there is no nonnegative solution,
- b) if  $\lambda \in ((\frac{\pi}{2R})^2, \lambda_1^*(m))$  there is a unique positive solution  $u_{\lambda, V_0}$ . Moreover  $\frac{\partial u_{\lambda, V_0}}{\partial n}(\pm R) < 0$  and  $\|u_{\lambda, V_0}\|_{L^\infty(-R, R)} = \left(\frac{V_0}{\lambda}\right)^{1/(1-m)} \gamma^{-1}(\sqrt{\lambda}R)$ ,
- c) if  $\lambda = \lambda_1^*(m)$  there is only one positive solution  $u_{\lambda_1^*(m), V_0}$ . Moreover

$$u'_{\lambda_1^*(m), V_0}(\pm R) = 0$$

and

$$\|u_{\lambda_1^*(m), V_0}\|_{L^\infty(-R, R)} = \left(\frac{2V_0}{\lambda_1^*(m)(1+m)}\right)^{1/(1-m)},$$

- d) if  $\lambda > \lambda_1^*(m)$ , there is a family of nonnegative solutions which are generated by extending by zero the function  $u_{\lambda_1^*(m), V_0}$  outside  $(-R, R)$  (and which we label again as  $u_{\lambda_1^*(m), V_0}$ ). In particular, if  $\lambda = \lambda_1^*(m)\omega$  with  $\omega > 1$  we have a family  $S_1(\lambda)$  of compact support nonnegative solutions with connected support defined by

$$u_{\lambda, V_0}(x) = \frac{1}{\omega^{1/(1-m)}} u_{\lambda_1^*(m), V_0}(\sqrt{\omega}x - z)$$

where the shifting argument  $z$  is arbitrary among the points  $z \in (-R, R)$  such that support  $u_{\lambda, V_0}(\cdot) \subset (-R, R)$ . Moreover, for  $\lambda > \lambda_1^*(m)$  large enough we can build, similarly, a subset of  $S_j(\lambda)$  of compact support nonnegative solutions with the support formed by  $j$ -components, with  $j \in \{1, 2, \dots, N\}$ , for some suitable  $N = N(\lambda)$  and then the set of nontrivial and nonnegative solutions of  $P(\lambda)$  is formed by  $S(\lambda) = \bigcup_{j=1}^N S_j(\lambda)$ . In any case those solutions are such that

$$\begin{aligned} \|u_{\lambda, V_0}\|_{L^\infty(-R, R)} &= \frac{1}{\omega^{1/(1-m)}} \|u_{\lambda_1^*(m), V_0}\|_{L^\infty(-R, R)} \\ &= \frac{1}{\omega^{1/(1-m)}} \left( \frac{2V_0}{\lambda_1^*(m)(1+m)} \right)^{1/(1-m)} \end{aligned}$$

for any  $\omega = \frac{\lambda}{\lambda_1^*(m)} > 1$ .

*Proof of Theorem 2.1.* It is easy to show, multiplying by  $u$  and integrating by parts, that there are nontrivial solutions only if  $\lambda > \lambda_1 = \frac{\pi^2}{4R^2}$ , the first eigenvalue to problem (3), i.e. (2) when  $V_0 = 0$ . To show the qualitative behaviour of solutions of problem (2), we make the change of variables

$$u_{\lambda, V_0}(x) = \left( \frac{V_0}{\lambda} \right)^{1/(1-m)} u(\sqrt{\lambda}x)$$

where  $u$  is now the solution of the renormalized problem

$$P(L) \begin{cases} -u'' = f(u) & \text{in } (-L, L), \\ u(\pm L) = 0, \end{cases} \tag{13}$$

where  $f(u) = u - u^m$  and  $L = \sqrt{\lambda}R$ . We introduce

$$F(r) = \int_0^r f(s) ds = \frac{r^2}{2} - \frac{r^{m+1}}{m+1}$$

and note that  $f(s) < 0$  if  $0 < s < 1 := r_f$  and  $f(s) > 0$  if  $1 < s$ . On the other hand  $F(s) < 0$  if  $0 < s < r_F = (2/(1+m))^{1/(1-m)}$  and  $F(s) > 0$  for  $s > r_F$ .

