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On the strong oscillatory behavior of all solutions to some second order evolution equations

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Dedicated to Professor João Paulo de Carvalho Dias on the occasion of his 70th birthday

Abstract. Let l be any positive number. For any non-negative potential $p \in L^{\infty}(0,l)$, we show that for any solution u of $u_{tt} + u_{xxxx} + p(x)u = 0$ in $\mathbb{R} \times (0,l)$ with $u = u_{xx} = 0$ on $\mathbb{R} \times \{0,l\}$, and for any form $\zeta \in \left(H^2(0,l) \cap H_0^1(0,l)\right)'$, the function $t \to \langle \zeta, u(t) \rangle$ has a zero in each closed interval I of $\mathbb{R}$ with length $|I| \geq \frac{\pi}{3} l^2$. A similar result of uniform oscillation property on each interval of length at least equal to $2l$ is established for all weak solutions of the equation $u_{tt} - u_{xx} + a(t)u = 0$ in $\mathbb{R} \times (0,l)$ with $u = 0$ on $\mathbb{R} \times \{0,l\}$ where a is a nonnegative essentially bounded coefficient. These results apply in particular to any finite linear combination of evaluations of the solution u at arbitrary points of $(0,l)$.

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1. Introduction

This paper is a worked out version of a concise preprint which was written in 1994 and that I did not try to publish at that time since I was expecting possibly better results. Since in 20 years essentially nothing new happenned concerning pointwise oscillation of vibrating systems, I decided to publish the results in a more developped style. Let us first consider the basic equation

$$u'' + Au(t) = 0, \tag{1.1}$$

where V is a real Hilbert space, $A \in L(V, V')$ is a symmetric, positive, coercive operator and there is a second real Hilbert space H for which $V \hookrightarrow H = H' \hookrightarrow V'$ where the imbedding on the left is compact. In this case it is well known, cf. e.g. [1], [2], [9] that all solutions $u \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ of (1.1) are almost periodic: $\mathbb{R} \to V$ with mean-value 0. Then for any form $\zeta \in V'$, the function $g(t) := \langle \zeta, u(t) \rangle$ is a real-valued continuous almost periodic function with mean-value 0. Assuming the existence of a sequence of intervals $J_n = [a_n, b_n]$ with length $|J_n| = |b_n - a_n|$ tending to infinity and such that

$$\forall t \in J_n, \qquad g(t) \geq 0,$$

the almost periodic character of g implies that $g \equiv 0$. Indeed the sequence of almost periodic translates $h_n(\cdot) = g\left(\frac{a_n+b_n}{2} + \cdot\right)$ has at least a uniformly convergent subsequence on the whole line. The limit h is almost periodic, nonnegative with mean-value 0, which is well known to imply, as in the periodic case, that $h \equiv 0$. Now g is the uniform limit on $\mathbb{R}$ of translates of h, implying $g \equiv 0$.

As a consequence, we find that either $g \equiv 0$, or there exists $M > 0$ such that on each interval J with $|J| \geq M$, g takes negative values and (by symmetry of the argument) positive values. We shall say that M is a strong oscillation length for the numerical function g.

Definition 1.1. We say that a number $M > 0$ is a strong oscillation length for a numerical function $g \in L^1_{loc}(\mathbb{R})$ if the following alternative holds: either $g(t) = 0$ almost everywhere, or for any interval J with $|J| \geq M$, we have

$$\operatorname{meas}\{t \in J, f(t) > 0\} > 0 \quad \text{and} \quad \operatorname{meas}\{t \in J, f(t) < 0\} > 0.$$

As a consequence of the previous argument we have

Proposition 1.2. *Under the above conditions on H, V and A, for any solution $u \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ of (1.1) and for any $\zeta \in V'$, the function $g(t) := \langle \zeta, u(t) \rangle$ has some finite strong oscillation number $M = M(u, \zeta)$.*

