Portugal. Math. (N.S.)
Vol. 72, Fasc. 4, 2015, 309–355
DOI 10.4171/PM/1970

Portugaliae Mathematica

Dispersive effects and high frequency behaviour for the Schrödinger equation in star-shaped networks

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(Communicated by Enrique Zuazua)

Abstract. We prove the time decay estimates $L^1(\mathscr{R}) \to L^\infty(\mathscr{R})$, where $\mathscr{R}$ is an infinite star-shaped network, for the Schrödinger group $e^{it(-d^2/dx^2+V)}$ for real-valued potentials V satisfying some regularity and decay assumptions. Further we show that the solution for initial conditions with a lower cutoff frequency tends to the free solution, if the cutoff frequency tends to infinity.

Mathematics Subject Classification: 34B45, 47A60, 34L25, 35B20, 35B40

Keywords: Dispersive estimate, Schrödinger operator, nonlinear Schrödinger equation, star-shaped network

1. Introduction

A characteristic feature of the Schrödinger equation is the loss of the localization of wave packets during evolution, the dispersion. This effect can be measured by L^∞-time decay, which implies a spreading out of the solutions, due to the time invariance of the L^2-norm. The well known fact that the free Schrödinger group in $\mathbb{R}^n$ considered as an operator family from L^1 to L^∞ decays exactly as $c \cdot t^{-n/2}$ follows easily from the explicit knowledge of the kernel of this group [23], p. 60. For Schrödinger operators in one and three space dimensions with potentials decaying sufficiently rapidly at infinity, similar estimates have been proved in [16] for the projection of the group on the subspace corresponding to the absolutely continuous spectrum (without optimality). This approach uses an expansion in generalized eigenfunctions together with estimates developed in inverse scattering theory [15]. We also refer to [27] for the Schrödinger equation on the half-line with Dirichlet boundary conditions at 0.

In this paper we derive analogous L^∞-time decay estimates for Schrödinger equations with decaying potentials on a one dimensional star shaped network.

Further we state a perturbation result showing that high energy solutions behave almost as free solutions. For this purpose we furnish an explicit estimate of the difference in terms of the lower cutoff frequency, the potential and time. This result seems to be new even on the line.

Before a precise statement of our main results, let us introduce some notation which will be used throughout the rest of the paper.

Let R_i, $i = 1, \dots, N$, be N ($N \in \mathbb{N}, N \geq 2$) disjoint sets identified with $(0, +\infty)$ and put $\mathscr{R} := \bigcup_{k=1}^{N} \overline{R}_k$. We denote by $f = (f_k)_{k=1,\dots,N} = (f_1, \dots, f_N)$ the functions on $\mathscr{R}$ taking their values in $\mathbb{C}$ and let f_k be the restriction of f to R_k.

Define the Hilbert space $\mathscr{H} = \prod_{k=1}^{N} L^2(R_k)$ with inner product $\big((u_k), (v_k)\big)_{\mathscr{H}} = \sum_{k=1}^{N} (u_k, v_k)_{L^2(R_k)}$ and introduce the following transmission conditions:

$$(u_k)_{k=1,\dots,N} \in \prod_{k=1}^{N} C(\overline{R_k}) \text{ satisfies } u_i(0) = u_k(0) \qquad \forall i, k = 1, \dots, N, \tag{1.1}$$

$$(u_k)_{k=1,\dots,N} \in \prod_{k=1}^{N} C^1(\overline{R_k}) \text{ satisfies } \sum_{k=1}^{N} \frac{du_k}{dx}(0^+) = 0. \tag{1.2}$$

Let $H_0 : \mathscr{D}(H_0) \to \mathscr{H}$ be the linear operator on $\mathscr{H}$ defined by:

$$\mathscr{D}(H_0) = \Big\{ (u_k) \in \prod_{k=1}^{N} H^2(R_k); (u_k) \text{ satisfies } (1.1), (1.2) \Big\},$$

$$H_0(u_k) = (H_{0,k} u_k)_{k=1,\dots,N} = \left(-\frac{d^2 u_k}{dx^2} \right)_{k=1,\dots,N} = -\Delta_{\mathscr{R}}(u_k).$$

This operator H_0 is self-adjoint and its spectrum $\sigma(H_0)$ is equal to $[0, +\infty)$ (see [4] for more details).

For any $s \in \mathbb{R}$, let us denote by $L^1_s(\mathscr{R})$ the space of all complex-valued measurable functions $\phi = (\phi_1, \dots, \phi_N)$ defined on $\mathscr{R}$ such that

$$\|\phi\|_{L^1_s(\mathscr{R})} := \int_{\mathscr{R}} |\phi(x)| \langle x \rangle^s \, dx = \sum_{k=1}^{N} \int_{R_k} |\phi_k(x)| \langle x \rangle^s \, dx < \infty,$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. This space is a Banach space with the norm $\| \cdot \|_{L^1_s(\mathscr{R})}$.

Let $V \in L^1_1(\mathscr{R})$. Denote by H the self-adjoint realization of the operator $-\dfrac{d^2}{dx^2} + V$ together with the transmission conditions (1.1) and (1.2) on $L^2(\mathscr{R})$. From Chapter 2 of [12], we deduce that its spectrum satisfies

$$\sigma(H) = [0, +\infty) \cup \{\text{a finite number of negative eigenvalues}\}.$$

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We first verify that the free Schrödinger group on the star-shaped network $\mathcal{R}$ satisfies the following dispersive estimate (see Section 3)

$$\|e^{itH_0}\|_{L^1(\mathcal{R})\to L^\infty(\mathcal{R})} \le C|t|^{-1/2}, \qquad t \neq 0.$$

Our goal is then to assume non restrictive assumptions on the potential V in terms of decay or regularity in order to get a similar decay for the Schrödinger equation with potential V. More precisely, we will prove the following theorem.

1.1 Theorem. *Let $V \in L^1_\gamma(\mathcal{R})$ be real valued, with $\gamma > 5/2$ and assume that* (4.38) *below holds. Then for all $t \neq 0$,*

$$\|e^{itH}P_{ac}(H)\|_{L^1(\mathcal{R})\to L^\infty(\mathcal{R})} \le C|t|^{-1/2} \tag{1.3}$$

where C is a positive constant and $P_{ac}(H)$ is the projection onto the absolutely continuous spectral subspace.

The assumption (4.38) is satisfied by a large choice of potentials (see Lemma 4.3 below). It allows to built the kernel of the resolvent (see Definition 4.11 and Theorem 4.12) and takes into account the ramification character of the problem.

At a first attempt, we have assumed that $V \in L^1_\gamma(\mathcal{R})$, with $\gamma > 5/2$ (in order to be able to apply some appropriate estimates on the derivatives of the Jost functions, see for instance Corollary 4.9), while a similar result probably holds under the assumption that $V \in L^1_2(\mathcal{R})$ (see [16] in the case $N = 2$). This decay of the potential implies that we do not need the so-called non resonance at zero energy assumption (see [16], pp. 163–164). For potentials $V \in L^1_1(\mathcal{R})$ such an assumption would appear but it is a difficult and delicate question. Furthermore, up to our knowledge, if $V \in L^1_1(\mathcal{R})$, it is unknown how to built the kernel of the resolvent.

As a consequence, we have the following $L^p - L^{p'}$ estimate.

1.2 Corollary ($L^p - L^{p'}$ estimate). *Under the assumptions of Theorem* 1.1, *for $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$ we have for all $t \neq 0$,*

$$\|e^{itH}P_{ac}(H)\|_{L^p(\mathcal{R})\to L^{p'}(\mathcal{R})} \le C|t|^{-1/p+1/2}, \tag{1.4}$$

where $C > 0$ is a constant.

Moreover we have the following Strichartz estimates which have been used in the context of the nonlinear Schrödinger equation to obtain well-posedness results.

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1.3 Corollary (Strichartz estimates). *Let the assumptions of Theorem* 1.1 *be satisfied. Then for* $2 \le p, q \le +\infty$ *and* $\frac{1}{p} + \frac{2}{q} = \frac{1}{2}$ *we have for all t,*

$$\|e^{itH} P_{ac}(H) f\|_{L^q(\mathbb{R}, L^p(\mathscr{R}))} \le C \|f\|_2, \qquad \forall f \in L^p(\mathscr{R}) \cap L^2(\mathscr{R}), \tag{1.5}$$

where $C > 0$ *is a constant.*

As a direct consequence, see [14], we have the following well-posedness result for a nonlinear Schrödinger equation with potential. Let $p \in (0, 4)$ and suppose that V satisfies the assumptions of Theorem 1.1. Then, for any $u_0 \in L^2(\mathscr{R})$, there exists a unique solution

$$u \in C\big(\mathbb{R}; L^2(\mathscr{R})\big) \cap \bigcap_{(q,r) \text{ admissible}} L^q_{loc}\big(\mathbb{R}; L^r(\mathscr{R})\big)$$

of the equation

$$\begin{cases} iu_t - \Delta_{\mathscr{R}} u + Vu \pm |u|^p u = 0, & t \ne 0, \\ u(0) = u^0. \end{cases} \tag{1.6}$$

Recall that a pair (q, r) is called admissible if (q, r) satisfies that $2 \le r, q \le +\infty$ and $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$.

1.4 Remark. Another direct consequence of the dispersive estimate (1.3) or of the $L^p - L^{p'}$ estimate (1.4) is that we can construct, as in [26], the scattering operator for the nonlinear Schrödinger equation with potential.

While proving Theorem 1.1 we obtain as results of independent interest the L^∞-time decay for the high frequency part of the group and a high frequency perturbation estimate:

1.5 Theorem. *Under the assumptions of Theorem* 1.1 *we have*

$$\|e^{itH} \chi_{\lambda_0}(H)\|_{1,\infty} \le \left(A + B \frac{\|V\|_1}{\sqrt{\lambda_0}} \right) |t|^{-1/2}, \qquad t \ne 0, \tag{1.7}$$

$$\|e^{itH} \chi_{\lambda_0}(H) - e^{itH_0} \chi_{\lambda_0}(H_0)\|_{1,\infty} \le B \frac{\|V\|_1}{\sqrt{\lambda_0}} |t|^{-1/2}, \qquad t \ne 0. \tag{1.8}$$

Here χ_{λ_0} *is smoothly cutting off the frequencies below* λ_0. *Expressions of* A, B *in terms of the cutoff function but independent of* λ_0 *are given in Theorem* 5.11.

In particular we have for any $f \in L^1(\mathscr{R})$ *that*

$$e^{itH} \chi_{\lambda_0}(H) f \to e^{itH_0} \chi_{\lambda_0}(H_0) f \qquad \text{for } \lambda_0 \to \infty$$

uniformly on $\mathscr{R}$ *for every fixed* $t > 0$.

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The perturbation estimate allows the simultaneous control of the smallness of the difference between perturbed and unperturbed group in terms of the cutoff frequency, the L^1-Norm of the potential and time.

Because the reflection and refraction of wave packets for the unperturbed Schrödinger equation on the star shaped network is known ([2] for the case of 3 branches), the above perturbation estimate furnishes an approximate spatial information on the propagation of high frequency wave packets with explicit control of the error. Note that the high frequency perturbation estimate seems to be new even for the Schrödinger equation with potential on the line and represents in this case an improvement of [16]. In [16], estimate (1.7) is furnished, but without explicit control of the dependence of the coefficient of $|t|^{-1/2}$ on λ_0. Without this control estimate (1.8) is not useful to prove the convergence of the solution to the free solution.

The paper is organized as follows. The second section deals with a counterexample which shows that the decay of the Schrödinger operator from $L^1(\mathscr{R})$ to $L^\infty(\mathscr{R})$ as $|t|$ goes to infinity is not guaranteed for all infinite networks. In Section 3, we prove the dispersive estimate for the free Schrödinger operator on star-shaped networks and we give some direct applications. The expansion in generalized eigenfunctions needed for the proof of Theorem 1.1, is given in Section 4. In the last section we give the proof of the main results of the paper (Theorems 1.1 and 1.5).