For  $\mu > r_F$  we define the mapping  $\gamma : [r_F, +\infty) \rightarrow \mathbb{R}$  given by (11). Now we use the following fact whose proof is exactly as in [9] and [12]: a function  $u$  is a positive solution of problem  $P(L)$  if and only if

$$\frac{1}{\sqrt{2}} \int_{u(x)}^\mu \frac{dr}{(F(\mu) - F(r))^{1/2}} = |x|, \quad \text{for } |x| \leq L,$$

where  $\mu := \|u\|_{L^\infty}$  (such that  $\mu \in (r_F, \infty)$ ) and  $L > 0$  are related by the equation  $\gamma(\mu) = L$ . Moreover

$$u'(\pm L) = \mp \sqrt{2F(\mu)}. \quad (14)$$

Thus, we get that  $u'(\pm R) = 0$  corresponds to the case in which the maximum of the solution is  $r_F$ .

The main properties of  $\gamma(\mu)$  are collected in the auxiliary properties i)–iv). The proof of properties i) and ii) is exactly the same presented in [9] and [12] for a similar case. For the proof of (iii), as in [12], we have

$$\gamma'(\mu) = \int_0^\mu \frac{\theta(\mu) - \theta(r)}{(F(\mu) - F(r))^{3/2}} dr$$

where  $\theta(t) = 2F(t) - tf(t) = -\frac{1-m}{1+m}t^{m+1}$  and differentiating we get for any  $t > 0$

$$\theta'(t) = -(1-m)t^m < 0.$$

Hence  $\gamma'(\mu) < 0$  for any  $\mu > r_F$ .

Finally, to prove (iv) we note that

$$\gamma(\mu) \leq \frac{\mu}{2} \int_0^1 \frac{dt}{\left(\frac{\mu^2}{2}(1-t^2)\right)^{1/2}} = \int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}. \quad (15)$$

Moreover, we have

$$\gamma(\mu) = \frac{\mu}{\sqrt{2}} \int_0^1 \frac{dt}{\left(\frac{\mu}{\sqrt{2}}\left((1-t^2) - \frac{1}{\mu^{1-m}}(1-t^{m+1})\right)\right)} \quad (16)$$

and if  $\mu \rightarrow +\infty$  by using Lebesgue's Theorem we get

$$\lim_{\mu \rightarrow +\infty} \gamma(\mu) = \frac{\pi}{2}.$$

Now we define  $L_0 = \frac{\pi}{2}$  and  $L^*$  given by

$$L^* = \gamma(r_F) = \frac{1}{\sqrt{2}} \int_0^{r_F} \frac{dr}{(F(\mu) - F(r))^{1/2}} = \frac{1}{\sqrt{2}} \int_0^{r_F} \frac{dr}{\left(\frac{r^{m+1}}{m+1} - \frac{r^2}{2}\right)^{1/2}}.$$

We know that  $u'(\pm R) = 0$  corresponds to the value  $L^*$  and that the maximum of the solution is  $r_F$ . So, the function, qualitatively, function  $\gamma$  is described by the following Figure 2:

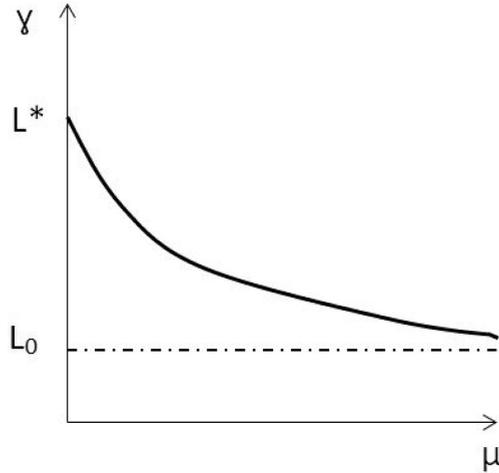


Figure 2. The function  $\gamma$ .