In the previous papers [3], [4], [8] we have focused our attention to obtaining a strong oscillation length independent of the solution and the observation in various cases, including non-linear perturbations of equation (1.1). A basic example is the vibrating string equation

$$u_{tt} - u_{xx} + g(t, u) = 0 \quad \text{in } \mathbb{R} \times (0, l), \qquad u = 0 \quad \text{on } \mathbb{R} \times \{0, l\} \tag{1.2}$$

where $l > 0$ and $g(t, \cdot)$ is an odd non-decreasing function of u for all t. Here the function spaces are $H = L^2(0, l)$ and $V = H^1_0(0, l)$. Since any function of V is continous, a natural form $\zeta \in V'$ is the Dirac mass δ_{x_0} for some $x_0 \in (0, l)$. It turns out that $2l$ is a strong oscillation length independent of the solution and the observation point x_0, but at the time of [3], [4] we did not consider the case of a general $\zeta \in V' = H^{-1}(0,l)$. The research done in [3], [4] was motivated first by the consideration of the special case $g = 0$, the ordinary vibrating string. Since in this case all solutions are $2l$-periodic with mean-value 0 functions with values in V, it is clear that $2l$ is a strong oscillation length independent of the solution and the observation point x_0. The slightly more complicated $g(t,u) = au$ with $a > 0$ is immediately more difficult since the general solution is no longer time-periodic, it is only almost periodic in t. The time-periodicity is too unstable and for an almost periodic function, the determination of strong oscillation lengths is not easy in general, as was exemplified in [8]. The oscillation result of [3], [4] is consequently not so immediate even in the linear case. In the nonlinear case, it becomes even more interesting because the solutions are no longer known to be almost periodic. The existence of non-trivial periodic solutions (cf. [10]) refers to very special solutions and for an equation very similar to (1.2), it was established in [7] that non-recurrent (in particular, not almost periodic) exceptional solutions do exist. The oscillation results from [3], [4] were later extended to more general string equations by Uesaka [11], [12].

The plan of the paper is as follows: In Section 2, we state a slight improvement of a result from [8] and we give some examples of application. Section 3 is devoted to a new kind of results which cannot follow from the method of [8] since we fall in the limiting case of the method. However a construction relying on the results from [3], [4] will allow us to obtain a strong oscillation length in the case $g(t,u) = a(t)u$.

2. Rapidly oscillating vibrating systems

Let H, V and A be as in the introduction and let $u \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ be a solution of (1.1). In general it is not known whether for any $\zeta \in V'$, the function $g(t) := \langle \zeta, u(t) \rangle$ has a strong oscillation length independent of the solution u. Actually, for the ordinary wave equation in a two dimensional domain, this is even unknown if $\zeta \in H$, while an intermediate result valid for $\zeta \in [D(A^{1/4})]'$ has been proved in [8] when the eigenvalues of A have a sufficient growth. In fact the method of [8] gives the following

Proposition 2.1. *Let $\{\lambda_n\}_{n \in \mathbb{N}}$ be the sequence of eigenvalues of A repeated according to multiplicity and setting $\mu_n := \{\lambda_n\}^{1/2}$, assume that*

$$T = 2\pi \sum_n \frac{1}{\mu_n} < \infty$$

Then for any solution $u \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ of (1.1) and any $\zeta \in V'$, the function $g(t) := \langle \zeta, u(t) \rangle$ has a strong oscillation length equal to T.