The main lines of our arguments are the following. The counter example (Section 2) uses explicit formulas for eigenfunctions of the laplacian on infinite trees from [22]. The L^∞-time decay of the free Schrödinger group on a star shaped network is reduced to the corresponding estimate on $\mathbb{R}$ using an appropriate change of variables (Section 3). The task of finding a complete family of generalized eigenfunctions for the Schrödinger operator with potential on the star shaped network is reduced to the case of the real line by separating the branches and extending the equations on $\mathbb{R}$ with vanishing potential. The generalized eigenfunctions on $\mathbb{R}$ resulting from techniques from [15] are then combined to families on the network by introducing correction terms to establish the transmission conditions. Using results of [15] for the real line case, we derive estimates showing the dependence of the generalized eigenfunctions on the potential. This enables us to prove a limiting absorption principle and then to derive an expansion of the Schrödinger group on the star in these generalized eigenfunctions (Section 4) following [5], [6]. The proof of the L^∞-time decay is divided in the low frequency and high frequency part, essentialy following the lines of [16]. For the high frequency components, the potential appears as a small perturbation: the resolvent of the Schrödinger operator can be expanded in a Neumann type series in terms of the resolvent of the free Schrödinger operator. By inserting this in Stones formula and exchanging the integration over the frequencies and the summation of the

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Neumann series, one reduces the estimate to the free case. For the low frequency components one uses the expansion in generalized eigenfunctions derived in Section 4, especially the qualitative knowledge of the dependence of the generalized eigenfunctions on the potential. This enables us to construct a representation of the solution as the free Schrödinger group acting on a well chosen (artificial) initial condition, which encodes the influence of the potential. Then one concludes using the results on the line.

Our approach does not furnish optimal results, as for example the estimate in [23], p. 60 for the free Schrödinger group or the results of [7]. This is due to the fact, that the use of Neumann type series and qualitative estimates from inverse scattering theory are to rough for this purpose. We conjecture that optimal estimates could be achieved in terms of an asymptotic expansion of first order following the lines of [7], where this problem has been solved for initial conditions in energy bands for the Klein Gordon equation with constant but different potentials on a star shaped network. It might be useful to find a way to represent solutions for general potentials by approximating these potentials by step functions, inspired by [13].

Note that the general perturbation theory for semigroups [18], Ch. 9, Thm. 2.12, p. 502 is applicable but not useful for our purposes: it yields that the difference between the (semi-)groups generated by the Schrödinger operator with potential and the free one grows at most proportionally to t, which engulfs the time decay at infinity. Nevertheless it furnishes additional information for small t.

The Trotter product formula [23], Thm. X.51, p. 245 is also applicable, but cannot establish L^∞-time decay either: it consists of an approximation of the perturbed group by long alternating compositions of values of the free Schrödinger group e^{itH_0} and the group of multiplication operators with e^{itV} but for small values of t. Thus even the explicit knowledge of the kernel of the free Schrödinger group is not useful for time-decay, because the factor $t^{-1/2}$ becomes effective only for large t.

The direct application of the variation of constants formula leads to the same phenomenon as the perturbation for semigroups: without a refined study of the superposition of the waves generated by the potential, the rough estimation of the integral term leads to a bound growing as a constant times t.

In [9] the authors prove dispersive estimates for Schrödinger equations on infinite trees with semi-infinite ends with Kirchhoff conditions at the nodes. The equations do not have a potential, but the operator has piecewise constant coefficients with finitely many discontinuities on each branch. The coefficients are bounded between two values. Here the difficulty comes from the necessity to give a recursive formula for the infinitely many terms of the resolvent of the operator. The inverse of the Wronskian is estimated using the theory of almost periodic functions. In [8], [11] the authors study the dispersion for the Schrödinger equation on the line with irregular coefficients.

In [1] the authors consider Schrödinger equations with attractive cubic nonlinearities on a star-shaped network with three branches. At the node they consider Kirchhoff-conditions, δ- or δ'-conditions. They indicate that the equation arises in quantum field theory, in the description of the Bose-Einstein condensates and electromagnetic pulse propagation in optical fibers. The Kirchhoff condition corresponds to a simple coupling ("beam splitter"), whereas the δ-condition describes the interaction with a point-potential. The authors obtain charge and energy conservation laws and deduce from these facts conditions for global in time existence of solutions. Further they treat the existence life time of solitary waves and prove that their transmission and reflection at the node is governed by the associated linear laws, due to the shortness of the interaction time with a point-shaped potential. However the authors do not consider variable potentials on the branches as it is done in our paper. Therefore the linear part of their paper has no substantial intersection with our setting but might motivate further studies.

In [2] an analogous setting as in [1] is considered, but with nonlinearities of order $2\mu + 1$ and only the δ-potential of strength α at the node. The existence of stationary solitons in both the attractive ($\alpha < 0$) and repulsive ($\alpha > 0$) case is proved. Again there is no significant interference with our results.

In [10] the authors consider free (linear) Schrödinger equations on tree-shaped networks with δ-potentials at the nodes. As a special case appears the star-shaped network with a delta-potential at the center. In this setting a $L^1 - L^\infty$-decay estimate is proved. Due to the fact that the δ-potential plays the role of a transmission condition, the methods are those for a problem with constant coefficients, and therefore there is only a marginal interference with our results. Nevertheless the result is instructive. The authors add the existence and uniqueness of a global in time solution of the same problem with a (attractive or repulsive) power nonlinearity of order $p + 1$.

The paper [20] deals with the general question of constructing generalized eigenfunctions of all possible self adjoint extensions of the Laplacian on networks with semi infinite ends. The result is formulated in terms of a so called scattering matrix, which indicates the reflected and transmitted flow for the stationary problem. For complicated networks the authors construct a product formula linking the scattering matrices of sub networks to the scattering matrix of the original network. The results of this article could serve to generalize our results to star shaped networks with general transmission conditions.

The article [21] considers discrete analogs of nonlinear Schrödinger equations on star-shaped networks including the existence of solitons, constants of motion and the calculus of transmission probabilities.

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Finally [25] treats the stationary (cubic) nonlinear Schrödinger equation for simple but more general networks as the star shaped ones as trees or helices. Explicit solution formulas are obtained.

The last two papers are instructive for further developments of our approach.

Acknowledgements. The authors thank the referees for many valuable remarks which helped us to improve the paper significantly.

2. A counterexample

Consider the infinite network $\mathscr{R} = \bigcup_{n\in\mathbb{N}} e_n$, where each edge $e_n = (n, n+1)$ with the set of vertices $\mathscr{V} = \bigcup_{n\in\mathbb{N}} v_n$, where $v_n = \{n\}$. For a fixed sequence of positive real numbers $\alpha = (\alpha_n)_{n\in\mathbb{N}}$, we define the Hilbert space $L^2(\mathscr{R}, \alpha)$ as follows

$$L^2(\mathscr{R}, \alpha) = \Big\{u = (u_n)_{n\in\mathbb{N}} : u_n \in L^2(e_n)\ \forall n \in \mathbb{N} \text{ such that} \\ \sum_{n\in\mathbb{N}} \alpha_n \int_{e_n} |u_n(x)|^2\, dx < \infty\Big\},$$

equipped with the inner product

$$(u, v) = \sum_{n\in\mathbb{N}} \alpha_n \int_{e_n} u_n(x) v_n(x)\, dx, \qquad \forall u, v \in L^2(\mathscr{R}, \alpha).$$

Similarly for all $k \in \mathbb{N}^*$, we set

$$H^k(\mathscr{R}, \alpha) = \{u = (u_n)_{n\in\mathbb{N}} \in L^2(\mathscr{R}, \alpha) : (u_n^{(\ell)})_{n\in\mathbb{N}} \in L^2(\mathscr{R}, \alpha)\ \forall \ell \in \{1, 2, \ldots, k\}\},$$

where $u_n^{(\ell)}$ means the ℓ derivative of u_n with respect to x.

Now we consider the Laplace operator $-\Delta_\alpha$ (depending on α) as follows:

$$\mathscr{D}(-\Delta_\alpha) = \{u = (u_n)_{n\in\mathbb{N}} \in H^2(\mathscr{R}, \alpha) : \text{satisfying (2.9), (2.10), (2.11) below}\},$$

$$u_0(0) = 0, \tag{2.9}$$

$$u_n(n+1) = u_{n+1}(n+1), \qquad \forall n \in \mathbb{N}, \tag{2.10}$$

$$\alpha_n \frac{du_n}{dx}(n+1) = \alpha_{n+1} \frac{du_{n+1}}{dx}(n+1), \qquad \forall n \in \mathbb{N}. \tag{2.11}$$

For all $u \in \mathscr{D}(-\Delta_\alpha)$, we set

$$-\Delta_\alpha u = \left(-\frac{d^2 u_n}{dx^2}\right)_{n\in\mathbb{N}}.$$

By Section 1.5 of [22], this operator is a non negative self-adjoint operator in $L^2(\mathcal{R}, \alpha)$.

Moreover in Theorem 1.13 of [22] it was shown the

2.1 Theorem. *For all $k \in \mathbb{N}^*$, $-k^2\pi^2$ is a simple eigenvalue of $-\Delta_\alpha$ if and only if*

$$s = \sum_{n \in \mathbb{N}} \frac{1}{\alpha_n} < \infty. \tag{2.12}$$

In that case the associated orthonormal eigenvector $\varphi^{[k]} = (\varphi^{[k]})_{n \in \mathbb{N}}$ is given by

$$\varphi_n^{[k]} = \sqrt{\frac{2}{s}} \frac{(-1)^{(n-1)k}}{\alpha_n} \sin\big(k\pi(x-n)\big), \qquad \forall x \in e_n,\ n \in \mathbb{N}.$$

Now assuming that (2.12) holds, then for any $k \in \mathbb{N}^*$ we consider the solution u of the Schrödinger equation

$$\begin{cases} \partial_t u - i\Delta_\alpha u = 0, \\ u(t=0) = \varphi^{[k]}, \end{cases}$$

or equivalently solution of

$$\begin{cases} \partial_t u_n - i\partial_x^2 u_n = 0, & \text{in } e_n \times \mathbb{R}, \\ u_0(0,t) = 0, & \text{on } \mathbb{R}, \\ u_n(n+1,t) = u_{n+1}(n+1,t) & \text{on } \mathbb{R},\ \forall n \in \mathbb{N}, \\ \alpha_n u_n'(n+1,t) = \alpha_{n+1} u_{n+1}'(n+1,t) & \text{on } \mathbb{R},\ \forall n \in \mathbb{N}, \\ u(t=0,\cdot) = \varphi^{[k]} & \text{on } \mathcal{R}. \end{cases}$$

This solution is given by $u(t) = e^{-itk^2\pi^2}\varphi^{[k]}$. Moreover simple calculations show that

$$\|u(t)\|_{\infty,\mathcal{R}} = \sqrt{\frac{2}{s}} \sup_{n \in \mathbb{N}} \frac{1}{\alpha_n} \big\|\sin\big(k\pi(\cdot - n)\big)\big\|_{\infty, e_n} = \sqrt{\frac{2}{s}} \sup_{n \in \mathbb{N}} \frac{1}{\alpha_n},$$

which is independent of t and then does not tend to zero as $|t|$ goes to infinity. On the other hand $u(t=0,\cdot)$ belongs to $L^1(\mathcal{R})$, since we have

$$\|u(t)\|_{L^1(\mathcal{R})} = \sqrt{\frac{2}{s}} \sum_{n \in \mathbb{N}} \frac{1}{\alpha_n} \big\|\sin\big(k\pi(\cdot - n)\big)\big\|_{L^1(e_n)} \le \sqrt{2s}.$$

In other words, we have proved the

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2.2 Theorem. *If* (2.12) *holds, then the norm of the Schrödinger operator* $e^{it\Delta_\alpha}$ *from* $L^1(\mathcal{R})$ *to* $L^\infty(\mathcal{R})$ *does not tend to zero as* $|t|$ *goes to infinity.*

This counterexample shows that the decay of the norm of the Schrödinger operator from $L^1(\mathcal{R})$ to $L^\infty(\mathcal{R})$ as $|t|$ goes to infinity is not guaranteed for all infinite networks. Hence the remainder the paper is to give some examples where such a case occurs.

Let us notice that our non dispersive property comes from the infinite numbers of discontinuities of the coefficient, since for a finite number of discontinuities or BV coefficient with a small variation of the coefficients, the dispersive property holds, see [8], [11].

3. Dispersive estimate for free Schrödinger operator on star-shaped networks

In this section we state the L^∞-time decay estimate for the free Schrödinger equation (and some consequences) on star shaped networks. For completeness we give the proof, although it is essentially the same as in [1], [17].