If now we go back with our change of variables we get

$$\|u_{\lambda, V_0}\|_{L^\infty(\Omega)} = \left(\frac{V_0}{\lambda}\right)^{1/(1-m)} r_F$$

and we obtain finally the bifurcation diagram given by the first branch of Figure 3 below, where solutions for  $\lambda > \lambda^*$  are compact supported solutions originated as in [9] and [12] from the extension by zero of the flat solution  $u_{\lambda^*}$  satisfying

$$\begin{cases} -u_{\lambda^*}'' + V_0|u_{\lambda^*}|^{m-1}u_{\lambda^*} = \lambda^*u_{\lambda^*} & \text{in } (-R, R), \\ u_{\lambda^*}(\pm R) = u'_{\lambda^*}(\pm R) = 0. \end{cases} \tag{17}$$

The rest of details are completely similar to the similar parts of the papers [9] and [12]. ■

**Remark 2.2.** Once that we know that for  $\lambda > \lambda_1^*(m)$  we have that

$$u_{\lambda, V_0}(x) = \frac{1}{\omega^{1/(1-m)}} u_{\lambda_1^*(m), V_0}(\sqrt{\omega}x - z)$$

we can express in terms of  $\lambda$  other norms (different than the  $L^\infty$ -norm). For instance we have that

$$\|u'_{\lambda, V_0}\|_{L^\infty(-R, R)} = C\lambda^{-(1+m)/(2(1-m))}$$

for a suitable constant  $C > 0$  independent of  $\lambda$ . This improves the conclusion  $\|u_{\lambda, V_0}\|_{H^1_0(-R, R)} \rightarrow 0$  as  $\lambda \rightarrow +\infty$  proved in Theorem 1 of [18]. ■

We shall end this section by studying the branches corresponding to nodal solutions, i.e. changing sign a finite number of times. We state with detail the case of the branch which bifurcates from the infinity from the second eigenvalue  $\lambda_2$  of the linear problem (3) and it is left to the reader to get similar statements for other branches. We recall that when  $\Omega = (-R, R)$ , then  $\lambda_2 = \pi^2/R^2$ . We shall study the branch of solutions which are nonnegative on  $(0, R)$  and nonpositive on  $(-R, 0)$  (the reverse case is, obviously, similar). Let us call to these solutions as “nonnegative-nonpositive solutions” (and by similarity with the definitions presented in the Introduction we can talk of positive-negative solutions, flat positive-negative solutions and nonnegative-nonpositive solutions with compact support).

**Theorem 2.3.** *We define  $\lambda_2^*(m) = 2\lambda_1^*(m)$  then we have:*

- a) *if  $\lambda \in (0, (\frac{\pi}{R})^2)$  there is no positive-negative solution,*
- b) *if  $\lambda \in ((\frac{\pi}{R})^2, \lambda_2^*(m))$  there is a unique positive-negative solution  $u_{\lambda, V_0}$ . Moreover*

$$\frac{\partial u_{\lambda, V_0}}{\partial n}(R) < 0, \quad \frac{\partial u_{\lambda, V_0}}{\partial n}(-R) > 0$$

and

$$\|u_{\lambda, V_0}\|_{L^\infty(-R, R)} = \left(\frac{V_0}{\lambda}\right)^{1/(1-m)} \gamma^{-1}(\sqrt{\lambda}R/2),$$

- c) *if  $\lambda = \lambda_2^*(m)$  there is only one positive-negative solution  $u_{\lambda_2^*(m), V_0}$ . Moreover*

$$u'_{\lambda_2^*(m), V_0}(\pm R) = 0$$

(i.e.  $u_{\lambda_2^*(m), V_0}$  is a flat positive-negative solution) and

$$\|u_{\lambda_2^*(m), V_0}\|_{L^\infty(-R, R)} = \left(\frac{2V_0}{\lambda_2^*(m)(1+m)}\right)^{1/(1-m)},$$