Proof. This is just a readjustment of the proof of Theorem 4.1 in [8]. Indeed, although the condition $u \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ does not imply the absolute convergence of the solution series in V, the function $g(t) := \langle \zeta, u(t) \rangle$ belongs to the closure in $C_b(\mathbb{R})$ of finite sums of τ_j-periodic functions with mean-value 0 where $\tau_j = 2\pi \frac{1}{\mu_j}$. Corollary 1.6 of [8] clearly applies to this situation and therefore if $g(t) := \langle \zeta, u(t) \rangle$ is nonnegative on a closed interval J with length $\geq T$, it has to vanish identically on J. Then, due to the pairwise orthogonality in V of the eigenfunctions occuring in the Fourier-Bohr expansion of $u(t)$, we have the special property that u is the limit in $C_b(\mathbb{R})$ of the *truncated* series, so that the functions g_k can be constructed exactly as in the proof of Lemma 2.8 from [8]. By the same argument, we obtain $g(t) := \langle \zeta, u(t) \rangle \equiv 0$ on $\mathbb{R}$. This concludes the proof. □

Corollary 2.2. *Let l be any positive number. For any non-negative potential $p \in L^\infty(0, l)$, and for any solution $u \in C\big(\mathbb{R}, H^2(0, l) \cap H_0^1(0, l)\big) \cap C^1\big(\mathbb{R}, L^2(0, l)\big)$ of*

$$u_{tt} + u_{xxxx} + p(x)u = 0 \quad \text{in } \mathbb{R} \times (0, l), \qquad u = u_{xx} = 0 \quad \text{on } \mathbb{R} \times \{0, l\}$$

and for any $\zeta \in \big(H^2(0, l) \cap H_0^1(0, l)\big)'$, the function $t \to \langle \zeta, u(t) \rangle$ has a strong oscillation length equal to $T = \frac{\pi}{3} l^2$.

Proof. We apply Proposition 2.1 with $H = L^2(0, l)$, $V = H^2(0, l) \cap H_0^1(0, l)$ and A defined by

$$Au = u_{xxxx} + p(x)u, \qquad \forall u \in V$$

Here the eigenvalues of A are simple and by Courant-Fisher's principle it follows easily that $\lambda_n \geq \left(\frac{n\pi}{l}\right)^4$, hence we can apply Proposition 2.1 with

$$T = 2\pi l^2 \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} = 2\pi l^2 \times \frac{1}{\pi^2} \times \frac{\pi^2}{6} = \frac{\pi}{3} l^2$$ □

Corollary 2.3. *Let l be any positive number. For any non-negative potential $p \in L^\infty(0, l)$, and for any solution $u \in C\big(\mathbb{R}, H_0^2(0, l)\big) \cap C^1\big(\mathbb{R}, L^2(0, l)\big)$ of*

$$u_{tt} + u_{xxxx} + p(x)u = 0 \quad \text{in } \mathbb{R} \times (0, l), \qquad u = u_x = 0 \quad \text{on } \mathbb{R} \times \{0, l\}$$

and for any $\zeta \in H^{-2}(0,l)$*, the function* $t \to \langle \zeta, u(t) \rangle$ *has a strong oscillation length equal to* $T = 2\pi l^2 \sum_n \frac{1}{\sigma_n^2}$*, where* σ_n *is the* n^{th} *positive root of the equation* $\cos\sigma \cosh\sigma = 1$.

Proof. We apply Proposition 2.1 with $H = L^2(0,l)$, $V = H_0^2(0,l)$ and A defined by

$$Au = u_{xxxx} + p(x)u, \qquad \forall u \in V$$

Here the eigenvalues of A are simple and by Courant-Fisher's principle it follows easily that $\lambda_n \geq \nu_n$ where ν_n is the n^{th} eigenvalue of the operator L defined by

$$Lu = u_{xxxx}, \qquad \forall u \in V$$

It is well known that $\nu_n = \left(\frac{\sigma_n}{l}\right)^4$ and the difference $\sigma_{n+1} - \sigma_n$ tends to π as n tends to infinity, in particular $\sum_n \frac{1}{\sigma_n^2} < \infty$ and we can apply Proposition 2.1. The rest is obvious. $\square$

Remark 2.4. Both Corollaries 2.2 and 2.3 are applicable to the functional ζ defined by

$$\langle \zeta, \varphi \rangle = \sum_{j \in J} \alpha_j \varphi(x_j) + \sum_{k \in K} \beta_k \varphi_x(y_k)$$

where the sets J, K are finite and the points x_j, y_k lie in $(0,l)$. This way we obtain a common strong oscillation length of all expressions of the form

$$\sum_{j \in J} \alpha_j u(t, x_j) + \sum_{k \in K} \beta_k u_x(t, y_k)$$

valid for all solutions in the natural energy space.