3.1 Theorem (Dispersive estimate). *For all* $t \neq 0$,

$$\|e^{itH_0}\|_{L^1(\mathcal{R})\to L^\infty(\mathcal{R})} \leq C|t|^{-1/2}, \tag{3.13}$$

where $C > 0$ *is a constant.*

Proof. Let v_j, $j = 1, \dots, N$, a solution of the following problem

$$\begin{cases} \partial_t v_j = -i\partial_x^2 v_j, \quad \mathbb{R}^+ \times \mathbb{R}^+, \\ v_j(t,0) = v_1(t,0), \quad \displaystyle\sum_{j=1}^N \partial_x v_j(t,0) = 0, \quad \mathbb{R}^+, \\ v_j(0,x) = v_j^0(x), \quad \mathbb{R}^+. \end{cases}$$

If we denote by $w_1 = \sum_{j=1}^N v_j$ and $w_j = v_j - v_1$, $\forall$, $j = 2, \dots, N$.

Then w_1 satisfies

$$\begin{cases} \partial_t w_1 = -i\partial_x^2 w_1, \quad \mathbb{R}^+ \times \mathbb{R}^+, \\ \partial_x w_1(t,0) = 0, \quad \mathbb{R}^+, \\ w_1(0,x) = \displaystyle\sum_{j=1}^N v_j^0(x), \quad \mathbb{R}^+, \end{cases}$$

and w_j, $j = 2, \dots, N$, satisfies the following problem

$$\begin{cases} \partial_t w_j = -i\partial_x^2 w_j, & \mathbb{R}^+ \times \mathbb{R}^+, \\ w_j(t,0) = 0, & \mathbb{R}^+, \\ w_j(0,x) = v_j^0(x) - v_1^0(x), & \mathbb{R}^+. \end{cases}$$

By an odd reflection transformation applied to w_1, we obtain $\tilde{w}_1(t,x) = \begin{cases} w_1(t,x), & x > 0, \\ -w_1(t,-x), & x < 0, \end{cases}$ which verifies

$$\begin{cases} \partial_t \tilde{w}_1 = -i\partial_x^2 \tilde{w}_1, & \mathbb{R}^2, \\ \tilde{w}_1(0,x) = \sum_{j=1}^{N} \tilde{v}_j^0(x), & \mathbb{R}, \end{cases}$$

where $\tilde{v}_j^0 = \begin{cases} v_j^0(x), & x > 0, \\ -v_j^0(-x), & x < 0, \end{cases}$ $j = 1, \dots, N$. So, according to the dispersive estimate for Schrödinger operator on the line (see [18] or [23] for more details), we have

$$\|w_1\|_{L^\infty(\mathbb{R}^+)} \le \|\tilde{w}_1\|_{L^\infty(\mathbb{R})} \le C|t|^{-1/2} \Big\| \sum_{j=1}^{N} \tilde{v}_j^0 \Big\|_{L^1(\mathbb{R})}, \quad \forall (v_j^0) \in L^2(\mathscr{R}) \cap L^1(\mathscr{R}), \tag{3.14}$$

where $C > 0$ is a constant.

Which implies

$$\|w_1\|_{L^\infty(\mathbb{R}^+)} \le 2C|t|^{-1/2} \Big\| \sum_{j=1}^{N} v_j^0 \Big\|_{L^1(\mathbb{R}^+)}, \qquad \forall (v_j^0) \in L^2(\mathscr{R}) \cap L^1(\mathscr{R}).$$

For $j = 2, \dots, N$, we notice that w_j is solution of the free Schrödinger equation on the half-line, hence by Theorem 2.1 of [27], we get

$$\|w_j\|_{L^\infty(\mathbb{R}_x^+)} \le C|t|^{-1/2} \|v_j^0 - v_1^0\|_{L^1(\mathbb{R}^+)}, \qquad \forall (v_j^0) \in L^2(\mathscr{R}) \cap L^1(\mathscr{R}), \tag{3.15}$$

where $C > 0$ is a constant.

Since, $v_j = w_j + v_1$, $\forall j = 2, \dots, N$ and $v_1 + \sum_{j=2}^{N} (w_j + v_1) = w_1 \Rightarrow v_1 = \frac{1}{N} w_1 - \frac{1}{N} \sum_{j=2}^{N} w_j$.

Thus (3.14)–(3.15) imply that

$$\|v_1\|_{L^\infty(\mathbb{R}^+)} \le \frac{4C}{N}|t|^{-1/2}\sum_{j=2}^{N}(\|v_j^0\|_{L^1(\mathbb{R}^+)} + \|v_1^0\|_{L^1(\mathbb{R}^+)}),$$
$$\forall (v_j^0) \in L^2(\mathscr{R}) \cap L^1(\mathscr{R}), \tag{3.16}$$

where $C > 0$ is a constant.

According to the above we have

$$\|v_1\|_{L^\infty(\mathbb{R}^+)} \le 4C|t|^{-1/2}\sum_{j=1}^{N}\|v_j^0\|_{L^1(\mathbb{R}^+)}, \tag{3.17}$$

and

$$\begin{aligned}\|v_j\|_{L^\infty(\mathbb{R}^+)} &\le \|w_j\|_{L^\infty(\mathbb{R}^+)} + \|v_1\|_{L^\infty(\mathbb{R}^+)}\\ &\le 2C|t|^{-1/2}(\|v_j^0\|_{L^1(\mathbb{R}^+)} + \|v_1^0\|_{L^1(\mathbb{R}^+)})\\ &\quad + 4C|t|^{-1/2}\sum_{j=1}^{N}\|v_j^0\|_{L^1(\mathbb{R}^+)}, \qquad \forall (v_j^0) \in L^2(\mathscr{R}) \cap L^1(\mathscr{R}),\end{aligned} \tag{3.18}$$

$\Rightarrow$

$$\|v_j\|_{L^\infty(\mathbb{R}^+)} \le 8C|t|^{-1/2}\sum_{j=1}^{N}\|v_j^0\|_{L^1(\mathbb{R}^+)}, \qquad \forall (v_j^0) \in L^2(\mathscr{R}) \cap L^1(\mathscr{R}),\ \forall j \ge 2. \tag{3.19}$$

Finally we obtain for all $t \neq 0$, $(v_j^0) \in L^2(\mathscr{R}) \cap L^1(\mathscr{R})$,

$$\|(v_j)\|_{L^\infty(\mathscr{R})} \le 8C|t|^{-1/2}\sum_{j=1}^{N}\|v_j^0\|_{L^1(\mathbb{R}^+)} = 8C|t|^{-1/2}\|(v_j^0)\|_{L^1(\mathscr{R})}, \tag{3.20}$$

which implies (3.13). □

As a direct consequence of the dispersive estimate for the free Schrödinger operator on a star-shaped network, we can obtain the following Strichartz estimates (for a direct proof, see [17])

3.2 Corollary ($L^p - L^{p'}$ estimate). *For $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$ we have for all $t \neq 0$,*

$$\|e^{itH_0}\|_{L^p(\mathscr{R}) \to L^{p'}(\mathscr{R})} \le C|t|^{-1/p+1/2}, \tag{3.21}$$

where $C > 0$ is a constant.

Proof. According to (3.13) we have

$$\sup_{t\neq 0}|t|^{1/2}\|e^{itH_0}f\|_\infty \le C\|f\|_1, \qquad \forall f \in L^1(\mathscr{R})\cap L^2(\mathscr{R}).$$

Interpolating with the L^2 bound $\|e^{itH_0}f\|_2 = \|f\|_2$, leads to

$$\sup_{t\neq 0}|t|^{-1/2+1/p}\|e^{itH_0}f\|_{p'} \le C\|f\|_p, \qquad \forall f \in L^1(\mathscr{R})\cap L^2(\mathscr{R}), \tag{3.22}$$

where $1 \le p \le 2$. It is well-known that via T^*T argument (3.22) gives rise to the class of Strichartz estimates

$$\|e^{itH_0}f\|_{L_t^q(L_x^p)} \le C\|f\|_2, \qquad \forall \frac{2}{q}+\frac{1}{p}=\frac{1}{2},\ 2<q\le+\infty,\ 2\le p\le\infty. \tag{3.23}$$

The endpoint $q=2$ is not captured by this approach but by the approach develloped by Keel and Tao in [19]. So the estimate (3.23) is valid for all $2 \le p, q \le +\infty$ satisfying $\frac{2}{q}+\frac{1}{p}=\frac{1}{2}$ and we have also,

$$\Big\|\int_{\mathbb{R}} e^{-itH_0}F(s,.)\,ds\Big\|_{L^2(\mathscr{R})} \le C\|F\|_{L^{q'}(\mathbb{R},L^{p'}(\mathscr{R}))},$$

$$\Big\|\int_0^t e^{i(t-s)H_0}F(s)\,ds\Big\|_{L^q(\mathbb{R},L^{r'}(\mathscr{R}))} \le C\|F\|_{L^{r'}(\mathbb{R},L^{s'}(\mathscr{R}))},$$

for all admissible pairs (q,p) and (r,s) satisfying $\frac{2}{q}+\frac{1}{p}=\frac{1}{2}$, $2\le q,p\le+\infty$. □

Corollary 1.3 can be proved in the same way.

According to (3.23) and [14], we have for $p \in (0,4)$, that for any $u_0 \in L^2(\mathscr{R})$ the equation

$$iu_t - \Delta_{\mathscr{R}}u \pm |u|^p u = 0, \qquad t\neq 0,\ u=u_0,\ t=0,$$

admits a unique solution $u \in C\big(\mathbb{R},L^2(\mathscr{R})\big)\cap\bigcap_{(q,r)\ \text{admissible}} L^q_{loc}\big(\mathbb{R},L^r(\mathscr{R})\big)$.

For similar results about nonlinear Schrödinger equation on graphs, we refer to [2], [21], [25].

4. Expansion in generalized eigenfunctions

The goal of this section is to find an explicit expression for the kernel of the resolvent of the operator H on the star-shaped network defined in Section 1. First we separate the branches by extending the potential of the Schrödinger operator by zero on $(-\infty, 0)$. Using [15], we construct N families of generalized eigenfunctions of the resulting N Schrödinger operators on $\mathbb{R}$, which we recombine on the network. This approach can be compared with the ones developed for Klein-Gordon equations in $\mathscr{R}$ by [5], [6].

For each $j = 1, \dots, N$, we recall that R_j is identified to $(0, +\infty)$ and denote by V_j the restriction of V to R_j. Consider R_j as a subset of $\mathbb{R}$ and denote by $\tilde{V}_j$ the extension of V_j by 0 outside R_j.

Now according to [15] (see also [26], [27]) for all $z \in \mathbb{C}^+ := \{z_1 \in \mathbb{C} : \Im z_1 \geq 0\}$, there exist two functions $f_{j,\pm}(z, \cdot)$ that satisfy the differential equation

$$-f''_{j,\pm}(z, x) + \tilde{V}_j(x) f_{j,\pm}(z, x) = z^2 f_{j,\pm}(z, x) \quad \text{ on } \mathbb{R}, \tag{4.24}$$

and that have the asymptotic behaviour

$$|f_{j,\pm}(z, x) - e^{\pm izx}| \to 0 \quad \text{ as } x \to \pm\infty. \tag{4.25}$$

According to Section 1 of [15] (see also [26], p. 45) we write

$$f_{j,\pm}(z, x) = e^{\pm izx} m_{j,\pm}(z, x),$$

to remove the oscillations of $f_{j,\pm}$ at infinity. The functions $m_{j,\pm}$ are the unique solutions of the Volterra integral equations:

$$m_{j,+}(z, x) = 1 + \int_x^{+\infty} \frac{e^{2iz(y-x)} - 1}{2iz} \tilde{V}_j(y) m_{j,+}(z, y)\, dy, \tag{4.26}$$

$$m_{j,-}(z, x) = 1 + \int_{-\infty}^{x} \frac{e^{2iz(y-x)} - 1}{2iz} \tilde{V}_j(y) m_{j,-}(z, y)\, dy, \tag{4.27}$$

and are called Jost functions (see [15], [24]). Recall that Lemma 1 of [15] (see also (2.5) of [26]) implies that

$$|m_{j,+}(z, x)| \leq C, \quad \forall x \in [0, \infty),\ z \in \mathbb{C}^+, \tag{4.28}$$

$$|m_{j,-}(z, x)| \leq 1 + C\frac{1 + x}{1 + |z|}, \quad \forall x \in [0, \infty),\ z \in \mathbb{C}^+, \tag{4.29}$$

for some $C > 0$. Accordingly as $f_{j,\pm}(z, x) = e^{\pm izx} m_{j,\pm}(z, x)$, we get

$$|f_{j,+}(z, x)| \leq C, \quad \forall x \in [0, \infty),\ z \in \mathbb{C}^+, \tag{4.30}$$

$$|f_{j,-}(z, x)| \leq C(1 + x) e^{\Im zx}, \quad \forall x \in [0, \infty),\ z \in \mathbb{C}^+. \tag{4.31}$$