- d) *if  $\lambda > \lambda_2^*(m)$ , there is a family of nonnegative-nonpositive solutions which are generated by extending by zero the function  $u_{\lambda_2^*(m), V_0}$  outside  $(-R, R)$  (and which we label again as  $u_{\lambda_1^*(m), V_0}$ ). In particular, if  $\lambda = \lambda_1^*(m)\omega$  with  $\omega > 1$ , we have a family  $\mathcal{S}_1(\lambda)$  of compact support nonnegative solutions with connected support defined by*

$$u_{\lambda, V_0}(x) = \frac{1}{\omega^{1/(1-m)}} u_{\lambda_2^*(m), V_0}(\sqrt{\omega}x - z)$$

where the shifting argument  $z$  is arbitrary among the points  $z \in (-R, R)$  such that support  $u_{\lambda, V_0} \subset (-R, R)$ . Moreover, for  $\lambda > \lambda_2^*(m)$  large enough we can build, similarly, a subset of  $S_j(\lambda)$  of compact support nonnegative solutions with the support formed by  $j$ -components, with  $j \in \{1, 2, \dots, N\}$ , for some suitable  $N = N(\lambda)$  and then the set of nontrivial and nonnegative solutions of  $P(\lambda)$  is formed by  $S(\lambda) = \bigcup_{j=1}^N S_j(\lambda)$ . In any case those solutions satisfy that

$$\begin{aligned} \|u_{\lambda, V_0}\|_{L^\infty(-R, R)} &= \frac{1}{\omega^{1/(1-m)}} \|u_{\lambda_2^*(m), V_0}\|_{L^\infty(-R, R)} \\ &= \frac{1}{\omega^{1/(1-m)}} \left( \frac{2V_0}{\lambda_2^*(m)(1+m)} \right)^{1/(1-m)} \end{aligned}$$

for any  $\omega = \frac{\lambda}{\lambda_2^*(m)} > 1$ . In addition there exists a continuum of solutions taking the value  $u_{\lambda, V_0} = 0$  in intermediate points between the set of points where  $u_{\lambda, V_0} < 0$  and where  $u_{\lambda, V_0} > 0$ .

*Proof of Theorem 2.3.* The crucial point is the elementary remark that if  $v$  is a nonnegative solution of problem

$$\begin{cases} -v'' + V_0 v^m = \lambda v & \text{in } (0, R), \\ v(0) = v(R) = 0, \end{cases} \tag{18}$$

then the function  $u$  defined by

$$u(x) = \begin{cases} v(x) & \text{if } x \in (0, R), \\ v(-x) & \text{if } x \in (-R, 0), \end{cases}$$

is a nonnegative-nonpositive solution of problem (2). Indeed, thanks to the symmetry of function  $v$  we know that  $v'(0) = -v'(R)$  and, in particular the function  $u$  is at least  $C^1$  and  $u'$  does not develop any singularity at the origin  $x = 0$ . As a matter of fact, the explicit construction in Theorem 2.1 allows to see that  $u \in C^2(-R, R)$  and that it is a classical solution of the equation (2) in the whole interval  $(-R, R)$ .

Now, the positive flat solution  $v^*(y)$ ,  $y \in (0, R)$  corresponds to the value of the parameter  $\hat{\lambda}^*(m)$

$$\hat{\lambda}^*(m) = \frac{1}{2(R/2)^2} \left( \int_0^{r^*} \frac{dr}{(F(\mu) - F(r))^{1/2}} \right)^2 = 2\lambda_1^*(m)$$

(this is an obvious adaptation of part b) of the proof of Theorem 2.1). Analogously, the first eigenvalue of the linear problem (3) when  $\Omega = (0, R)$  is given by

