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# Existence and uniqueness results for the pressureless Euler-Poisson system in one spatial variable

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Dedicated to Professor João Paulo de Carvalho Dias on the occasion of his 70th birthday

**Abstract.** We study the Euler-Poisson system describing the evolution of a fluid without pressure effect and, more generally, also treat a class of nonlinear hyperbolic systems with an analogous structure. We investigate the initial value problem by generalizing a method first introduced by LeFloch in 1990 and based on Volpert's product and Lax's explicit formula for scalar conservation laws. We establish several existence and uniqueness results when one component of the system (the density) is measure-valued and the second one (the velocity) has bounded variation. Existence is proven for general initial data, while uniqueness is guaranteed only when the initial data does not generate rarefaction centers. Our proof proceeds by solving first a nonconservative version of the problem and constructing solutions with bounded variation, while the solutions of the Euler-Poisson system is then deduced by differentiation.

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## 1. Introduction

We study here a class of nonlinear hyperbolic systems in one space dimension which includes, in particular, the pressureless Euler-Poisson system

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2) = \kappa \rho E, \\ &\partial_x E = \rho. \end{aligned} \tag{1.1}$$

Here, $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ denotes the velocity of the fluid, $\rho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ its density, and $E$ the electric field, normalized so that $\lim_{x \to +\infty} E(t, x) = 0$ for each time $t \geq 0$. Here $\kappa$ is a given constant whose sign determines the (attractive or repulsive) nature of the underlying force. More generally, we consider the systems

$$\partial_t \rho + \partial_x \big(\rho f'(u)\big) = 0, \tag{1.2a}$$

$$\partial_t(\rho u) + \partial_x \big(\rho f'(u) u\big) = \rho h\Big(\int_{-\infty}^{x} \rho \, dy\Big), \tag{1.2b}$$

with $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$. In the above, $h : \mathbb{R} \to \mathbb{R}$ is a given Lipschitz continuous function and $f : \mathbb{R} \to \mathbb{R}$ satisfies the following two assumptions:

(A1) $f : \mathbb{R} \to \mathbb{R}$ is a smooth, strictly convex function.

(A2) $\lim_{|u| \to +\infty} \frac{f(u)}{|u|} = +\infty$.

Our objective is to establish several existence and uniqueness results in a suitable class of weak solutions, for both the above systems and their nonconservative formulations which are obtained by formally integrating in space.

In our investigation, we closely follow LeFloch [10], which treated the same problem but without electric field. The presence of an electric field significantly modify the analysis, although the strategy in [10] can still be closely followed. In particular, as observed in [10], it is very convenient to introduce a nonconservative version of the system under consideration and seek for solutions of bounded variation—defined by relying on the so-called Volpert's product [5], [9], [12], [14]—and then to recover the original system by differentiation. It then naturally follows that solutions can be measure-valued.

We do not try to review here the vast literature existing on the pressureless Euler system and we simply refer to [2], [3], [4], [6] and the numerous references cited therein. As far as the Euler-Poisson system is concerned, we recall that Tadmor and Wei [13] recently proposed a theory of weak solutions based on a variational approach. Note also that transport equations with discontinuous coefficients have been found to be useful for the analysis of the linear stability of shock waves [1].

An outline of this paper is as follows. In Section 2, we present some elementary properties of bounded variation (BV) functions and Volpert's product. In Section 3, we study the following hyperbolic system in a nonconservative form (formally derived by integration of (1.2))

$$\partial_t w + f'(u) \partial_x w = 0, \tag{1.3a}$$

$$\partial_t u + \partial_x f(u) = h(w). \tag{1.3b}$$

Weak solutions to (1.3) are defined in the sense of Volpert's product, and we establish the existence and uniqueness of bounded variation solution when the initial data $u(0, \cdot) = u_0$ has bounded variation and the initial data $w(0, \cdot) = w_0$ is Lipschitz continuous.

In Section 4, we return to the conservative formulation (1.2), but first substitute (1.2b) by (1.3b), which, in some sense, contains "more information" (for the dynamics of the velocity variable in the vacuum regions) than (1.2b). We note that a solution to (1.2a) and (1.3b) is also a solution of (1.2). Finally, in Section 5, we generalize our existence theory above by allowing both initial data $u_0$ and $w_0$ to be bounded variation functions and by defining the composition of a BV function and a strictly monotone function.

## 2. Background material

**2.1. Change of variable formulas.** Throughout, we work with functions of bounded variation (BV) and we recall that the total variation of a function $v : [a, b] \to \mathbb{R}$ is defined as

$$TV_{[a,b]}v := \sup \sum_{i=1}^{n} |v(x_i) - v(x_{i-1})|,$$

where the supremum is taken over all finite partitions $a = x_0 < x_1 < \cdots < x_n = b$. The interval $[a, b]$ can be decomposed in two disjoint sets: the set $\mathscr{C}$ of points of continuity of $v$ and the set $\mathscr{J}$ of points of jump. The set $\mathscr{J}$ is at most countable, while left- and right-hand limits $v(x\pm)$ exist for each $x \in [a, b]$ and are distinct if $x \in \mathscr{J}$. For instance, monotone functions have (locally) bounded variation.