Remark 2.5. a) Corollary 2.2 is obviously non optimal since if $p = 0$, all solutions are periodic with period $2\pi\left(\frac{l}{\pi}\right)^2 = \frac{2}{\pi} l^2$ and mean-value 0: in this case the smallest oscillation time has been overestimated by a factor at least $\frac{\pi^2}{6}$. On the other hand, the simple harmonic solution associated with the smallest eigenvalue has precisely an oscillation time equal to $\frac{1}{\pi} l^2$, so that we are not dramatically far away from optimality, especially if p is small in L^∞.

b) It is known that in fact, $\sigma_n \geq n$ for all n. Hence Corollary 2.2 is also valid with T replaced by $\frac{\pi}{3} l^2$. This can be interpreted saying that the clamped boundary conditions provide a stronger elastic response than simply supported boundary conditions, a phenomenon which is known in other contexts.

c) The method of this section does not provide any oscillation result when p becomes time dependent. Actually, even the existence of zeroes of the function $u(t,\xi)$ is presently unknown for u a smooth solution of the equation

$$u_{tt} + u_{xxxx} + p(t)u = 0 \quad \text{in } \mathbb{R}\times(0,l), \qquad u = u_{xx} = 0 \quad \text{on } \mathbb{R}\times\{0,l\}$$

when $p \geq 0$ is smooth function of t only and $\xi \in (0,l)$.

3. Strong oscillations for a perturbed string equation

Theorem 3.1. *Let l be any positive number. For any non-negative potential $a \in L^{\infty}_{loc}(\mathbb{R})$, and for any solution $u \in C\big(\mathbb{R}, H_0^1(0,l)\big) \cap C^1\big(\mathbb{R}, L^2(0,l)\big)$ of*

$$u_{tt} - u_{xx} + a(t)u = 0 \quad \text{in } \mathbb{R}\times(0,l), \qquad u = 0 \quad \text{on } \mathbb{R}\times\{0,l\} \tag{3.1}$$

and for any $\zeta \in H^{-1}(0,l)$, the function $t \to \langle \zeta, u(t)\rangle$ has a strong oscillation length equal to $T = 2l$.

Proof. It follows at once from the Lax-Milgram theorem that any $\zeta \in H^{-1}(0,l)$ can be written as $\zeta = -h_x$ for some $h \in L^2(0,1)$. Then for any $t \in \mathbb{R}$ and any solution $u \in C\big(\mathbb{R}, H_0^1(0,l)\big) \cap C^1\big(\mathbb{R}, L^2(0,l)\big)$ of (4.1) we have

$$\langle \zeta, u(t)\rangle = v(t,0)$$

where v is defined on $\mathbb{R}\times\mathbb{R}$ by the formula

$$v(t,x) = \langle \zeta, U(t,\cdot + x)\rangle = \int_0^l h(y)U_y(t,x+y)\,dy$$

and $U : \mathbb{R}^2 \to \mathbb{R}$ is the odd and $2l$-periodic extension of u with respect to the space variable x. We observe that $z(t,x,y) = U_y(t,x+y) \in C\big(\mathbb{R}^2, L^2(0,l)\big)$ and the continuity is uniform for t bounded and x arbitrary. It follows that $v \in C(\mathbb{R}^2)$. On the other hand, by construction we clearly have

$$v_{tt} - v_{xx} + a(t)v = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^2)$$

and for each t given, the function $v(t,\cdot)$ is $2l$-periodic in x with mean-value 0. We are therefore in a good position to apply the method introduced in [3], consisting in using the positivity preserving of the wave operator inside a characteristic triangle after exchanging x and t. More precisely we shall establish the following Lemma