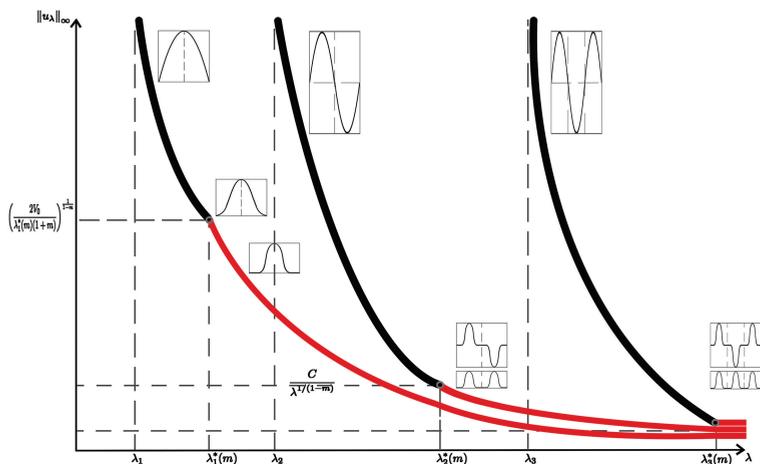


Figure 3. Bifurcation diagram.

$\pi^2/R^2$  (which coincides with  $\lambda_2$ , the second eigenvalue  $\lambda_2$  of the linear problem (3) when  $\Omega = (-R, R)$ ). Thus, by using Theorem 2.1 we get the qualitative reproduction of the first branch of nonnegative solutions of the problem (18) but now going asymptotically to infinity when  $\lambda \searrow \lambda_2$  and generating a flat positive-negative solution for  $\lambda = \hat{\lambda}^*(m)$  (which may be denoted by  $\lambda_2^*(m)$ ). Since the associate solution  $v_{\lambda_2^*(m), V_0}$  is “flat” near its boundary ( $v'_{\lambda_2^*(m), V_0}(0) = v'_{\lambda_2^*(m), V_0}(R) = 0$ ) we get that its extension by zero generates nonnegative-nonpositive solutions with compact support for  $\lambda > \lambda_2^*(m)$ . ■

**Remark 2.4.** Note that if  $m \nearrow 1$  then  $\lambda_i^*(m) \nearrow +\infty$  (since the integral becomes divergent near  $r = 0$ ). Contrarily, if  $m \searrow 0$  then

$$\lambda_1^*(m) = \frac{1}{2R^2} \left( \int_0^{(2/(1+m))^{1/(1-m)}} \frac{dr}{(F(\mu) - F(r))^{1/2}} \right)^2$$

converges to

$$\lambda_1^*(0) = \frac{1}{2R^2} \left( \int_0^2 \frac{dr}{(F_0(\mu) - F_0(r))^{1/2}} \right)^2,$$

where  $F_0(r) := \frac{r^2}{2} - r = r\left(\frac{r}{2} - 1\right)$ . ■

### 3. The general case $N > 1$ . Asymptotic bifurcation

In this section we sketch the employ of asymptotic bifurcation in order to get positive or non-negative solutions to our problem in the case of a general domain in  $\mathbb{R}^N$ .

We consider first the problem

$$\begin{cases} -\Delta u + |u|^{m-1}u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where  $\Omega$  is an open bounded set in  $\mathbb{R}^N$ . Let us start with the case  $m > 1$ . Here  $\lambda$  is a real parameter. For  $m = 2$  this is a version of the well-known *logistic equation*.

It is also well-known that nontrivial solutions exists only for  $\lambda > \lambda_1$ . There are several proofs of the existence for any  $\lambda > \lambda_1$  of a unique positive solution. One of them is to apply some global bifurcation theorem by Rabinowitz showing that there is an unbounded continuum of positive solutions bifurcating from the first eigenvalue  $\lambda = \lambda_1$  of the linearized problem at the origin for (19). The result follows in this case from global bifurcation together with some (easy to find) a priori estimates for positive solutions.

In the one-dimensional case  $\Omega = (0, 1)$  it was proved in [20], [21] that all (simple) eigenvalues of the linearized problem (at the origin) are bifurcation points and that the well-known nodal properties of eigenfunctions are preserved all along the bifurcating continua. It was also proved by Böhme [3] and Marino [17] that in the variational case all eigenvalues are actually bifurcating points for (19), independently of its multiplicity.