We will use the following notion of inverse of a non-decreasing and right-continuous function $F(x) : [a, b] \to [A, B]$. Its *generalized inverse* is defined by

$$g(y) = \inf\{x : F(x) > y\},$$

which is obviously non-decreasing. Similarly, we can define the inverse of left-continuous/right-continuous and non-decreasing/non-increasing functions.

We will also need change of variable formula. Given any right-continuous BV function $v$ defined on an interval $(a, b)$, there exists a unique finite Borel measure $\mu_v$ associated to $v$, such that

$$v(x) - v(a+) = \mu_v\big((a, x]\big), \qquad v(a+) - v(a) = \mu_v(a).$$

We denote $\mu_v$ by $dv$ and decompose it into an absolutely continuous part denoted by $v'\,dx$, an atomic part $d_a v$, and a singular part $d_s v$. We thus have $dv = v'(x)\,dx + d_a v + d_s v$.

Given such a function $v$, we use the notation $L^1(dv)$ for the set of Borel functions that are integrable with respect to the measure $dv$. From [14], we have the following result.

**Proposition 2.1.** *Let $u : [a,b] \to \mathbb{R}$ be a right-continuous function of bounded variation and let $X : [a,b] \to [c,d]$ be a continuous, non-decreasing (and not necessary strictly increasing) function with $X(a) = c$, $X(b) = d$.*

1. *If $X^{-1}$ denotes the generalized inverse of $X$, then for all $g \in L^1\big(d(u \circ X)\big)$*

$$\int_c^d g(s) d\big(u \circ X(s)\big) = \int_a^b g \circ X^{-1}\, du(x).$$

2. *For any function $g \in L^1\big(d(u \circ X)\big)$, one has*

$$\int_c^d (g \circ X)(s) d\big(u \circ X(s)\big) = \int_a^b g(x)\, du(x).$$

**2.2. Nonconservative products.** More generally, the total variation of an integrable function $v = v(x)$ defined on a domain $\Omega \subset R^N$ (for $N \geq 1$) is defined by

$$TV_\Omega v = \sup\left\{\int_\Omega v \nabla \cdot \varphi / \varphi : \Omega \to \mathbb{R}, \text{ smooth, compactly supported, } \|\varphi\|_{L^\infty(\Omega)} \leq 1\right\}.$$

When this total variation is finite, the first order derivatives $\frac{\partial u}{\partial x_i}$ are finite Borel measures. Let us recall the following regularity results for BV functions, as proven first in Volpert [14]. For a function u in $BV(\Omega, \mathbb{R})$, it happens that up to a set of vanishing Hausdorff measure, each point $x \in \Omega$ is regular, that is, is either a point of approximate continuity or a point of approximate jump. With obvious notation, we write $\Omega = \mathscr{C} \cup \mathscr{J} \cup \mathscr{N}$. To be more specific, recall that the formula

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\{(x - x_0, \nu) > 0\} \cap B_\varepsilon(x_0)} |u(x) - l_\nu u(x_0)|\, dx = 0$$

holds at a point of approximate jump, where $\nu$ is a unit normal at $x_0$, $B_\varepsilon(x_0)$ denotes the ball of radius $\varepsilon > 0$ centered at $x_0$ and $l_{\pm\nu} u(x_0) =: u_\pm(x_0)$ denote the left- and right-hand traces. At a point of approximate continuity, the normal $\nu$ is irrelevant and the traces $u_\pm(x_0)$ coincides.

We now turn to the notion of averaged superposition.

**Definition 2.2.** Given $g \in C^1(\mathbb{R})$ and $u \in L^\infty(\Omega, \mathbb{R}) \cap BV(\Omega, \mathbb{R})$, the averaged superposition of the function $u$ by $g$ is defined as

$$\hat{g}(u)(x) = \begin{cases} g\big(u(x)\big), & \text{at an approximate continuity point,} \\ \int_0^1 g\big((1-\lambda)u_-(x) + \lambda u_+(x)\big)\, d\lambda, & \text{at a jump point.} \end{cases}$$

The following result was proven by Volpert [14].

**Proposition 2.3.** *Let $u$, $v$ be two functions in $L^\infty(\Omega, \mathbb{R}) \cap BV(\Omega, \mathbb{R})$ and $g$ be in $C^1(\mathbb{R})$. Then, the function $\hat{g}(u)$ above is measurable and integrable with respect to each Borel measure $\frac{\partial u}{\partial x_i}$, so that the nonconservative product $\hat{g}(u)\frac{\partial v}{\partial x_i}$ makes sense as a finite Borel measure.*

As was first proposed by LeFloch [9], [10], Volpert's product is useful in order to define a notion of weak solutions to systems in nonconservative form such as

$$\partial_t u + \hat{A}(u)\partial_x u = 0, \tag{2.1}$$

where $u$ is the unknown function and $A = A(u)$ is a matrix-valued map of $u$.