Lemma 3.2. *Let $v \in C(\mathbb{R}^2)$ be any solution of*

$$v_{tt} - v_{xx} + a(t,x)v = 0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^2) \tag{3.2}$$

where $a \geq 0$, $a \in L^\infty_{loc}(\mathbb{R}^2)$ with $a(t,x) = a(t,-x)$ almost everywhere. Assuming

$$\forall t \in \mathbb{R}, \qquad \int_{-l}^{+l} v(t,x)\,dx = 0$$

then for any compact interval $J \in \mathbb{R}$ with $|J| \geq 2l$, either the function $v(t,0)$ takes both positive and negative values on J, or denoting by m the middle of J, there exists an open neighborhood W of $\{m\} \times (-l,+l)$ such that

$$\forall (t,x) \in W, \qquad v(t,x) + v(t,-x) = 0$$

Proof. We set

$$\forall (t,x) \in \mathbb{R}^2, \qquad v(t,x) + v(t,-x) = w(t,x)$$

It is clear that w is also a solution of (3.2) and by hypothesis

$$\forall t \in \mathbb{R}, \qquad \int_{-l}^{+l} w(t,x)\,dx = 0$$

By a translation in t it is sufficient to consider the case $J = [-l,+l]$ and prove the result with $m = 0$. Let us assume for instance that

$$\forall t \in J, \qquad v(t,0) \geq 0$$

and let us introduce the open characteristic square

$$\mathscr{S} := \{(t,x) \in \mathbb{R}^2, |t| + |x| < l\}.$$

It is not difficult, using the even character of w in x, to check the formula

$$\begin{aligned} &\forall (t,x) \in \mathscr{S}, \\ &w(t,x) = \frac{1}{2}\left[w(t+x,0) + w(t-x,0) + \int_0^x d\sigma \int_{t-(x-\sigma)}^{t-(x+\sigma)} a(s,\sigma)w(s,\sigma)\,ds\right] \\ &\qquad =: z(t,x). \end{aligned} \tag{3.3}$$

Indeed, z is easily seen to be a continuous solution of

$$z_{tt} - z_{xx} + a(t,x)w = 0 \qquad \text{in } \mathscr{D}'(\mathscr{S})$$

with $z(t,0) = w(t,0)$ for all $t \in J$ and $z(t,x) = z(t,-x)$ for all $(t,x) \in \mathscr{S}$. Therefore $f = w - z \in C(\mathscr{S})$ is a solution of

$$f_{tt} - f_{xx} = 0 \qquad \text{in } \mathscr{D}'(\mathscr{S})$$

for which $f(t,0) = 0$ for all $t \in J$ and $f(t,x) = f(t,-x)$ for all $(t,x) \in \mathscr{S}$. The standard theory of distributions gives $f \equiv 0$ in $\mathscr{S}$. Now, considering the forward characteristic triangle

$$\mathscr{T} := \{(t,x) \in \mathbb{R}^2, x \geq 0, |t| + x < l\}$$

by using as in [4] a Gronwall lemma with respect to the increasing variable $x \in [0,l]$ for the function

$$\psi(x) := \int_{-l+x}^{l-x} w^-(t,x)\,dx$$

we can show easily that $w \geq 0$ in $\mathscr{T}$. In particular

$$\forall (t,x) \in \mathscr{T}, \qquad w(t,x) \geq \frac{1}{2}[w(t+x,0) + w(t-x,0)]$$

and for $t = 0$ we find

$$\forall x \in [0,l], \qquad w(0,x) \geq \frac{1}{2}[w(x,0) + w(-x,0)]$$

Since w is even in x, the same property holds true in J and since the integral of $w(0,x)$ on J is 0, by integrating on J we find