Property (4.25) implies the existence of functions $T_j, R_{j,1}, R_{j,2}$, $j = 1, \dots, N$, called transmission and reflection coefficients, such that

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$$f_{j,+}(x,r) \sim \frac{1}{T_j(r)} e^{irx} + \frac{R_{j,2}(r)}{T_j(r)} e^{-irx}, \qquad x \to -\infty$$

$$f_{j,-}(x,r) \sim \frac{1}{T_j(r)} e^{-irx} + \frac{R_{j,1}(r)}{T_j(r)} e^{irx}, \qquad x \to \infty$$

for $r \in \mathbb{R}$. For future purposes, for all real numbers r, we need the scattering matrix $S_j(r) \in \mathbb{C}^{2\times 2}$ associated with (4.24) given by

$$S_j(r) = \begin{pmatrix} T_j(r) & R_{j,2}(r) \\ R_{j,1}(r) & T_j(r) \end{pmatrix}$$

and that is continuous on $\mathbb{R}$. According to [15], T_j has a meromorphic extension to $\mathbb{C}^+$ (with a finite numbers of simple poles that are non zero purely imaginary numbers) that is given by (see [15], p. 145)

$$\frac{1}{T_j(z)} = 1 - \frac{1}{2iz}\int_{-\infty}^{+\infty} \tilde{V}_j(y) m_{j,+}(z,y)\,dy \qquad \forall z \in \mathbb{C}^+. \tag{4.32}$$

Since $\tilde{V}_j$ has its support in $(0,+\infty)$, by Remark 10 of [15] $R_{j,2}$ admits also a meromorphic extension on $\mathbb{C}^+\backslash\mathbb{R}$ (with the same poles as the ones of T_j) that is given by (compare [15], p. 145 when z is real)

$$\frac{R_{j,2}(z)}{T_j(z)} = \frac{1}{2iz}\int_{-\infty}^{+\infty} e^{2izy}\tilde{V}_j(y) m_{j,+}(z,y)\,dy \qquad \forall z \in \mathbb{C}^+. \tag{4.33}$$

Due to the fact that $\tilde{V}_j$ is zero on $(-\infty,0)$, the generalized eigenfunctions $f_{j,\pm}$ of the Schrödinger operators on the line have the following properties.

4.1 Lemma. *For all $z \in \mathbb{C}^+$, $z \neq 0$, we have*

$$f_{j,-}(z,x) = e^{-izx} \qquad \forall x \le 0, \tag{4.34}$$

$$f_{j,+}(z,x) = \frac{1}{T_j(z)} e^{izx} + \frac{R_{j,2}(z)}{T_j(z)} e^{-izx} \qquad \forall x \le 0. \tag{4.35}$$

In particular, it holds

$$f_{j,-}(z,0) = 1, \tag{4.36}$$

$$f_{j,+}(z,0) = \frac{1+R_{j,2}(z)}{T_j(z)}. \tag{4.37}$$

Proof. From the expression (4.27), we directly get (4.34) and (4.36). The situation is more complicated for $f_{j,+}$. Indeed from the expression (4.26), we see that

$$m_{j,+}(z,x) = 1 + \int_0^{+\infty} \frac{e^{2iz(y-x)} - 1}{2iz} \tilde{V}_j(y) m_{j,+}(z,y)\,dy, \qquad \forall x \le 0.$$

This is equivalent to

$$\begin{aligned} m_{j,+}(z,x) &= 1 - \frac{1}{2iz}\int_0^{+\infty} \tilde{V}_j(y) m_{j,+}(z,y)\,dy + \frac{e^{-2izx}}{2iz}\int_0^{+\infty} e^{2izy}\tilde{V}_j(y) m_{j,+}(z,y)\,dy \\ &= 1 - \frac{1}{2iz}\int_{-\infty}^{+\infty} \tilde{V}_j(y) m_{j,+}(z,y)\,dy \\ &\quad + \frac{e^{-2izx}}{2iz}\int_{-\infty}^{+\infty} e^{2izy}\tilde{V}_j(y) m_{j,+}(z,y)\,dy, \qquad \forall x \le 0. \end{aligned}$$

Hence according to the expression of $\frac{1}{T_j(z)}$ and $\frac{R_{j,2}(z)}{T_j(z)}$ given in (4.32) and (4.33), we obtain (4.35). According to this identity we trivially have

$$f_{j,+}(z,0) = \frac{1 + R_{j,2}(z)}{T_j(z)}. \qquad \square$$

For our next considerations, we need that

$$f_{j,+}(z,0) \neq 0,$$

at least for all $z \in \mathbb{C}^+$ close to the real axis.

Therefore we make the following assumption:

$$1 + \int_0^{+\infty} x V_j(x) m_{j,+}(0,x)\,dx \neq 0, \qquad \forall j = 1, \dots, N, \tag{4.38}$$

that allows to obtain the next result.

4.2 Lemma. *If the assumption* (4.38) *holds, then there exists* $\kappa > 0$ *small enough and two positive constants* C_1, C_2 *such that*

$$C_1 \le |f_{j,+}(z,0)| \le C_2 \qquad \forall z \in B_\kappa, \tag{4.39}$$

where $B_\kappa = \{z_1 \in C^+ : 0 \le \Im z_1 \le \kappa\}$.

Proof. Recall that

$$f_{j,+}(z,0) = \frac{1 + R_{j,2}(z)}{T_j(z)}.$$

By (4.32) and (4.33) we see that (see property IV of Theorem 1 in [15], p. 147) there exist $R, C > 0$ such that

$$|T_j(z) - 1| + |R_{j,2}(z)| \le \frac{C}{|z|}, \qquad \forall |z| > R. \tag{4.40}$$

Hence (4.39) holds for all $|z| > R_0$, with R_0 large enough.

Now for $|z|$ small, we remark that $\frac{1+R_{j,2}(z)}{T_j(z)}$ is different from zero for all $z \in \mathbb{R}\backslash\{0\}$ by using the properties II and V of Theorem 1 in [15], p. 146. Furthermore using (4.32) and (4.33), one easily checks that

$$\lim_{z\to 0} \frac{1 + R_{j,2}(z)}{T_j(z)} = 1 + \int_0^{+\infty} tV_j(t)m_{j,+}(0,t)\,dt. \tag{4.41}$$

Consequently our assumption garantess that the continuous function $f_{j,+}(\cdot, 0)$ is different fom zero on the whole compact $[-R_0, R_0]$ and therefore (4.39) holds for all real numbers $z \in [-R_0, R_0]$. By the continuity of $f_{j,+}(\cdot, 0)$ on $B_{\delta'}$ for δ' small enough, we deduce that (4.39) holds for all $z \in B_\kappa \cap \{z_1 \in \mathbb{C} : \Re z_1 \in [-R_0, R_0]\}$, by choosing κ small enough. □

The assumption (4.38) is technical but it is satified by a large choice of potentials. Let us list some specific examples.

4.3 Lemma. 1. *In the generic case, namely if*

$$\int_0^{+\infty} V_j(x)m_{j,+}(0,x)\,dx \neq 0,$$

then we have

$$1 + \int_0^{+\infty} xV_j(x)m_{j,+}(0,x)\,dx \neq 0, \tag{4.42}$$

if V_j is non negative or if

$$\int_0^{+\infty} x|V_j(x)|\,dx \le \rho$$

where ρ is the unique positive number such that $\rho e^\rho = 1$.

2. *In the exceptional case, namely if*

$$\int_0^{+\infty} V_j(x)m_{j,+}(0,x)\,dx = 0,$$

then (4.42) *always holds.*

Proof. In the exceptional case, by Theorem 1 of [15], there exists a constant $C \in (0,1)$ such that

$$|R_{j,2}(r)| \le C, \qquad \forall r \in \mathbb{R}.$$

Hence

$$\lim_{\substack{r \to 0 \\ r \in \mathbb{R}}} \left| \frac{1 + R_{j,2}(r)}{T_j(r)} \right| \ge 1 - C,$$

which implies that (4.42) holds.

In the generic case and if V_j is non negative, then $m_{j,+}(0,\cdot)$ is a non negative function and therefore (4.42) directly holds.

In the generic case and if V_j has no sign, then the considerations of Lemma 1 of [15], p. 133 shows that

$$|m_{j,+}(0,0)| \ge 1 - \gamma_j e^{\gamma_j},$$

where $\gamma_j = \int_0^{+\infty} t|V_j(t)|\,dt$. Hence if $1 - \gamma_j e^{\gamma_j} > 0$, we deduce that $m_{j,+}(0,0)$ is different from zero. This yields the conclusion since

$$m_{j,+}(0,0) = f_{j,+}(0,0) = \lim_{z \to 0} f_{j,+}(z,0). \qquad \square$$

Note that $V_j = 0$ is an exceptional case.

We now prove that $R_{j,2}(z)$ is continuous and uniformly bounded in B_κ if $\kappa > 0$ small enough (suggested by Remark 10 of [15]).

4.4 Lemma. *For all $j = 1, \dots, N$, there exists a positive constant C_j such that*

$$|R_{j,2}(z)| \le C_j, \qquad \forall z \in B_\kappa, \tag{4.43}$$

for $\kappa > 0$ small enough.

Proof. By Theorem 1 of [15], there exists $C_1 > 0$ such that

$$|T_j(z)| \le C_1, \qquad \forall z \in B_\kappa,$$

for $\kappa > 0$ small enough. Hence by (4.33) we deduce that (4.43) holds for all $|z| > \epsilon$, for any $\epsilon > 0$.

For z in the ball $|z| \le \epsilon$, we distinguish the generic case from the exceptional one. In the generic case, by part V of Theorem 1 of [15], p. 150, we know that

$$T_j(z) = \alpha_j z + o(z), \qquad \text{for } z \to 0$$

with $\alpha_j \ne 0$ and again using (4.33) we deduce that (4.43) for $|z| \le \epsilon$.

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In the exceptional case, by (4.33) we may write

$$R_{j,2}(z) = \frac{T_j(z)}{2iz}\Big(\int_0^{+\infty} (e^{2izy} - 1) V_j(y) m_{j,+}(z, y)\, dy + \int_0^{+\infty} V_j(y)\big(m_{j,+}(z, y) - m_{j,+}(0, y)\big)\, dy\Big),$$

because $\int_0^{+\infty} V_j(t) m_{j,+}(0, t)\, dt = 0$. Therefore we obtain that

$$|R_{j,2}(z)| \le C_1\Big(\Big|\int_0^{+\infty} \frac{e^{2izy} - 1}{2iz} V_j(y) m_{j,+}(z, y)\, dy\Big| + \Big|\int_0^{+\infty} V_j(y) \frac{m_{j,+}(z, y) - m_{j,+}(0, y)}{2iz}\, dy\Big|\Big).$$

For the first term of this right hand side, due to (4.28) we can directly apply the dominated convergence theorem to conclude that

$$\int_0^{+\infty} \frac{e^{2izy} - 1}{2iz} V_j(y) m_{j,+}(z, y)\, dy \to \int_0^{+\infty} y V_j(y) m_{j,+}(0, y)\, dy \qquad \text{as } z \to 0.$$

Since this limit is finite, we deduce that

$$\Big|\int_0^{+\infty} \frac{e^{2izy} - 1}{2iz} V_j(y) m_{j,+}(z, y)\, dy\Big| \le C,$$

for $|z|$ small enough.

For the second term, we use the same argument. Namely since $\tilde{V}_j$ belongs to $L_2^1(\mathbb{R})$, by Remark 3 of [15], the derivative $\dot{m}_{k,+}$ of $m_{k,+}$ with respect to k exists and is continuous on $\mathbb{C}^+$. Moreover by Lemma 2.1 of [26], p. 46, there exists $C_2 > 0$ such that

$$|\dot{m}_{k,+}(z, y)| \le C_2, \qquad \forall x \ge 0. \tag{4.44}$$

Consequently by using the mean value theorem we have

$$\frac{m_{j,+}(z, y) - m_{j,+}(0, y)}{2iz} = \frac{\dot{m}_{k,+}(\theta z, y)}{2i},$$

for some $\theta \in (0, 1)$ and therefore

$$\Big|\frac{m_{j,+}(z, y) - m_{j,+}(0, y)}{2iz}\Big| \le \frac{C_2}{2}, \qquad \forall x \ge 0.$$

The application of dominated convergence theorem yields

$$\int_0^{+\infty} V_j(y)\frac{m_{j,+}(z,y)-m_{j,+}(0,y)}{2iz}\,dy \to \int_0^{+\infty} V_j(y)\dot{m}_{j,+}(0,y)\,dy \qquad \text{as } z\to 0.$$

The conclusion follows since this right-hand side is finite. □

We are now ready to give the different families of generalized eigenfunctions of H.