## 3. The nonconservative formulation

**3.1. Definition and existence theory.** We begin by introducing our notion of solutions.

**Definition 3.1.** A pair of functions $(w, u)$ in $L^\infty\big(\mathbb{R}_+, BV(\mathbb{R})\big)$ is said to be a weak solution to (1.3) if:

1. The component $u$ is a weak solution in the sense of distributions to

$$\partial_t u + \partial_x f(u) = h(w). \tag{3.1}$$

2. The component $w$ is a weak solution in the sense of Volpert-LeFloch to

$$\partial_t w + \widehat{f'}(u)\partial_x w = 0. \tag{3.2}$$

In this section, we are interested in initial data with regularity

$$u_0 \in BV(\mathbb{R}), \qquad w_0 \in W^{1,\infty}(\mathbb{R}).$$

We are going to rely on a generalization of Lax's explicit formula [7], [8], which is well-known for *homogeneous* conservation laws. Let $G : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ be given by

$$G(t, x, y) = \int_0^y \left(z - x + \int_0^t f'\big(u_0(z) + h\big(w_0(z)\big)s\big) ds\right) dz, \tag{3.3}$$

and denote by $\xi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ the minimizer associated with the function $y \mapsto G(t, x, y)$ (for $t$, $x$ fixed), which is characterized by the property

$$G\big(t, x, \xi(t,x)\big) = \inf_{y \in \mathbb{R}} G(t, x, y), \qquad \text{a.e. } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{3.4}$$

We observe that, when $t$ approaches 0, the function $G$ approaches $(1/2)\big((y-x)^2 - x^2\big)$ which trivially achieves its minimum (with respect to the variable $y$) at the point $x$, so that $\xi(0,x) = x$.

**Lemma 3.2.** *For each time $t$, the function $\xi(t,\cdot)$ defined in* (3.4) *is non-decreasing in $x$.*

*Proof.* We write

$$H_{t,x}(z) = z - x + \int_0^t f'\big(u_0(z) + h\big(w_0(z)\big)s\big)\, ds, \tag{3.5}$$

so that $G(t,x,y) = \int_0^y H_{t,x}(z)\, dz$. Let us assume that there exists two points $x_1 < x_2$ such that $\xi(t,x_1) > \xi(t,x_2)$. Since $\xi(t,x_1)$ is a minimizer of $G(t,x_1,y)$, we have

$$\int_{\xi(t,x_2)}^{\xi(t,x_1)} H_{t,x_1}(z)\, dz = G\big(t, x_1, \xi(t,x_1)\big) - G\big(t, x_1, \xi(t,x_2)\big) \le 0.$$

Moreover, we have

$$\int_{\xi(t,x_2)}^{\xi(t,x_1)} H_{t,x_2}(z)\, dz = \int_{\xi(t,x_2)}^{\xi(t,x_1)} H_{t,x_1}(z)\, dz + \int_{\xi(t,x_2)}^{\xi(t,x_1)} (x_1 - x_2)\, dz < 0.$$

Thus, $G\big(t, x_2, \xi(t,x_1)\big) < G\big(t, x_2, \xi(t,x_2)\big)$, which contradicts the definition of $\xi(t,x_2)$. □

We begin with the existence theory.

**Theorem 3.3** (Existence result for the nonconservative formulation). *Given any initial data $u_0$ in $BV(\mathbb{R})$ and $w_0$ in $W^{1,\infty}(\mathbb{R})$, the system* (3.1)–(3.2) *admits at least one weak solution $(u,w)$ in $L^\infty\big(\mathbb{R}_+, BV(\mathbb{R})\big)$ satisfying the initial data:*

$$w(0,\cdot) = w_0, \qquad u(0,\cdot) = u_0,$$

*which is given by the formula*

$$u(t,x) = b\big(t, x, \xi(t,x)\big) + h\big(w_0\big(\xi(t,x)\big)\big)t, \tag{3.6}$$

$$w(t,x) = w_0\big(\xi(t,x)\big), \tag{3.7}$$

*where $b(t, x, y)$ is defined by the implicit relation*

$$Y\big(t, x, y, b(t, x, y)\big) = 0,$$

*with (for all $(t, x, y, b)$)*

$$Y(t, x, y, b) := y - x + \int_0^t f'\big(b + h\big(w_0(y)\big)s\big)\, ds. \tag{3.8}$$

*Proof.* Since the function $f$ is convex and satisfies $\lim_{x \to +\infty} \frac{f(x)}{|x|} = +\infty$, the function $G(t, x, y)$ reaches its minimum at some point(s). According to Lemma 3.2, for each fixed $t$ the function $\xi(t, \cdot)$ is non-decreasing, and therefore $w$ is well-defined. We have also, for all $t > 0$,

$$\lim_{b \to \pm\infty} \int_0^t f'\big(b + h\big(w_0(y)\big)s\big)\, ds = \pm\infty.$$

Therefore, for every $t > 0$ and fixed $x$, $y$, the function $Y(t, x, y, b)$ will have at least one root $b$. Moreover, by the strict monotonicity of $f'$, this root is unique. Therefore, the expression proposed for $u$ is also well-defined.