$$\forall x \in J, \qquad w(0,x) = 0$$

Now by using formula (3), by using a Gronwall Lemma with respect to the increasing variable $x \in [0,l]$ for the function

$$\phi(x) := \int_{-l+x}^{l-x} w(t,x)\,dx$$

we obtain $w \equiv 0$ in $\mathscr{T}$. By applying the same procedure to the backward characteristic triangle

$$\mathscr{T}' := \{(t,x) \in \mathbb{R}^2, x \leq 0, |t| - x < l\}$$

we obtain $w \equiv 0$ in $\mathscr{S}$, which implies the conclusion of the Lemma, the neighborhood W of $\{m\} \times (-l, +l)$ being an open characteristic square centered at $(m, 0)$. □

End of the proof of Theorem 3.1. Assuming that $\langle \zeta, u(t) \rangle$ *does not* take both positive and negative values on some compact interval $J \in \mathbb{R}$ with $|J| \geq 2l$, Lemma 3.2 applied with a independent of x implies the existence of an open neighborhood W of $\{m\} \times (-l, +l)$ such that

$$\forall (t, x) \in W, \qquad w(t, x) = v(t, x) + v(t, -x) = 0$$

Assuming for simplicity, as previously, that $m = 0$, by periodicity and continuity it follows that

$$\forall x \in \mathbb{R}, \qquad w(0, x) = 0$$

On the other hand it is not difficult to deduce from the definition of v that $v \in C^1\big(\mathbb{R}, \mathscr{D}'(\mathbb{R})\big)$ with $v_t(t, x) = z_x(t, x)$ where

$$z(t, x) = \int_0^l h(y) U_t(t, x + y)\, dy$$

Hence for all real t, the function $v_t(t, x)$ is the x-derivative of a continuous function of x, and so is $w_t(t, x)$. By periodicity and continuity of w we now find

$$\operatorname{supp}\{w_t(0, x)\} \subset \{l\} + 2l\mathbb{Z}$$

This is easily seen to imply, since $w_t(0, x)$ is the x-derivative of a continuous function of x, that in fact

$$w_t(0, \cdot) = 0 \qquad \text{in } \mathscr{D}'(\mathbb{R})$$

Finally let ρ_ε be the standard approximation of the Dirac measure by smooth functions with support in $[-\varepsilon, \varepsilon]$ and let

$$w_\varepsilon(t, \cdot) := \rho_\varepsilon * w(t, \cdot)$$

It is clear that for all $\varepsilon > 0$, w_ε is a smooth solution of $u_{tt} - u_{xx} + a(t)\big)u$ in $\mathbb{R} \times \mathbb{R}$ with $w_\varepsilon(0, \cdot) = (w_\varepsilon)_t(t, \cdot) =$ on $\mathbb{R}$. The method of characteristics for solutions with locally finite energy now gives $w_\varepsilon \equiv 0$ on $\mathbb{R} \times \mathbb{R}$. By letting ε tend to 0 we finaly obtain $w \equiv 0$ on $\mathbb{R} \times \mathbb{R}$, thereby concluding the proof of Theorem 3.1. □

Remark 3.3. a) Lemma 3.2, formulated with a time-dependent potential a, implies more than Theorem 3.1: it contains as a special case the basic result of [4].

b) In the special case where

$$\zeta = \sum_{j \in J} \alpha_j \delta x_j$$

where the set J is finite and the points x_j lie in $(0, l)$, the function v appearing in the proof is just a linear combination of space translates of the extension U of u. The proof in this case becomes more natural, and the general case corresponds formally to an approximation by density of this special case. If we only want to prove the existence of zeroes on J, actually the density argument is exactly what we need. However the proof of the oscillation result requires the whole power of Lemma 3.2.