Problem (19) with  $0 < m < 1$  can be studied by using the results by Rabinowitz [22] (see also [1] and [4]) following and greatly extending previous classical work by Krasnoselski [16] for bifurcation at infinity.

For this we need a theorem concerning the existence of an asymptotic derivative for the nonlinear solution operator arising in the problem. More precisely, it is known that for any  $u \in L^2(\Omega)$  the problem

$$\begin{cases} -\Delta w + |w|^{m-1}w + w = u & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

has a unique solution  $w \in H^2(\Omega) \cap H_0^1(\Omega)$  and that the nonlinear solution operator given by  $w = Pu$ ,  $P : L^2(\Omega) \rightarrow L^2(\Omega)$ , is a compact monotone operator and that  $P$  is Fréchet differentiable at the infinity with  $A = P'(\infty)$ , where  $A$  is defined as the unique solution of the linear problem

$$\begin{cases} -\Delta Au + Au = u & \text{in } \Omega, \\ Au = 0 & \text{on } \partial\Omega, \end{cases} \tag{21}$$

and

$$\lim_{\|u\|_{L^2(\Omega)} \rightarrow +\infty} \frac{\|Pu - Au\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} = 0$$

(see [2], [5], [6]).

It follows from a result in [5] (Theorem 2.1) [6] and [2] that all eigenvalues for  $A$  are actually asymptotic bifurcation points following a partial result in [23]. Later, the authors learned that the result was previously proved in [19]. The corresponding results hold for the Sturm-Liouville nonlinear problem as well.

However, there is an interesting feature making a difference between ordinary (i.e., at the origin) and asymptotic bifurcation. Indeed, the nodal properties of solution are not preserved all along the bifurcating continua in the case of bifurcation at infinity. This remark is already in Rabinowitz's paper [22], more precisely a counterexample is given in ([22], Remark 2.12); nodal properties are only preserved in *some* neighborhood (at infinity) of the corresponding eigenvalues. From this point of view our results here (and in [10]) may be interpreted as particular examples showing how the nodal properties change along the corresponding branches in this precise instance.

These ideas were pursued in [13], where the existence of an unbounded continuum of non-negative solutions bifurcating at infinity at  $\lambda = \lambda_1$  was proved. But the question of flat and compact solutions was not raised there. More precisely the following result was proved in [13].

**Theorem 3.1.** *Under the above assumptions there exists an unbounded continuum of non-negative solutions to (19) bifurcating from infinity at  $\lambda_1$ .*

It is possible, reasoning as in [22] to show that solutions in some neighborhood of  $(\lambda_1, \infty)$  are actually positive ( $u > 0$ ) and also  $\frac{\partial u}{\partial n} < 0$  on  $\partial\Omega$ . It is easy to see that there is no ordinary bifurcation point; moreover, a simple argument gives the estimate

$$\|u_\lambda\|_{L^\infty(\Omega)} \geq \left(\frac{1}{\lambda}\right)^{1/(1-m)}$$

for any non-negative solution for the value  $\lambda$  of the parameter. Much more information is given in [11]. Concerning uniqueness it was only shown in [13] that if  $u$  and  $v$  are ordered solutions, i.e. if say,  $u \leq v$ , then  $u \equiv v$ . Again, some results can be found in [11].

*Added in proof.* After the completion of this paper the authors became aware of two papers by V. Ozolins, R. Lai, R. Caffisch, and S. Osher (Proc. Nat. Acad. Sci.

USA 110(46), 2013, 18368–18373 and Proc. Nat. Acad. Sci. USA 111 (5), 2014, 1691–1696) where they present a set of very interesting numerical results on the so called “compressed modes” functions. In the one dimensional case such functions are very similar to the nodal solutions considered here for the special case of  $m = 0$  (see our Remark 2.4).

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