4.5 Lemma. *Under the assumption* (4.38), *then for all* $z\in B_\kappa$, $z\neq 0$ *and all* $j\in\{1,\dots,N\}$, *there exist two generalized eigenfunctions* $F_{z^2}^{\pm,j}:\mathcal{R}\to\mathbb{C}$ *of* H *defined by*

$$F_{z^2}^{\pm,j}(x) := F_{z^2,k}^{\pm,j}(x) \qquad \forall x\in\overline{R_k},$$

where $F_{z^2,k}^{\pm,j}$ *is in the form*

$$\begin{cases} F_{z^2,j}^{\pm,j}(x) = c_{j,\pm,1}(z)f_{j,\pm}(z,x)+c_{j,\pm,2}(z)f_{j,\mp}(z,x),\\ F_{z^2,k}^{\pm,j}(x) = d_{j,k,\pm}(z)f_{k,\mp}(z,x), \quad \forall k\neq j,\end{cases} \tag{4.45}$$

and $c_{j,\pm,1}(z)$, $c_{j,\pm,2}(z)$*and* $d_{j,k,\pm}(z)$ *are given by* (*modulo* N)

$$c_{j,\pm,1}(z) = \frac{f_{j+1,\mp}(z,0)}{W_{j,\pm}(z)}\Big(f'_{j,\mp}(z,0)+f_{j,\mp}(z,0)\sum_{k\neq j}\frac{f'_{k,\mp}(z,0)}{f_{k,\mp}(z,0)}\Big),$$

$$c_{j,\pm,2}(z) = -\frac{f_{j+1,\mp}(z,0)}{W_{j,\pm}(z)}\Big(f'_{j,\pm}(z,0)+f_{j,\pm}(z,0)\sum_{k\neq j}\frac{f'_{k,\mp}(z,0)}{f_{k,\mp}(z,0)}\Big),$$

$$d_{j,k,\pm}(z) = \frac{f_{j+1,\mp}(z,0)}{f_{k,\mp}(z,0)}, \qquad \forall k\neq j,$$

$W_{j,\pm}(z)$ *is the Wronskian relatively to* $f_{j,\pm}$, *namely*

$$W_{j,\pm}(z) = f_{j,\pm}(z,x)f'_{j,\mp}(z,x)-f_{j,\mp}(z,x)f'_{j,\pm}(z,x),$$

that is constant in x and different from 0 (*since* $z\neq 0$).

Proof. We look for generalized eigenfunctions in the form (4.45), the constants $c_{j,\pm,1}(z)$, $c_{j,\pm,2}(z)$ and $d_{j,k,\pm}(z)$ will be fixed below in order to guarantee the continuity of $F_{z^2}^{\pm,j}$ at 0 and the Kirchoff law. This will show that $F_{z^2}^{\pm,j}$ are generalized eigenfunctions of H since $F_{z^2,k}^{\pm,j}$ satisfies

$$-\frac{d^2}{dx^2}F_{z^2,k}^{\pm,j}(x)+\tilde{V}_j(x)F_{z^2,k}^{\pm,j}(z,x) = z^2F_{z^2,k}^{\pm,j} \qquad \text{on } R_k.$$

Since each branch j plays the same rule, we can take $j = 1$ and write $c_{1,\pm,1}(z) = c_1$, $c_{1,\pm,2}(z) = c_2$ and $d_{1,k,\pm}(z) = d_k$. The continuity at 0 is equivalent to

$$c_1 f_{1,\pm}(z,0) + c_2 f_{1,\mp}(z,0) = d_k f_{k,\mp}(z,0) \qquad \forall k \neq 1,$$

while the Kirchoff law is equivalent to

$$c_1 f'_{1,\pm}(z,0) + c_2 f'_{1,\mp}(z,0) + \sum_{k=2}^{N} d_k f'_{k,\mp}(z,0) = 0.$$

Since by Lemma 4.1 $f_{k,\mp}(z,0)$ is different from 0, we will get

$$d_k = \frac{d_2 f_{2,\mp}(z,0)}{f_{k,\mp}(z,0)}, \qquad \forall k \neq 1,$$

and the continuity and the Kirchoff law reduce to

$$\begin{cases} c_1 f_{1,\pm}(z,0) + c_2 f_{1,\mp}(z,0) = d_2 f_{2,\mp}(z,0), \\ c_1 f'_{1,\pm}(z,0) + c_2 f'_{1,\mp}(z,0) = -d_2 f_{2,\mp}(z,0) \sum_{k=2}^{N} \frac{f'_{k,\mp}(z,0)}{f_{k,\mp}(z,0)}. \end{cases}$$

This 2×2 linear system in c_1 and c_2 has a unique solution since its determinant is exactly $W_{1,\pm}(z)$. The resolution of this system leads to the conclusion with the choice $d_2 = 1$. □

4.6 Remark. The choice (4.45) was guided by the simple case when $N = 2$ and $V_k = 0$, $k = 1, 2$. In that case, we recover the standard generalized eigenfunctions, namely

$$F^{\pm,1}_{z^2,1}(x) = e^{\pm izx}, \qquad \forall x > 0,$$

as well as

$$F^{\pm,1}_{z^2,2}(x) = e^{\mp izx}, \qquad \forall x > 0.$$

According to Lemma 4.1, we see that

$$c_{j,+,1}(z) = -\frac{izN}{W_{j,+}(z)},$$

which is always different from 0 if $z \in \mathbb{C}^+$, $z \neq 0$, while

$$c_{j,-,1}(z) = \frac{izf_{j+1,+}(z,0)}{W_{j,-}(z)} \sum_{k=1}^{N} \frac{1 - R_{k,2}(z)}{1 + R_{k,2}(z)},$$

is not clearly different from zero. This is investigated in the next Lemma

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4.7 Lemma. *Under the assumption* (4.38), *there exists* $\kappa > 0$ *small enough such that*

$$s(z) := \sum_{k=1}^{N} \frac{1 - R_{k,2}(z)}{1 + R_{k,2}(z)},$$

satisfies

$$|s(z)| \geq C, \qquad \forall z \in B_\kappa, \tag{4.46}$$

for some $C > 0$.

Proof. Clearly s is continuous on $B_\kappa \backslash \{0\}$ for κ small enough, hence we first analyze the behaviour of s near $z = 0$.

For $z \in B_\kappa \backslash \{0\}$ and $k \in \{1, \ldots, N\}$, we write

$$s_k(z) := \frac{1 - R_{k,2}(z)}{1 + R_{k,2}(z)} = \frac{1 - R_{k,2}(z)}{T_k(z)} \frac{T_k(z)}{1 + R_{k,2}(z)}.$$

The absolute value of the second factor is uniformly bounded from below on B_κ thanks to Lemmas 4.1 and 4.2.

For the first factor, we distinguish between the generic and the exceptional case: In the exceptional case,

$$|T_k(z)| \geq c_k, \qquad \forall z \in B_\kappa,$$

for some $c_k > 0$ (and κ small enough) and therefore s_k is continuous on B_κ.

In the generic case, using (4.32) and (4.33), we may write

$$\frac{1 - R_{k,2}(z)}{T_k(z)} = 1 - \int_0^{+\infty} \frac{1 + e^{2izy}}{2iz} \tilde{V}_k(y) m_{k,+}(z, y)\, dy \qquad \forall z \in \mathbb{C}^+,\ z \neq 0.$$

As underlined before, the derivative $\dot{m}_{k,+}$ of $m_{k,+}$ with respect to k exists, is continuous on $\mathbb{C}^+$and satisfies (4.44). Accordingly, using the mean value theorem and the dominated convergence theorem, we get for all $z \neq 0$ small enough

$$\frac{1 - R_{k,2}(z)}{T_k(z)} = 1 - \frac{v_k}{iz} + r_k(z),$$

where r_k is a continuous function at $z = 0$ and $v_k = \int_0^{+\infty} V_k(t) m_{k,+}(0, t)\, dt$ (that is different from zero because we are in the generic case).

In the same manner we can refine (4.41) and prove that

$$\frac{1 + R_{k,2}(z)}{T_k(z)} = \gamma_k + z r_k^{(1)}(z),$$

where $r^{(1)}$ is a continuous function at $z = 0$ and $\gamma_k = 1 + \int_0^{+\infty} t V_k(t) m_{k,+}(0, t)\, dt$ that is a real number different from 0 by our hypothesis (4.38). Consequently for z small enough we will get

$$\frac{T_k(z)}{1 + R_{k,2}(z)} = \gamma_k^{-1} + z r_k^{(2)}(z), \tag{4.47}$$

where $r^{(2)}$ is a continuous function at $z = 0$.

The two previous expansions show that for all $z \neq 0$ small enough

$$s_k(z) = -\frac{v_k}{i \gamma_k z} + r_k^{(3)}(z),$$

where $r^{(3)}$ is a continuous function at $z = 0$.

In summary, we have obtained that for all $z \neq 0$ small enough

$$s(z) = -\frac{1}{iz} \sum_{k \text{ generic}} \frac{v_k}{\gamma_k} + r(z),$$

where r is a continuous function at $z = 0$.

Now we can distinguish two cases:

i) If $\sum_{k \text{ generic}} \frac{v_k}{\gamma_k} = 0$, then s is continuous at $z = 0$, and therefore s is continuous on B_κ.

ii) If $K := \sum_{k \text{ generic}} \frac{v_k}{\gamma_k} \neq 0$, then s blows up at $z = 0$ and therefore there exists δ_0 small enough such that

$$|s(z)| \geq \frac{K}{2|z|}, \qquad \forall |z| < \delta_0. \tag{4.48}$$

Now for $|z|$ large, by (4.40) we have

$$\lim_{|z| \to +\infty} s_k(z) = 1,$$

hence there exists R_0 large enough such that

$$\Re s(z) \geq \frac{N}{2}, \qquad \forall z \in B_\kappa : |z| > R_0. \tag{4.49}$$

For small value of $|z|$, we first restrict ourselves on the real line. First we notice that

$$\Re s_k(z) = \Re \frac{1 - R_{k,2}(z)}{1 + R_{k,2}(z)} = \frac{1 - |R_{k,2}(z)|^2}{|1 + R_{k,2}(z)|^2}.$$

But according to parts II and V of Theorem 1 of [15],

$$|R_{k,2}(z)| < 1, \qquad \forall z \in \mathbb{R},\, z \neq 0,$$

and therefore

$$\Re s_k(z) > 0, \qquad \forall z \in \mathbb{R},\, z \neq 0.$$

Now thanks to (4.41) and to the relation

$$1 - |R_{k,2}(z)|^2 = |T_k(z)|^2,$$

valid for all real numbers z, we deduce that

$$\lim_{\substack{z \to 0 \\ z \in \mathbb{R}}} \frac{1 - |R_{k,2}(z)|^2}{|1 + R_{k,2}(z)|^2} = \frac{1}{\gamma_k^2},$$

where $\gamma_k = 1 + \int_0^{+\infty} tV_j(t)m_{j,+}(0,t)\,dt$ that by hypothesis is a real number different from 0.

This shows that

$$\lim_{\substack{z \to 0 \\ z \in \mathbb{R}}} \Re s(z) = \sum_{k=1}^{N} \frac{1}{\gamma_k^2},$$

and consequently as $\Re s$ is a continuous function on $\mathbb{R}$ that is different from zero for all real numbers, due to (4.49), it satisfies

$$\Re s(z) \geq C, \qquad \forall z \in \mathbb{R}, \tag{4.50}$$

for some $C > 0$.