c) In [5]–[7], the authors investigated the problem

$$u_{tt} - u_{xx} + cu \int_0^l u^2(t,x)\,dx = 0 \quad \text{in } \mathbb{R} \times (0,l), \qquad u = 0 \quad \text{on } \mathbb{R} \times \{0,l\} \tag{3.4}$$

which can be viewed as a simplified model to understand the more difficult (from the point of view of classification of trajectories) equation

$$u_{tt} - u_{xx} + u^3 = 0 \quad \text{in } \mathbb{R} \times (0,l), \qquad u = 0 \quad \text{on } \mathbb{R} \times \{0,l\} \tag{3.5}$$

Solutions of (3.4) are special cases of (4.1) with $a(t) = c \int_0^l u^2(t,x)\,dx$. By applying Theorem 3.1 we obtain

Corollary 3.4. *For any solution* $u \in C(\mathbb{R}, H_0^1(0,l)) \cap C^1(\mathbb{R}, L^2(0,l))$ *of* (3.4) *and for any* $\zeta \in H^{-1}(0,l)$, *the function* $t \to \langle \zeta, u(t) \rangle$ *has a strong oscillation length equal to* $2l$.

Remark 3.5. The method of this section does not seem provide any interesting oscillation result when a becomes space dependent.

4. Some extensions and open questions

4.1. More regular solutions

The oscillation theorems stated and proved in Sections 2–3 concern the existence of uniform oscillation times for linear observation of the solutions lying in the dual of the natural space for the equation. If the solution is more regular, the question naturally arises of whether the oscillation properties are valid for more singular observation operators. In the two main examples considered in Sections 2–3, such results are easily derived from invariance properties under some elementary operations. For instance, we have

Proposition 4.1. *Let l be any positive number. Let us introduce $D_1 = \{z \in H^2(0,l) \cap H_0^1(0,l), z_{xx} \in H^2(0,l) \cap H_0^1(0,l)\}$. For any non-negative potential $p \in L^\infty(0,l)$, and for any solution $u \in C(\mathbb{R}, D_1) \cap C^1(\mathbb{R}, H^2(0,l) \cap H_0^1(0,l))$ of*

$$u_{tt} + u_{xxxx} + p(x)u = 0 \quad \text{in } \mathbb{R} \times (0,l), \qquad u = u_{xx} = 0 \quad \text{on } \mathbb{R} \times \{0,l\}$$

and for any $\zeta \in (H^2(0,l) \cap H_0^1(0,l))'$, the function $t \to \langle \zeta, u_t(t) \rangle$ has a strong oscillation length equal to $T = \frac{\pi}{3} l^2$.

Remark 4.2. This result is not very strong, since when $\phi = \phi(t) \in C^1(\mathbb{R})$, between two zeroes of ϕ there is a zero of ϕ'. Oscillations of u are therefore more interesting that oscillations of u_t. However Proposition 4.1 provides the additional result that $\langle \zeta, u_t(t) \rangle$ vanishes identically if it has a constant sign on some interval of length $\geq 2l$.

Remark 4.3. For the string equation, in [3] it was shown that the functions $u_x(t,0)$ and $u_x(t,l)$, which are locally square integrable in t (the so-called hidden regularity property) even for initial data in the natural energy space, are both strongly oscillatory. It is natural to ask what happens to $u_x(t,x_0)$ when x_0 is an interior point. A partial answer is given by the following property

Proposition 4.4. *Let l be any positive number. For any non-negative potential $a \in L^\infty_{loc}(\mathbb{R})$, and for any solution $u \in C(\mathbb{R}, H^2(0,l) \cap H_0^1(0,l)) \cap C^1(\mathbb{R}, H_0^1(0,l))$ of*

$$u_{tt} - u_{xx} + a(t)u = 0 \quad \text{in } \mathbb{R} \times (0,l), \qquad u = 0 \quad \text{on } \mathbb{R} \times \{0,l\} \tag{4.1}$$

and for any $\zeta \in (H^2(0,l) \cap H_0^1(0,l))'$, the function $t \to \langle \zeta, u(t) \rangle$ has a strong oscillation length equal to $T = 2l$.