We next consider the equation (3.2). Thanks to ([5], Theorem A2), we see that $w$ belongs to $L^\infty\big(\mathbb{R}_+, BV(\mathbb{R})\big)$ whenever $w_0$ (only) belongs to $W^{1,\infty}(\mathbb{R})$. Since $\xi$ has bounded variation, so that we can introduce the decomposition $\mathbb{R}_+ \times \mathbb{R} = \mathscr{C} \cup \mathscr{J} \cup \mathscr{N}$ associated with $\xi$, as was introduced in Section 2. Consider first the set of approximate continuity $\mathscr{C}$, and define the function $F$ by

$$F'(y) := f'\big(u_0(y) + h\big(w_0(y)\big)t\big)$$

for all $t$, $y$. Thanks to Proposition 2.1, $F$ is well-defined. We combine $\xi$ and $F$ into the formula

$$\xi(t, x) = \lim_{N \to +\infty} \xi_N(t, x), \qquad \xi_N(t, x) = \frac{\int -y \exp(-NG)\, dy}{\int \exp(-NG)\, dy}.$$

Similarly, we write

$$F(t, x) = \lim_{N \to +\infty} F_N(t, x), \qquad F_N(t, x) = \frac{\int F(y) \exp(-NG)\, dy}{\int \exp(-NG)\, dy}.$$

In the set $\mathscr{C}$, we have

$$\partial_t w = w_0'(\xi)\partial_t \xi, \tag{3.9}$$

$$\widehat{f'}(u)\partial_x w = w_0'(\xi) f'\big(b\big(t, x, \xi(t, x)\big) + h\big(w_0\big(\xi(t, x)\big)\big)t\big)\partial_x \xi. \tag{3.10}$$

Now, we introduce the function

$$V_N = -\ln \int \exp(-NG)\, dy$$

and compute

$$\xi_N = -\frac{1}{N}\partial_x V_N, \qquad F_N = \frac{1}{N}\partial_t V_N.$$

Therefore, the equation $\partial_t \xi_N + \partial_x F_N = 0$ holds true. Letting $N \to +\infty$ and changing $F'$ back to $f'$, we get

$$\partial_t \xi + f'\big(u_0(\xi) + h\big(w_0(\xi)\big)t\big)\partial_x \xi = 0. \tag{3.11}$$

Hence, on the set of approximate continuity points, we have

$$H_{t,x}\big(\xi(t,x)\big) = 0,$$

with $H_{t,x}(z)$ defined in (3.5). The fact that $b$ is uniquely defined ensures that

$$b\big(t, x, \xi(t,x)\big) = u_0\big(\xi(t,x)\big).$$

Combining (3.5), (3.10) with (3.11), we obtain the equation (3.2).

Turning our attention to the set of jump points, we consider the Borel measure defined by

$$\mu := \partial_t w + \widehat{f'}(u)\partial_x w. \tag{3.12}$$

Taking a point in $\mathscr{J}$ denoted by $(t_*, x_*)$, we find according to Volpert's definition

$$\mu\{(t_*, x_*)\} = -\sigma(w_+ - w_-) + \int_0^1 f'\big(u_- + \lambda(u_+ - u_-)\big)\, d\lambda (w_+ - w_-).$$

On the other hand, Rankine-Hugoniot jump condition yields us the shock speed

$$\sigma := \frac{f(u_+) - f(u_-)}{u_+ - u_-} = \int_0^1 f'\big(u_- + \lambda(u_+ - u_-)\big)\, d\lambda.$$

We thus conclude that the mass of the corresponding Borel measure $\mu\{(t_*, x_*)\} = 0$ vanishes. Combining our results for the set of approximate continuity points and the set of jump points, we have thus derived the equation (3.2).

We need next to discuss the equation (3.1). As done before, we introduce the function

$$b_N(t,x) = \frac{\int b\big(t,x,\xi(t,x)\big)\exp(-NG)\,dy}{\int \exp(-NG)\,dy}.$$

and $b(t,x) = \lim_{N\to+\infty} b_N(t,x)$. In particular, we have

$$\lim_{N\to+\infty} b_N(0,x) = u_0(x).$$

As for the derivation of the equation (3.2), we have here

$$\partial_t b + f'\big(b(t,x,\xi) + h\big(w_0(\xi)\big)t\big)\partial_x b = 0$$

and, since

$$\partial_t w + f'\big(b\big(t,x,\xi(t,x)\big) + h(w)t\big)\partial_x w = 0,$$

we deduce

$$\begin{aligned}
&\partial_t\big(h\big(w_0\big(\xi(t,x)\big)\big)t\big) + f'\big(b\big(t,x,\xi(t,x)\big) + h\big(w_0\big(\xi(t,x)\big)\big)t\big)\partial_x\big(h\big(w_0\big(\xi(t,x)\big)\big)t\big)\\
&\quad = h'(w_0)w_0'\big(\xi(t,x)\big)t\big(\partial_t w + f'\big(b\big(t,x,\xi(t,x)\big) + h(w)t\big)\partial_x w\big) + h(w)\\
&\quad = h(w).
\end{aligned}$$

Therefore, we have obtained (3.1) and this completes the proof of Theorem 3.3. □

**3.2. Uniqueness theory.** For given initial data $u_0 \in BV(\mathbb{R})$ and $w_0 \in W^{1,\infty}(\mathbb{R})$, we have constructed a solution in $L^\infty\big(\mathbb{R}_+, BV(\mathbb{R})\big)$ given by the explicit formula (3.6)–(3.7) based the (generalized) characteristic $\xi = \xi(t,x)$. As shown in [8], solutions $u = u(t,x)$ to homogeneous conservation laws satisfy the entropy inequality (for all $x_1 < x_2$ and all times $t > 0$)

$$\frac{u(t,x_2) - u(t,x_1)}{x_2 - x_1} \le k(t), \tag{3.13}$$

where $k = k(t) > 0$ is some function determined by the initial data and may blows up at $t = 0$. In particular, at each jump point, we have $f'(u_-) > \sigma > f'(u_+)$, where $\sigma$ denotes the shock speed. Relying on (3.13), we are going to prove a uniqueness result for the system (3.1)–(3.2) under the assumptions that the map $h$ is non-increasing and the initial data $u_0$ also satisfies the entropy condition.