In the first case mentioned before, namely if $K = 0$, then by the uniform continuity of $\Re s$ on the compact set $B_\kappa \cap \{z_1 \in \mathbb{C} : 0 \leq z_1 \leq R_0\}$, where R_0 is the parameter introduced above, we deduce that

$$\Re s(z) \geq C/2, \qquad \forall z \in B_{\kappa'} \cap \{z_1 \in \mathbb{C} : |z_1| \leq R_0\}, \tag{4.51}$$

if κ' is chosen small enough. In that case the conclusion directly follows from (4.49) and (4.51).

In the case when $K \neq 0$, we use the uniform continuity of $\Re s$ on the compact set $B_\kappa \cap \left\{z_1 \in \mathbb{C} : \frac{\delta_0}{2} \leq |z_1| \leq R_0\right\}$ (where R_0, δ_0 are the parameter introduced above), and (4.50) to conclude that

$$\Re s(z) \geq C/2, \qquad \forall z \in B_{\kappa'} \cap \left\{z_1 \in \mathbb{C} : \frac{\delta_0}{2} \leq |z_1| \leq R_0\right\}, \tag{4.52}$$

if κ' is chosen small enough.

In this second case the conclusion follows from (4.48), (4.49) and (4.51). □

4.8 Corollary. *Under the assumption* (4.38), *for* $\kappa > 0$ *small enough there exist two positive constants* c_1, c_2 *such that*

$$|c_{j,-,1}(z) W_{j,-}(z)| \geq c_1 |z|, \qquad \forall z \in B_\kappa, \tag{4.53}$$

$$|c_{j,-,2}(z)| \leq c_2 |s(z)|, \qquad \forall z \in B_\kappa. \tag{4.54}$$

Proof. As

$$c_{j,-,1}(z) = \frac{iz f_{j+1,+}(z,0)}{W_{j,-}(z)} s(z),$$

by the previous Lemma and Lemma 4.2, we deduce that (4.53) holds.

By its definition and Lemma 4.1, we may write

$$c_{j,-,2}(z) = iz \frac{f_{j+1,+}(z,0)}{W_{j,-}(z)} \left(1 + \sum_{k \neq j} \frac{1 - R_k(z)}{1 - R_k(z)}\right),$$

hence thanks to the definition of $s(z)$, we obtain

$$c_{j,-,2}(z) = iz \frac{f_{j+1,+}(z,0)}{W_{j,-}(z)} \left(\frac{2R_{j,2}(z)}{1 + R_{j,2}(z)} + s(z)\right).$$

Now recalling that

$$W_{j,-}(z) = -W_{j,+}(z) = -\frac{2iz}{T_j(z)},$$

we can write

$$c_{j,-,2}(z) = -\frac{f_{j+1,+}(z,0)}{2} \left(\frac{2R_{j,2}(z) T_j(z)}{1 + R_{j,2}(z)} + s(z) T_j(z)\right). \tag{4.55}$$

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By Lemmas 4.1, 4.2, 4.4 and 4.7 we deduce that there exists $C_1 > 0$ such that

$$|c_{j,-,2}(z)| \leq C_1\big(1 + |s(z)|\big) \leq \left(\frac{C_1}{C} + C_1\right)|s(z)|,$$

with the constant C from (4.46). □

4.9 Corollary. *Under the assumption* (4.38), *and if* $V_k \in L^1_\gamma(0,\infty)$ *with* $\gamma > 5/2$, *for all* $k = 1, \dots, N$, *then for all* $R > 0$, s^{-1} *belongs to* $H^1(-R, R)$.

Proof. With the notation from the previous Lemma, we see that r_k is given by

$$r_k(z) = \int_0^{+\infty} \frac{V_k(y)}{2iz}\big(2m_{k,+}(0,y) - (1 + e^{2izy})m_{k,+}(z,y)\big)\,dy,$$

and is continuous on $\mathbb{R}$. Moreover for $z \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$ we easily see that r_k is differentiable at z and that

$$\dot{r}_k(z) = -\int_0^{+\infty} \frac{V_k(y)}{2iz^2}\big(2m_{k,+}(0,y) - g_{k,+}(z,y) + z\dot{g}_{k,+}(z,y)\big)\,dy.$$

where for shortness we have set

$$g_{k,+}(z,y) := (1 + e^{2izy})m_{k,+}(z,y).$$

But the mean value theorem implies that

$$g_{k,+}(z,y) = 2m_{k,+}(0,y) + z\dot{g}_{k,+}(\theta z, y),$$

for some $\theta \in (0,1)$ and therefore

$$\dot{r}_k(z) = \int_0^{+\infty} \frac{V_k(y)}{2iz}\big(\dot{g}_{k,+}(\theta z, y) - \dot{g}_{k,+}(z,y)\big)\,dy, \qquad \forall z \in \mathbb{R}^*.$$

As

$$\dot{g}_{k,+}(z,y) = 2iye^{2izy}m_{k,+}(z,y) + (1 + e^{2izy})\dot{m}_{k,+}(z,y),$$

the previous identity can be equivalently written

$$\begin{aligned}\dot{r}_k(z) = \int_0^{+\infty} V_k(y)\bigg(&ym_{k,+}(\theta z, y)\frac{e^{2i\theta zy} - e^{2izy}}{z}\\ &+ ye^{2izy}\frac{m_{k,+}(\theta z, y) - m_{k,+}(z,y)}{z} + \frac{e^{2i\theta zy} - e^{2izy}}{2iz}\dot{m}_{k,+}(z,y)\\ &+ (1 + e^{2i\theta zy})\frac{\dot{m}_{k,+}(\theta z, y) - \dot{m}_{k,+}(z,y)}{2iz}\bigg)\,dy, \qquad \forall z \in \mathbb{R}^*.\end{aligned}$$

Again by the mean value theorem we get

$$\begin{aligned}\dot{r}_k(z) = \int_0^{+\infty} V_k(y)\Big(&2iy^2 m_{k,+}(\theta z, y)e^{2i\theta' zy}(\theta - 1)\\ &+ ye^{2izy}\dot{m}_{k,+}(\theta'' z, y)(\theta - 1)\\ &+ ye^{2i\theta' zy}(\theta - 1)\dot{m}_{k,+}(z, y)\\ &+ (1 + e^{2i\theta zy})\frac{\dot{m}_{k,+}(\theta z, y) - \dot{m}_{k,+}(z, y)}{2iz}\Big)\, dy, \qquad \forall z \in \mathbb{R}^*,\end{aligned}$$

for some $\theta', \theta'' \in (\theta, 1)$. Note that we cannot apply the mean value theorem to the last term since $\dot{m}_{k,+}$ is not differentiable. But according to Lemma 2.2 of [26] we have

$$|\dot{m}_{k,+}(z, y) - \dot{m}_{k,+}(0, y)| \le C|z|^{\gamma-2}, \qquad \forall y \ge 0, \tag{4.56}$$

for some $C > 0$ independent of z and y. This estimate, (4.28) and (4.44) lead to

$$|\dot{r}_k(z)| \le C \int_0^{+\infty} |V_k(y)|(y^2 + y + |z|^{\gamma-2})\, dy, \qquad \forall z \in \mathbb{R}^*.$$

for some $C > 0$. Hence according to our hypothesis on V_k, we get

$$|\dot{r}_k(z)| \le C_1(1 + |z|^{\gamma-3}), \qquad \forall z \in \mathbb{R}^*,$$

for some $C_1 > 0$.

This estimate and the continuity of r_k imply that r_k belong to $H^1(-R, R)$ for any $R > 0$ due to the hypothesis $\gamma > 5/2$.

In the same way we need to precise the splitting (4.47) on the real line (actually near 0). For that purpose, we consider

$$g_k(z) := \frac{m_{k,+}(z, 0) - m_{k,+}(0, 0)}{z}, \qquad \forall z \in \mathbb{R},$$

and show that g_k belongs to $H^1(-R, R)$ for any $R > 0$. First g_k is continuous at 0 because $m_{k,+}(z, 0)$ is in $C^1(\mathbb{R})$. Second by Leibniz's rule we have

$$\dot{g}_k(z) = \frac{\dot{m}_{k,+}(z, 0)z - \big(m_{k,+}(z, 0) - m_{k,+}(0, 0)\big)}{z^2}$$

and therefore by the mean value theorem we get

$$\dot{g}_k(z) = \frac{\dot{m}_{k,+}(z,0) - \dot{m}_{k,+}(\theta z,0)}{z},$$

for some $\theta \in (0,1)$ and we conclude by (4.56).

But we see that

$$\frac{\left(m_{k,+}(z,0)\right)^{-1}}{z} = \frac{\left(m_{k,+}(0,0)\right)^{-1}}{z} + h_k(z) = \frac{1}{\gamma_k z} + h_k(z)$$

with

$$h_k(z) = \frac{m_{k,+}(z,0) - m_{k,+}(0,0)}{z m_{k,+}(z,0) m_{k,+}(0,0)} = \frac{g_k(z)}{m_{k,+}(z,0) m_{k,+}(0,0)}.$$

According to the previous considerations, g_k belongs to $H^1(-R,R)$, for any $R > 0$ and since $m_{k,+}(\cdot,0)$ belongs to $C^1(\mathbb{R})$ and is uniformly bounded from below (due to Lemmas 4.1 and 4.2), $\frac{1}{m_{k,+}(\cdot,0)}$ is also in $C^1(\mathbb{R})$. Therefore h_k also belongs to $H^1(-R,R)$, for any $R > 0$.

Coming back to s, recalling that

$$s(z) = \sum_{k=1}^{N} \left(1 + \frac{i v_k}{z} + r_k(z)\right) \left(m_{k,+}(z,0)\right)^{-1},$$

we have finally shown that

$$s(z) = i\frac{K}{z} + r_s(z),$$

where r_s belongs to $H^1(-R,R)$, for any $R > 0$.

Now we distinguish the case $K = 0$ to the other one: In the first case, we have that $s = r_s$ belongs to $H^1(-R,R)$, for any $R > 0$ and since s is uniformly bounded from below by the previous Lemma, we deduce that $\frac{1}{s}$ belongs to $H^1(-R,R)$, for any $R > 0$.

If $K \neq 0$, then

$$\frac{1}{s(z)} = \frac{z}{iK + z r_s(z)},$$

that is a continuous function in $\mathbb{R}$ and moreover for $z \in \mathbb{R}^*$, we have after elementary calculations

$$\frac{d}{dz}\frac{1}{s}(z) = \frac{iK - z^2 \dot{r}_s(z)}{\left(iK + z r_s(z)\right)^2}.$$

Since this right-hand side is in $L^2(-R,R)$, for any $R>0$ (because the denominator is different from zero near $z=0$, while by the previous Lemma, for any $z\in\mathbb{R}^*$ $s(z)\geq C$ is equivalent to $|iK+zr_s(z)|\geq C|z|$), we still conclude that $\frac{1}{s}$ belongs to $H^1(-R,R)$, for any $R>0$. □

4.10 Corollary. *Under the assumption* (4.38), *and if* $V_k\in L^1_\gamma(0,+\infty)$ *with* $\gamma>5/2$, *then the function*

$$\mathbb{R}\to C: z\to \frac{c_{j,-,2}(z)}{f_{j+1,+}(z,0)s(z)},$$

belongs to $H^1(-R,R)$ *for all* $R>0$.

Proof. By (4.55), we see that

$$\frac{c_{j,-,2}(z)}{f_{j+1,+}(z,0)s(z)}=-\frac{1}{2}\left(\frac{2R_{j,2}(z)T_j(z)}{(1+R_{j,2}(z))s(z)}+T_j(z)\right)=-\frac{1}{2}\left(\frac{2R_{j,2}(z)}{f_{j,+}(z,0)s(z)}+T_j(z)\right).$$

But according to Remark 10 of [15], T_j is analytic in a neighbourhood of the real line, hence it is at least in $C^1(\mathbb{R})$. On the other hand $f_{j,+}(z,0)=m_{j,+}(z,0)$ is $C^1(\mathbb{R})$ due to Remark 3 of [15], hence $\frac{1}{f_{j,+}(z,0)}$ has the same property due to Lemma 4.2. Finally the identity (4.37) of Lemma 4.1 yields

$$R_{j,2}(z)=f_{j,+}(z,0)T_j(z)-1,$$

hence it also belongs to $C^1(\mathbb{R})$.