Remark 4.5. Proposition 4.4 is applicable to the functional ζ defined by

$$\langle \zeta, \varphi \rangle = \sum_{j \in J} \alpha_j \varphi(x_j) + \sum_{k \in K} \beta_k \varphi_x(y_k)$$

where the sets J, K are finite and the points x_j, y_k lie in $(0,l)$. This way we obtain a common strong oscillation length of all expressions of the form

$$\sum_{j\in J}\alpha_j u(t,x_j)+\sum_{k\in K}\beta_k u_x(t,y_k)$$

valid for all solutions $u\in C\big(\mathbb{R},H^2(0,l)\cap H_0^1(0,l)\big)\cap C^1\big(\mathbb{R},H_0^1(0,l)\big)$.

4.2. Relaxing the positivity condition on the potential

The oscillation theorems stated and proved in Sections 2–3 concern the case of a nonnegative potential p (resp. a), but they can be generalized easily when the negative part of the potential is uniformly smaller than the first eigenvalue of the principal part A_0 of A. In this case the oscillation length is larger. We leave the easy calculations to the reader (the formula is given in [8].)

4.3. Some open problems

1) Find sufficient conditions on $a=a(t,x)\geq 0$ for strong oscillation of the solutions to

$$u_{tt}-u_{xx}+a(t,x)u=0 \quad \text{in } \mathbb{R}\times(0,l), \qquad u=0 \quad \text{on } \mathbb{R}\times\{0,l\} \tag{4.2}$$

This is related to a possible generalization of the results from [4] to combinations of Dirac measures instead of just pointwise oscillations. Any result of this type would be interesting for the solutions of

$$u_{tt}-u_{xx}+g(u)=0 \quad \text{in } \mathbb{R}\times(0,l), \qquad u=0 \quad \text{on } \mathbb{R}\times\{0,l\} \tag{4.3}$$

2) Same problem for

$$u_{tt}+u_{xxxx}+p(t,x)u=0 \quad \text{in } \mathbb{R}\times(0,l), \qquad u=u_x=0 \quad \text{on } \mathbb{R}\times\{0,l\} \tag{4.4}$$

and

$$u_{tt}+u_{xxxx}+p(t,x)u=0 \quad \text{in } \mathbb{R}\times(0,l), \qquad u=u_{xx}=0 \quad \text{on } \mathbb{R}\times\{0,l\} \tag{4.5}$$

Here the situation is even worse than for the string equation since presently no pointwise oscillation result seems to be known for the solutions of

$$u_{tt}+u_{xxxx}+g(u)=0 \quad \text{in } \mathbb{R}\times(0,l), \qquad u=u_x=0 \quad \text{on } \mathbb{R}\times\{0,l\} \tag{4.6}$$

or

$$u_{tt}+u_{xxxx}+g(u)=0 \quad \text{in } \mathbb{R}\times(0,l), \qquad u=u_{xx}=0 \quad \text{on } \mathbb{R}\times\{0,l\} \tag{4.7}$$

3) The string equation represents a limiting case which is not covered in the general framework of Proposition 2.1. It has been treated by using the method of characteristics based on the propagation properties of the 1D wave equation. Similarly, the plate equation

$$u_{tt} + \Delta^2 u + p(x)u = 0 \quad \text{in } \mathbb{R} \times (0, l), \qquad u = \Delta u = 0 \quad \text{on } \mathbb{R} \times \{0, l\} \quad (4.8)$$

is in 2 dimensions a limiting case of Proposition 2.1 in the sense that the relevant series is in $l^{1+\varepsilon}$ for every $\varepsilon > 0$ but not in l^1. What happens in this case? Even the problem of pointwise oscillations is open, even for $p = 0$ and $\Omega = (0, a) \times (0, b)$ with $\frac{a^2}{b2} \notin \mathbb{Q}$.

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