**Proposition 3.4.** *Suppose that $\partial_x w \geq 0$ in the sense of distribution and that the map $w \mapsto h(w)$ is non-increasing. Suppose also that the initial data $u_0$ satisfies the entropy condition*

$$\frac{du_0(x)}{dx} \leq K_0 \tag{3.14}$$

*in the sense of distributions. Then, the solution u to system* (3.1)–(3.2) *also satisfies the entropy condition*

$$\frac{\partial u}{\partial x}(t, \cdot) \leq K_0 \tag{3.15}$$

*for each time t.*

We emphasize that our addititional assumption $\partial_x w \geq 0$ above is natural since we are primarily interested in returning (in the following section) to the conservative system for which we write $\partial_x w = \rho$ with $\rho \geq 0$.

*Proof.* Consider the equation

$$\partial_t u + \partial_x f(u) = \theta \partial_{xx}^2 u + h(w), \tag{3.16}$$

with $\theta > 0$ and let us differentiate (3.16) once with respect to $x$:

$$\partial_t(\partial_x u) + f'(u)\partial_{xx}^2 u + f''(u)(\partial_x u)^2 = \theta \partial_{xx}^2(\partial_x u) + h'(w)\partial_x w.$$

Since the flux $f$ is a strictly convex, we have $f'' > 0$ and, in addition, $h'(w)\partial_x w \leq 0$ holds since $h$ is non-increasing. Thus, we get

$$\partial_t(\partial_x u) + f'(u)\partial_{xx}^2 u \leq \theta \partial_{xx}^2(\partial_x u).$$

Writing $a = (\partial_x u)^+ = \max(0, \partial_x u)$, we find

$$\partial_t a + f'(u)\partial_x a - \theta \partial_{xx}^2 a \leq 0.$$

Considering the function $v = a - K_0$, we thus have

$$\partial_t v + f'(u)\partial_x v - \theta \partial_{xx}^2 v \leq 0.$$

By the maximum principle, we conclude that $a \leq K_0$. Finally, by letting $\theta \to 0$, we have proven that $\frac{\partial u(t,x)}{\partial x} \leq K_0$. □

The following technical observation will be useful.

**Lemma 3.5.** *Assume that both* $(w_1, u_1)$ *and* $(w_2, u_2)$ *are entropy solutions to* (3.1)–(3.2) *in* $L^\infty(\mathbb{R}_+, BV(\mathbb{R}))$. *Then the function* $w_2(t,x) - w_1(t,x)$ *has the same sign as* $u_2(t,x) - u_1(t,x)$ *on the set of points of approximate continuity.*

*Proof.* Assume that $u_2(t,x) \geq u_1(t,x)$ on some set of points of approximate continuity. Consider the equation

$$\partial_t w_i + f'(u_i)\partial_x w_i = \varepsilon \partial^2_{xx} w_i. \tag{3.17}$$

where $i = 1, 2$ and $\varepsilon > 0$. Noting that

$$\begin{aligned}
&\partial_t w_2 + f'(u_1)\partial_x w_2 - \varepsilon \partial^2_{xx} w_2 \\
&\quad = \partial_t w_2 + f'(u_2)\partial_x w_2 - \varepsilon \partial^2_{xx} w_2 + \big(f'(u_2) - f'(u_1)\big)\partial_x w_2 \\
&\quad \geq \partial_t w_2 + f'(u_2)\partial_x w_2 - \varepsilon \partial^2_{xx} w_2 = 0,
\end{aligned}$$

we see that $w_2$ is a sub-solution of (3.17) with speed $f'(u_1)$. Letting $\varepsilon \to 0^+$, we conclude that $w_2 \geq w_1$ and that $w_2 - w_1$ has the same sign has $u_2 - u_1$. □

**Theorem 3.6** (Uniqueness result for the nonconservative formulation). *Consider initial data* $u_0 \in BV(\mathbb{R})$ *and* $w_0 \in W^{1,\infty}(\mathbb{R})$ *and suppose that* $u_0$ *satisfies the entropy condition*

$$\frac{du_0}{dx} \leq K_0, \qquad \frac{dw_0}{dx} \geq 0$$

*for some constant* $K_0 > 0$. *Then, provided the map h is non-increasing, the Cauchy problem for the system* (3.1)–(3.2) *admits at most one entropy solution satisfying* $\partial_x w \geq 0$.

*Proof.* Let $(w_1, u_1)$ and $(w_2, u_2)$ be entropy solutions to (3.1)–(3.2) and let us introduce the decomposition $\mathbb{R}_+ \times \mathbb{R} = \mathscr{C} \cup \mathscr{J} \cup \mathscr{N}$ as in the proof of Theorem 3.3. Standard techniques yield the inequality

$$\frac{d}{dt}\int |u_2 - u_1|\, dx \leq \int_{\mathscr{C}} \operatorname{sgn}(u_2 - u_1)\big(h(w_2) - h(w_1)\big),$$

in which the right-hand side is non-positive since $h$ is non-increasing and $\operatorname{sgn}(u_2 - u_1) = \operatorname{sgn}(w_2 - w_1)$ (according to Lemma 3.5). Thus, we obtain $u_1(t,x) = u_2(t,x) := u(t,x)$ for a.e. $(t,x)$.