The conclusion follows from the previous Corollary and these regularity properties (the product of a C^1 function with a H^1 function is still in H^1). □

4.11 Definition (Kernel of the resolvent). Let the assumption (4.38) be satisfied, then for all $z\in B_\kappa$, $z\neq 0$, all $j\in\{1,\dots,N\}$, and all $x\in R_j$, we define (modulo N)

$$K(x,x',z^2)=\begin{cases}\frac{1}{W_j(z)}F^{-,j}_{z^2,j}(x)F^{-,j+1}_{z^2,j}(x'), & \text{for } x'\in R_j,\ x'>x,\\ \frac{1}{W_j(z)}F^{-,j+1}_{z^2,j}(x)F^{-,j}_{z^2,j}(x'), & \text{for } x'\in R_j,\ x'<x,\\ \frac{1}{W_j(z)}F^{-,j+1}_{z^2,j}(x)F^{-,j}_{z^2,k}(x'), & \text{for } x'\in R_k,\ k\neq j,\end{cases}$$

where $W_j(z)=c_{j,-,1}(z)d_{j+1,j,-}(z)W_{j,-}(z)$.

4.12 Theorem. *Let the assumption* (4.38) *be satisfied and let* $f\in\mathscr{H}$. *Then, for* $x\in\mathscr{R}$ *and* $z\in B_\kappa$ *such that* $\Im z>0$, *we have*

$$[R(z^2,H)f](x)=\int_{\mathscr{R}}K(x,x',z^2)f(x')\,dx'. \tag{4.57}$$

Proof. Fix $j \in \{1, \dots, N\}$, and z as in the statement. Then we notice that the Wronskian $W_j(z)$ between $F^{-,j}_{z^2,j}$ and $F^{-,j+1}_{z^2,j}$ is different from zero, namely by Lemma 4.5 we have

$$\begin{aligned} W_j(z) &= [F^{-,j}_{z^2,j}, F^{-,j+1}_{z^2,j}](x) \\ &= F^{-,j}_{z^2,j}(x)(F^{-,j+1}_{z^2,j})'(x) - (F^{-,j}_{z^2,j})'(x)F^{-,j+1}_{z^2,j}(x) \\ &= \big(c_{j,-,1}(z)f'_{j,-}(z,x) + c_{j,-,2}(z)f'_{j,+}(z,x)\big)d_{j+1,j,-}(z)f_{j,+}(z,x) \\ &\quad - \big(c_{j,-,1}(z)f_{j,-}(z,x) + c_{j,-,2}(z)f_{j,+}(z,x)\big)d_{j+1,j,-}(z)f'_{j,+}(z,x) \\ &= c_{j,-,1}(z)d_{j+1,j,-}(z)W_{j,-}(z). \end{aligned}$$

Hence by Lemma 4.2 and Corollary 4.8 this Wronskian is different from zero.

Consequently the same arguments than in Proposition 3.2 of [5] show that (4.57) holds. The main ingredient is that we can apply the dominated convergence theorem because the generalized eigenfunction $F^{-,j}_{z^2,k}$ is in $L^2(R_k)$ if $j \neq k$. □

4.13 Remark. The choice of the kernel comes from this Theorem because $F^{+,j}_{z^2,k}$ is not in $L^2(R_k)$ if $j \neq k$.

Here and below the complex square root is chosen in such a way that $\sqrt{r \cdot e^{i\phi}} = \sqrt{r}e^{i\phi/2}$ with $r > 0$ and $\phi \in [-\pi, \pi)$. Accordingly for any positive real number λ and any $\varepsilon > 0$, we will define

$$z_\varepsilon = \sqrt{\lambda + i\varepsilon}$$

that will be in $\mathbb{C}^+$.

4.14 Theorem (Limiting absorption principle). *Let the assumption* (4.38) *be satisfied. Let $\delta > 0$ be fixed. Then for all real numbers $\lambda > 0$, $0 < \varepsilon < \delta$ and $(x, x') \in \mathcal{R}^2$ we have*

1. $\lim_{\substack{\alpha \to 0 \\ \alpha > 0}} K(x, x', z_\alpha^2) = K(x, x', \lambda)$,
2. $|K(x, x', z_\varepsilon^2)| \leq \frac{C}{\sqrt{\lambda}} e^{\gamma(x+x')}$, *where* $0 < \gamma < \max\{1, \delta\}$.

Proof. The first part of the Theorem is direct since $\lambda + i\alpha$ tends to λ as $\alpha > 0$ tends to 0 and consequently

$$\sqrt{\lambda + i\alpha} \to \sqrt{\lambda},$$

as $\alpha > 0$ tends to 0. We further use the fact that the functions $f_{j,\pm}(\cdot, x)$ and $f'_{j,\pm}(\cdot, x)$ are continuous in $\mathbb{C}^+$ for any fixed $x \in \mathbb{R}$.

For the second part of the Theorem, we first use the estimates (4.30) and (4.31), this last one implying

$$|f_{j,-}(z_\varepsilon, x)| \le C(1+x)e^{\Im z_\varepsilon x} \le C(1+x)e^{\max\{1,\delta\}x}, \qquad \forall x \in [0, +\infty), \quad (4.58)$$

where we have used the property

$$\Im z_\varepsilon = |\Im\sqrt{\lambda + i\varepsilon}| \le \max\{1, \Im(\lambda + i\varepsilon)\} = \max\{1, \varepsilon\}.$$

Notice that by the definition $W_j(z) = c_{j,-,1}(z)d_{j+1,j,-}(z)W_{j,-}(z)$ and by Lemma 4.2 and Corollary 4.8, we get

$$|W_j(z)| \ge C|z|, \quad (4.59)$$

for some $C > 0$.

Now we distinguish between the following three cases:

1. If $x, x' \in R_j$ with $x' > x$, then

$$\begin{aligned}
&K(x, x', z_\varepsilon^2) \\
&\quad = \frac{1}{W_j(z_\varepsilon)} F^{-,j}_{z_\varepsilon^2,j}(x) F^{-,j+1}_{z_\varepsilon^2,j}(x') \\
&\quad = \frac{1}{W_j(z_\varepsilon)} \big(c_{j,-,1}(z_\varepsilon)f_{j,-}(z_\varepsilon,x) + c_{j,-,2}(z_\varepsilon)f_{j,+}(z_\varepsilon,x)\big)d_{j+1,j,-}(z_\varepsilon)f_{j,+}(z_\varepsilon,x') \\
&\quad = \frac{1}{W_{j,-}(z_\varepsilon)} f_{j,-}(z_\varepsilon,x)f_{j,+}(z_\varepsilon,x') + \frac{c_{j,-,2}(z_\varepsilon)}{iz_\varepsilon f_{j+1,+}(z_\varepsilon,0)s(z_\varepsilon)} f_{j,+}(z_\varepsilon,x)f_{j,+}(z_\varepsilon,x').
\end{aligned}$$

As there exists $c > 0$ such that

$$|W_{j,-}(z)| \ge c|z|, \qquad \forall z \in \mathbb{C}^+,$$

by Lemma 4.2 and Corollary 4.8, we obtain

$$|K(x, x', z_\varepsilon)| \le \frac{C}{|z_\varepsilon|}\big(|f_{j,-}(z_\varepsilon,x)| + |f_{j,+}(z_\varepsilon,x)|\big)|f_{j,+}(z_\varepsilon,x')|.$$

The estimates (4.30) and (4.58) then yields

$$|K(x, x', z_\varepsilon^2)| \le \frac{C}{|z_\varepsilon|}\big(1 + (1+x)e^{\max\{1,\delta\}x}\big). \quad (4.60)$$

2. If $x, x' \in R_j$ with $x' > x$, then

$$K(x, x', z_\varepsilon^2) = \frac{1}{W_j(z_\varepsilon)} F^{-,j+1}_{z_\varepsilon^2,j}(x) F^{-,j}_{z_\varepsilon^2,j}(x'),$$

and the above arguments (by simply exchanging the role of x and x') yields

$$|K(x,x',z_\varepsilon)| \leq \frac{C}{|z_\varepsilon|}\big(1+(1+x')e^{\max\{1,\delta\}x'}\big). \tag{4.61}$$

3. If $x \in R_j$ and $x' \in R_k$ with $k \neq j$, we have

$$\begin{aligned} K(x,x',z_\varepsilon^2) &= \frac{1}{W_j(z_\varepsilon)} F^{-,j+1}_{z_\varepsilon^2,j}(x) F^{-,j}_{z_\varepsilon^2,k}(x') \\ &= \frac{1}{W_j(z_\varepsilon)} d_{j+1,j,-}(z_\varepsilon) f_{j,+}(z_\varepsilon,x) d_{j+1,k,-}(z_\varepsilon) f_{k,+}(z_\varepsilon,x). \end{aligned}$$

Hence by Lemma 4.2 and the estimates (4.30) and (4.59), we obtain

$$|K(x,x',z_\varepsilon^2)| \leq \frac{C}{|z_\varepsilon|}. \tag{4.62}$$

The estimates (4.60), (4.61) and (4.62) imply the conclusion since $|z_\varepsilon| > \sqrt{\lambda}$. □

4.15 Theorem. *Take $f \in \mathscr{H}$ with a compact support and let $0 \leq a < b < +\infty$. Then for any continuous scalar function h defined on the real line and for all $x \in R_j$, we have*

$$\big(h(H)E(a,b)f\big)(x) = -\frac{1}{\pi}\int_{(a,b)} h(\lambda) \sum_{k=1}^{N} \int_{R_k} f(x') \Im K(x,x',\lambda)\, dx'\, d\lambda,$$

where E is the resolution of the identity of H.

Proof. The proof is similar to the one of Lemma 3.13 of [3] (see also Proposition 4.5 of [6]) and is therefore omitted. The main ingredients are the use of Stone's formula, Theorem 4.12 and the limiting absorption principle Theorem 4.14 (that allows to apply the dominated convergence theorem). □

4.16 Remark. Theorem 4.15 directly implies that

$$\sigma\big(HE[0,+\infty)\big) = \sigma_{ac}\big(HE[0,+\infty)\big) = [0,+\infty) \quad \text{and} \quad \sigma_{pp}(H) \subset (-\infty,0),$$

where σ_{ac} is the absolutely continuous spectrum and σ_{pp} the pure point spectrum. The additional informations that

$$\sigma\big(HE(-\infty,0)\big) = \sigma_{pp}(H)$$

and that this set is finite follow from Chapter 2 of [12].

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5. Proof of Theorems 1.1 and 1.5

The proof of the L^∞-time decay will be carried out by manipulating the solution formula in a way to reduce the problem to the well known case of the free Schrödinger equation on the line [23], p. 60.

We shall decompose an general initial conditions into a part with a spectral representation with compact support and a part with a sufficiently high lower cutoff energy (frequency). The technique will be different in the two cases.

5.1. High energy limit. For high energy (frequency) initial conditions, we can use an expansion (called Born series) of the resolvent of the Hamiltonian with potential in terms of the free resolvent (Proposition 5.1). To this end we use a formula for the free resolvent established in [5]. This leads to a corresponding expansion of the Schrödinger group via Stone's formula. Then we adapt a technique of [16] to extract the expression corresponding to the Schrödinger group on the line to the formulas of the transmission problem, see Theorem 5.11, part 1. While doing this, we improve the calculations of [16] in the sense that we find an explicit expression for the coefficient of the time decay in terms of the cutoff frequency and the potential. This explicit knowledge is essential to deduce from this the perturbation Theorem 5.11, part 3, using the fact that the free Schrödinger group is the first term of the expansion. The results of this section are of independent interest and Theorem 5.11, part 3 seems to be new even on the line.

5.1 Proposition. *Let* $R_0(\lambda + i\varepsilon) = \left(-\frac{d^2}{dx^2} - (\lambda + i\varepsilon)\right)^{-1}$ *and* $R_V(\lambda + i\varepsilon) = \big(H - (\lambda + i\varepsilon)\big)^{-1}$. *Then we have*

1. *the representation*

$$\lim_{\varepsilon\to 0,\, \varepsilon>0} [R_0(\lambda + i\varepsilon)f](x) = [R_0(\lambda + i0)f](x) = \int_{\mathscr{R}} K_0(x, x', \lambda + i0) f(x')\, dx'$$

for almost all $x \in \mathbb{R}$ and $f \in L^2(\mathscr{R})$ with

$$K_0(x, x', \lambda \pm i0) = \frac{\mp i}{N\sqrt{\lambda}} \begin{cases} \big(1 - \frac{N}{2}\big) e^{\pm i(x+x')\sqrt{\lambda}} + \frac{N}{2} e^{\pm i|x-x'|\sqrt{\lambda}}, & x' \in \overline{R_j}, \\ \big(1 - \frac{N}{2}\big) e^{\pm i(x+x')\sqrt{\lambda}} + \frac{N}{2} e^{\pm i(x-x')\sqrt{\lambda}}, & x' \in \overline{R_k},\ k \neq j, \end{cases} \tag{5.63}$$

2. *the estimate*

$$|K_0(x, x', \lambda \pm 0)| \leq \frac{N-1}{N\sqrt{\lambda}}, \qquad \forall (x, x') \in \mathscr{R}^2, \tag{5.64}$$

3. *the following expansion: suppose* $N \geq 2$, *let* $0 < q_* < 1$ *and* $\lambda > \lambda_* = \frac{4(N-1)^2\|V\|_1^2}{N^2 q_*^2}$. *Then*

$$\langle R_V(\lambda \pm i0)f, g\rangle = \sum_{k\geq 0}\big\langle R_0(\lambda \pm i0)\big(-VR_0(\lambda \pm i0)\big)^k f, g\big\rangle$$

for any $V, f, g \in L^1(\mathscr{R})$. *The* $+(-)$ *sign is valid, if* $\Im\lambda > 0$ (*respectively* $\Im\lambda < 0$).