Consider next the components $w_1$, $w_2$. By a standard regularization argument, we have

$$\upsilon = \partial_t |w_2 - w_1| + \widehat{f'}(u)\partial_x |w_2 - w_1| = 0.$$

Define the Borel measure

$$\upsilon = \partial_t |w_2 - w_1| + \partial_x \big(\widehat{f'}(u)|w_2 - w_1|\big). \tag{3.18}$$

On the set of points of approximate continuity, we have

$$\upsilon = \partial_t |w_2 - w_1| + \partial_x \big(\widehat{f'}(u)|w_2 - w_1|\big) = f''(u)\partial_x u |w_2 - w_1| \leq K|w_2 - w_1|,$$

by Proposition 3.4.

On the set of points of jump, say at a point $(t_*, x_*)$ we have

$$\begin{aligned}
\upsilon\big((t_*, x_*)\big) &= f'(u_+)|w_{2_+} - w_{1_+}| - f'(u_-)|w_{2_-} - w_{1_-}| \\
&\quad - \int_0^1 f'\big(u_- + \lambda(u_+ - u_-)\big)\, d\lambda (|w_{2_+} - w_{1_+}| - |w_{2_-} - w_{1_-}|) \\
&= |w_{2_+} - w_{1_+}|\Big(f'(u_+) - \int_0^1 f'\big(u_- + \lambda(u_+ - u_-)\big)\, d\lambda\Big) \\
&\quad + |w_{2_-} - w_{1_-}|\Big(\int_0^1 f'\big(u_- + \lambda(u_+ - u_-)\big)\, d\lambda - f'(u_-)\Big).
\end{aligned}$$

Since

$$\frac{f(u_+) - f(u_-)}{u_+ - u_-} = \int_0^1 f'\big(u_- + \lambda(u_+ - u_-)\big)\, d\lambda,$$

we find

$$\begin{aligned}
\upsilon\big((t_*, x_*)\big) &= |w_{2_+} - w_{1_+}|\bigg(f'(u_+) - \frac{f(u_-) - f(u_-)}{u_+ - u_-}\bigg) \\
&\quad + |w_{2_-} - w_{1_-}|\bigg(\frac{f(u_+) - f(u_-)}{u_+ - u_-} - f'(u_-)\bigg).
\end{aligned}$$

Consequently, $\upsilon\big((t_*, x_*)\big) \leq 0$ holds since $u_- > u_+$ and $f$ is convex.

We deduce the inequality

$$\upsilon \leq K|w_2 - w_1| \tag{3.19}$$

and, after integration in space and time,

$$\frac{d}{dt}\int |w_2 - w_1|\, dx \leq \int K|w_2 - w_1|.$$

Applying Gronwall's inequality, we get

$$\int |w_2(t,x) - w_1(t,x)|\, dx \leq e^{Kt}\int |w_2(0,x) - w_1(0,x)|\, dx,$$

which yields us the desired uniqueness property. □

## 4. The conservative formulation

**4.1. Definition and existence theory.** Following [9], the theoy in the previous section can be reformulated at the level of the *conservative* system

$$\begin{aligned} &\partial_t \rho + \partial_x\big(\rho f'(u)\big) = 0,\\ &\partial_t u + \partial_x f(u) = h\big(\rho(-\infty, x)\big), \end{aligned} \tag{4.1}$$

such that the component $u$ is a BV function in $x$, but the component $\rho$ is a measure in $x$. In particular, we can obtain uniqueness for the velocity component even in regions where the mass density vanishes. We omit the details.

Consider next the system

$$\begin{aligned} &\partial_t \rho + \partial_x\big(\rho f'(u)\big) = 0,\\ &\partial_t(\rho u) + \partial_x\big(\rho u f'(u)\big) = \rho h\big(\rho(-\infty, x)\big). \end{aligned} \tag{4.2}$$

Denote by $\mathscr{M}(\mathbb{R})$ the space of all bounded Borel measures. Clearly, uniqueness can no longer be expected in regions where $\rho$ vanishes, *unless* we strengthen the notion of weak solution, as we do below. We emphasize that for sufficiently regular solutions with non-vanishing density, (4.1) and (4.2) are equivalent. Furthermore, the jump relations at non-vanishing and bounded density can aso be checked to be equivalent. This motivates us to propose the following notion of solution.

**Definition 4.1.** A pair of functions $(\rho, u)$ satisfying

$$\rho \in L^\infty\big(\mathbb{R}_+, \mathscr{M}(\mathbb{R})\big), \qquad u \in L^\infty\big(\mathbb{R}_+, BV(\mathbb{R})\big)$$

is said to be a *precised entropy solution* to the system (4.2) if $(u, w)$ with

$$w(t, x) = \rho\big(t, (-\infty, x)\big)$$

(for all $x$ and almost every $t$) is a weak solution to (3.1)–(3.2) in the sense of Definition 3.1.

Our first result based on this definition is as follows.

**Theorem 4.2** (Existence result for the conservative formulation). *Given any* $u_0 \in BV(\mathbb{R})$ *and* $\rho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *with* $\rho_0 \geq 0$, *the system* (4.2) *admits a precised entropy solution in* $L^\infty\big(\mathbb{R}_+, BV(\mathbb{R})\big) \times L^\infty\big(\mathbb{R}_+, \mathcal{M}(\mathbb{R})\big)$ *satisfying the initial data*

$$\rho(0, \cdot) = \rho_0, \qquad u(0, \cdot) = u_0.$$

*It is given by the formula*

$$\rho(t, \cdot) = \partial_x\Big(\int_{-\infty}^{\xi(t,x)} \rho_0(s)\, ds\Big), \tag{4.3}$$

$$u(t, x) = b\big(t, x, \xi(t, x)\big) + h\Big(\int_{-\infty}^{\xi(t,x)} \rho_0(s)\, ds\Big) t, \tag{4.4}$$

*where* $\xi = \xi(t, x)$ *is the minimizer of* $G = G(t, x, y)$ *defined in* (3.4).

We have here $w_0(x) = \int_{-\infty}^{x} \rho_0(s)\, ds$ for $x \in \mathbb{R}$, and Theorem 4.2 is immediate in view of Theorem 3.3.

**4.2. Uniqueness theory.** Our next result is an immediate consequence of Theorem 3.6, stated as follows.

**Theorem 4.3** (Uniqueness result for the conservative formulation). *Let* $u_0 \in BV(\mathbb{R})$ *and* $\rho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *satisfying* $\rho_0 \geq 0$ *and the entropy condition*

$$\frac{du_0}{dx} \leq K_0 \tag{4.5}$$

*in the sense of distributions, and suppose that the map* $h$ *is non-increasing. Then,* (4.2) *admits at most one precised entropy solution* $(\rho, u)$ *in* $L^\infty\big(\mathbb{R}_+, BV(\mathbb{R})\big) \times L^\infty\big(\mathbb{R}_+, \mathcal{M}(\mathbb{R})\big)$. *satisfying* $\rho \geq 0$.

Since our notion of solution requires that the velocity satisfies an evolution equation, we have uniqueness of both $\rho$ and $u$—whereas only the uniqueness of the momentum $\rho u$ would be expected otherwise. Finally, we have reached the following final conclusion.

**Corollary 4.4.** *Let* $u_0 \in BV(\mathbb{R})$ *and* $\rho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *with* $\rho_0 \geq 0$ *be initial conditions satisfying the entropy condition* (4.5) *and suppose that the map* $h$ *is non-increasing. Then, the Cauchy problem for* (4.2) *has one and only one precised entropy solution in* $L^\infty(\mathbb{R}_+, BV(\mathbb{R})) \times L^\infty(\mathbb{R}_+, \mathcal{M}(\mathbb{R}))$.

## 5. Dealing with BV initial data

In Section 3, we have proved that (1.3) admits a weak solution when the initial data are $u_0 \in BV(\mathbb{R})$ and $w_0 \in W^{1,\infty}(\mathbb{R})$. Here, we further generalize this result and weaken the regularity requirement on the initial data by now taking both $u_0, w_0 \in BV(\mathbb{R})$.

**Lemma 5.1.** *Let* $y = y(x)$ *be a strictly increasing function and* $g \in BV(\mathbb{R})$, *then the composition* $g \circ y \pm (x)$ *is well-defined and is a BV function.*

*Proof.* Use the notation $(g \circ y)_\pm(x) := g\big(y(x)\pm\big)$. There is no difficulty to define the function $(g \circ y)_\pm(x)$ when $x$ is a point of continuity of $y$ or $y$ is a continuity point of $h$. Consider next the case where $x_*$ is a jump point for $y$, and $y_* \in [y(x_*-), y(x_*+)]$ is a also a jump point for the function $g$. Since the map $y$ is strictly monotone, $y(x_*\pm)$ are uniquely determined by the value of $x_*$. Therefore, it makes sense to define $(g \circ y)_\pm(x_*) := g\big(y(x_*\pm)\big)$. This provides us with a unique definition for the composition of the two functions. □

**Remark 5.2.** When the function $y$ is not strictly monotone, the definition of the composition may fail. Consider the simple example

$$g(y) = \begin{cases} g(y) = y, & y \leq 1, \\ g(y) = 2y, & y > 1 \end{cases} \tag{5.1}$$

$$y(x) = \begin{cases} y(x) = \frac{1}{2}x, & x < 1, \\ y(x) = 1, & 1 \leq x < 2, \\ y(x) = x, & x \geq 2. \end{cases} \tag{5.2}$$

At the point $x_* = 1$, we have $y(x_*-) = 1$ and $h\big(y(x_*-)\big) = 1$, but note that $y_*- = \frac{1}{2}$, so that $g(y_*-) \neq g\big(y(x_*-)\big)$, which is in contradiction with the notion of composition of functions.

**Lemma 5.3.** *Assume that*

$$\frac{du_0}{dx} \leq K_0, \qquad \textit{in the sense of distributions},$$

*for some constant* $K_0 \geq 0$ *and suppose that the map h is non-increasing. Then, the minimizer* $\xi = \xi(t, x)$ *of* $y \mapsto G(t, x, y)$ *is such that* $x \mapsto \xi(t, x)$ *is strictly increasing for almost every time t.*

*Proof.* Since $x \mapsto \xi(t, x)$ is increasing for every $t$ (according to Lemma 3.2), so on the set of jump points, we have $\xi(t, x-) < \xi(t, x+)$. We only have to focus on the set of approximate continuity points. We would like to use the identity $H_{t,x}\big(\xi(t, x)\big) = 0$, which holds since $\xi$ is a minimizer of $y \mapsto G(t, x, y)$. Suppose that there exist two points $(t, \underline{x})$ and $(t, \bar{x})$ with $\underline{x} < \bar{x}$ such that $\xi(t, \underline{x}) = \xi(t, \underline{x})$. By monotonicity, $\xi(t, \cdot)$ is a constant for all $x \in (\underline{x}, \bar{x})$. Derive $\xi$ with respect to $x$ on $(\underline{x}, \bar{x})$:

$$0 = \frac{\partial \xi}{\partial x} = \frac{1}{1 + \int_0^t f''\big(u_0\big(\xi(t, x)\big) + h\big(w_0\big(\xi(t, x)\big)\big)s\big)\big(h'(w_0)sw_0' + u_0'\big)\,ds},$$

from which we deduce that

$$\int_0^t f''\big(u_0\big(\xi(t, \bar{x})\big) + h\big(w_0\big(\xi(t, \bar{x})\big)\big)s\big)\big(h'(w_0)sw_0' + u_0'\big)\,ds = +\infty$$

since $\xi(t, x) \equiv \xi(t, \underline{x}) = \xi(t, \bar{x})$ on $(\underline{x}, \bar{x})$. However, Since $f'' > 0$ and $h$ is non-increasing, we have

$$\int_0^t f''\big(u_0\big(\xi(t, \bar{x})\big) + h\big(w_0\big(\xi(t, \bar{x})\big)\big)s\big)\big(h'(w_0)sw_0' + u_0'\big)\,ds\,ds \leq K_0 M t,$$

where $M > 0$ depends on $f$ and $t$. This contradicts the fact that $\frac{\partial \xi}{\partial x} = 0$ on $x \in (\underline{x}, \bar{x})$. □

**Theorem 5.4** (Existence theory for the nonconservative system with BV initial data). *Let* $u_0$ *and* $w_0$ *be in* $BV(\mathbb{R})$ *and satisfy the conditions in Lemma* 5.3. *Then the system* (3.1)–(3.2) *has at least one weak solution* $(u, w)$ *in* $L^\infty\big(\mathbb{R}_+, BV(\mathbb{R})\big)$ *satisfying the initial data that* $w(0, \cdot) = w_0$, $u(0, x\cdot) = u_0$ *given by*

$$w(t, x) = \begin{cases} w_0\big(\xi(t, x)+\big), & H_{t,x}(\xi) \leq 0, \\ w_0\big(\xi(t, x)-\big), & H_{t,x}(\xi) > 0 \end{cases} \tag{5.3}$$

$$u(t, x) = \begin{cases} b\big(t, x, \xi(t, x)+\big) + h(w)t, & H_{t,x}(\xi) \leq 0, \\ b\big(t, x, \xi(t, x)-\big) + h(w)t, & H_{t,x}(\xi) > 0, \end{cases} \tag{5.4}$$

*with* $b(t, x, \cdot)$ *satisfying* (3.8) *and* $H_{t,x}(\cdot)$ *defined in* (3.5).

In view of Lemmas 5.1 and 5.3, the solution is well-defined. Similarly, we have the following statement for the conservative formulation.

**Theorem 5.5** (Existence theory for the conservative system with general initial data). *Let $u_0$ be in $BV(\mathbb{R})$ and $\rho_0$ in $\mathscr{M}_{loc}(\mathbb{R})$ and assume that*

$$\frac{du_0}{dx} \leq K_0, \qquad \text{in the sense of distributions}$$

*for some constant $K_0 \geq 0$ and suppose that the map $h$ is non-increasing. Then the system* (4.2) *has a weak solution in $L^\infty\big(\mathbb{R}_+, BV(\mathbb{R})\big) \times L^\infty\big(\mathbb{R}_+, \mathscr{M}(\mathbb{R})\big)$ which satisfies the initial data*

$$\rho(0,\cdot) = \rho_0, \qquad u(0,\cdot) = u_0,$$

*and is given by*

$$\rho(t,\cdot) = \begin{cases} \rho_0\big((-\infty, \xi(t,x)+)\big), & H_{t,x}(\xi) \leq 0, \\ \rho_0\big((-\infty, \xi(t,x)-)\big), & H_{t,x}(\xi) > 0, \end{cases} \tag{5.5}$$

$$u(t,x) = \begin{cases} b\big(t,x,\xi(t,x)+\big) + h\big(\rho_0\big((-\infty,\xi(t,x)+)\big)\big)t, & H_{t,x}(\xi) \leq 0, \\ b\big(t,x,\xi(t,x)-\big) + h\big(\rho_0\big((-\infty,\xi(t,x)-)\big)\big)t, & H_{t,x}(\xi) > 0, \end{cases} \tag{5.6}$$

*with $b(t,x,\cdot)$ defined in* (3.8) *and $H_{t,x}(\cdot)$ in* (3.5).

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