Proof.
1.:
Direct consequence of [5].
2.:
Follows from 1.
3.:
From 2. and the assumption on V it follows

$$\|VR_0(\lambda \pm i0)f\|_1 \leq \frac{N-1}{N\sqrt{\lambda}}\|V\|_1\|f\|_1.$$

Due to (4.25) we see, that the Jost functions are bounded for fixed λ. Therefore one has

$$R_V(\lambda - i0)g \in L^\infty(\mathscr{R}) \qquad \text{for } \lambda > 0.$$

Hence

$$\begin{aligned}\big|\big\langle R_V(\lambda + i0)\big(VR_0(\lambda + i0)\big)^k f, g\big\rangle\big| &\leq \big\|\big(VR_0(\lambda + i0)\big)^k f\big\|_1 \|R_V(\lambda - i0)g\|_\infty \\ &\leq \left(\frac{N-1}{N\sqrt{\lambda}}\right)^k \|V\|_1^k \|f\|_1 \|R_V(\lambda - i0)g\|_\infty \\ &= q(\lambda)^k \|f\|_1 \|R_V(\lambda - i0)g\|_\infty\end{aligned}$$

with $q(\lambda) := \frac{N-1}{N\sqrt{\lambda}}$. Our assumption $\lambda_* < \lambda$ implies

$$q(\lambda) < \frac{4(N-1)}{N\sqrt{\lambda_*}}\|V\|_1 = q_* < 1.$$

Therefore the series from the statement of 3. converges. The equality comes from simple calculations. □

Note that the factor 4 in the definition of λ_* is not necessary in this Proposition, but will be necessary later on.

Now we shall estimate the L^1-norm of the Fourier transform of a frequency band cutoff function times all negative powers λ^{-n} of the frequency. These quantities measure the influence of the cutoff function on the terms of the expansion of the high frequency part of the solution. Note that in [16] it is claimed (only indicating the steps of a proof), that there exists a bound which is independent of n. This does not seem to be rigorously correct: writing down the details of the proof sketched in [16], we find an explicit bound in terms of certain norms of the cutoff function, but which grows linearly in n. But this growth has no influence on the convergence of the expansion of the solution. Nevertheless the explicitness of the estimate will allow us to give an upper bound of the coefficient of the time decay of the solution.

5.2 Definition. Let $\phi \in C^\infty(\mathbb{R})$ be such that $0 \le \phi(\lambda) \le 1$ and $\phi(\lambda) = 1$ if $|\lambda| \le 1$ and $\phi(\lambda) = 0$ if $|\lambda| \ge 2$. Let $\lambda_0 \ge 1$ and $L > 2\lambda_0$. Define

1. $\chi_{\lambda_0} \in C^\infty([1, \infty[)$ by $\chi_{\lambda_0}(\lambda) := 1 - \phi\left(\frac{\lambda}{\lambda_0}\right)$, $\lambda \ge 1$,
2. $\chi_{\lambda_0, L} \in C^\infty([1, \infty[)$ by $\chi_{\lambda_0, L}(\lambda) := \chi_{\lambda_0}(\lambda)\phi\left(\frac{\lambda}{\lambda_0}\right)$, $\lambda \ge 1$.

5.3 Theorem. *For $n \in \mathbb{N}$, $\lambda_0 \ge 1$, $L > 2\lambda_0$ it holds*:*

$$\|[\chi_{\lambda_0, L}(\lambda^2)\lambda^{-n}]^\vee\|_1 \le c(n)\lambda_0^{-n/2},$$

with $c(0) = N_1 + N_1^2$, $c(1) = 2(N_1 + N_1^2) + 32\sqrt{2}N_2$, $c(n) = \frac{4}{n-1} + 32\sqrt{2}N_2 n$, $n \ge 2$ and hence $c(n) \le Mn$, $n \ge 1$, where $N_1 = \|[\phi(\lambda^2)]^\vee\|_1$, $N_2 = \|\phi\|_{C^2(\mathbb{R})}$ and $M = 32\sqrt{2}\max\{N_1 + N_1^2, N_2\}$.

Proof. The proof follows from Theorem 5.7 and Propositions 5.8 and 5.10 below. □

5.4 Proposition. *Suppose $\lambda_0, L \ge 1$ and $2\lambda_0 < L$. Then we have for all $\lambda \in \mathbb{R}$*

$$|\chi_{\lambda_0, L}(\lambda^2)| \le \mathbb{1}_{\{\sqrt{\lambda_0} \le |\lambda| \le \sqrt{2L}\}}(\lambda),$$

$$\left|\frac{d}{d\lambda}\left(\chi_{\lambda_0, L}(\lambda^2)\right)\right| \le 2|\lambda|\,\|\phi\|_{C^1(\mathbb{R})}\left(\frac{1}{\lambda_0}\mathbb{1}_{\{\sqrt{\lambda_0} \le |\lambda| \le \sqrt{2\lambda_0}\}}(\lambda) + \frac{1}{L}\mathbb{1}_{\{\sqrt{L} \le |\lambda| \le \sqrt{2L}\}}(\lambda)\right),$$

$$\left|\frac{d^2}{d\lambda^2}\left(\chi_{\lambda_0, L}(\lambda^2)\right)\right| \le \|\phi\|_{C^2(\mathbb{R})}\left(\left(\frac{2}{\lambda_0} + \frac{4|\lambda|^2}{\lambda_0^2}\right)\mathbb{1}_{\{\sqrt{\lambda_0} \le |\lambda| \le \sqrt{2\lambda_0}\}}(\lambda) + \left(\frac{2}{L} + \frac{4|\lambda|^2}{L^2}\right)\mathbb{1}_{\{\sqrt{L} \le |\lambda| \le \sqrt{2L}\}}\right).$$

*We write shortly $f^\vee = \mathcal{F}^{-1}f$

Proof. Clearly we have

$$\frac{d}{d\lambda}\left(\phi\left(\frac{\lambda^2}{\alpha}\right)\right) = \frac{2\lambda}{\alpha}\phi'\left(\frac{\lambda^2}{\alpha}\right) \quad \text{and} \quad \frac{d^2}{d\lambda^2}\left(\phi\left(\frac{\lambda^2}{\alpha}\right)\right) = \frac{2\lambda}{\alpha}\phi'\left(\frac{\lambda^2}{\alpha}\right) + \frac{4\lambda^2}{\alpha^2}\phi''\left(\frac{\lambda^2}{\alpha}\right).$$

Further we have for λ_0, L and $\lambda \geq 1$

$$\frac{d}{d\lambda}(\chi_{\lambda_0,L}(\lambda^2)) = -\left[\phi\left(\frac{\lambda^2}{\lambda_0}\right)\right]'\phi\left(\frac{\lambda^2}{L}\right) + \left(1 - \phi\left(\frac{\lambda^2}{\lambda_0}\right)\right)\left[\phi\left(\frac{\lambda^2}{L}\right)\right]'$$

and

$$\frac{d^2}{d\lambda^2}(\chi_{\lambda_0,L}(\lambda^2)) = -\left[\phi\left(\frac{\lambda^2}{\lambda_0}\right)\right]''\phi\left(\frac{\lambda^2}{L}\right) - 2\left[\phi\left(\frac{\lambda^2}{\lambda_0}\right)\right]'\left[\phi\left(\frac{\lambda^2}{L}\right)\right]' + \left(1 - \phi\left(\frac{\lambda^2}{\lambda_0}\right)\right)\left[\phi\left(\frac{\lambda^2}{L}\right)\right]''.$$

We estimate for $\alpha \geq 1$ and $\lambda \in \mathbb{R}$:

$$\left|\phi\left(\frac{\lambda^2}{\alpha}\right)\right| \leq \|\phi\|_{C^0(\mathbb{R})}\mathbb{1}_{]-\infty,2]}\left(\frac{\lambda^2}{\alpha}\right) = \|\phi\|_{C^0(\mathbb{R})}\mathbb{1}_{\{|\lambda|\leq\sqrt{2\alpha}\}}(\lambda)$$

due to $\frac{\lambda^2}{\alpha} \leq 2 \Leftrightarrow |\lambda| \leq \sqrt{2\alpha}$. Similarly we have

$$\left|1 - \phi\left(\frac{\lambda^2}{\alpha}\right)\right| \leq \mathbb{1}_{[1,+\infty[}\left(\frac{\lambda^2}{\alpha}\right) = \mathbb{1}_{\{\sqrt{\alpha}\leq|\lambda|\}}(\lambda).$$

Further

$$\left|\frac{d}{d\lambda}\left(\phi\left(\frac{\lambda^2}{\alpha}\right)\right)\right| = \left|\frac{2\lambda}{\alpha}\phi'\left(\frac{\lambda^2}{\alpha}\right)\right| \leq \frac{2|\lambda|}{\alpha}\|\phi\|_{C^1(\mathbb{R})}\mathbb{1}_{\{\sqrt{\alpha}\leq|\lambda|\leq\sqrt{2\alpha}\}}(\lambda)$$

and

$$\left|\frac{d^2}{d\lambda^2}\left(\phi\left(\frac{\lambda^2}{\alpha}\right)\right)\right| \leq \left(\frac{2}{\alpha} + \frac{4|\lambda|^2}{\alpha^2}\right)\|\phi\|_{C^2(\mathbb{R})}\mathbb{1}_{\{\sqrt{\alpha}\leq|\lambda|\leq\sqrt{2\alpha}\}}.$$

The three stated estimates directly follow from the previous properties. □

5.5 Proposition. *Let $n \in \mathbb{N}^*$ and let $\lambda_0, L \geq 1$ with $2\lambda_0 \leq L$. Then, recalling that $N_2 = \|\phi\|_{C^2(\mathbb{R})}$,*

$$\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee(\tau)\tau^2\|_\infty \leq 16\sqrt{2}N_2\lambda_0^{(-n-1)/2}n.$$

Proof. By standard properties of the Fourier transform we have

$$\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee(\tau)\tau^2\|_\infty = \big\|\big[\big(\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}\big)''\big]^\vee\big\|_\infty \le \big\|\big(\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}\big)''\big\|_1.$$

Hence by Leibniz's rule and the previous proposition, we find that

$$\begin{aligned}
&\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee(\tau)\tau^2\|_\infty \\
&\quad\le \int_{-\infty}^{+\infty}\left|\frac{d^2}{d\lambda^2}\big(\chi_{\lambda_0,L}(\lambda^2)\big)\right||\lambda|^{-n}\,d\lambda \\
&\qquad + 2\int_{-\infty}^{+\infty}\left|\frac{d}{d\lambda}\chi_{\lambda_0,L}(\lambda^2)\right|n|\lambda|^{-n-1}\,d\lambda + \int_{-\infty}^{+\infty}|\chi_{\lambda_0,L}(\lambda^2)|n(n+1)|\lambda|^{-n-2}\,d\lambda \\
&\quad\le \int_{-\infty}^{+\infty}\Bigg[\left(\frac{2}{\lambda_0}+\frac{4|\lambda|^2}{\lambda_0^2}\right)\|\phi\|_{C^2(\mathbb{R})}\mathbb{1}_{\{\sqrt{\lambda_0}\le|\lambda|\le\sqrt{2\lambda_0}\}}(\lambda) \\
&\qquad\qquad + \left(\frac{2}{L}+\frac{4|\lambda|^2}{L^2}\right)\|\phi\|_{C^2(\mathbb{R})}\mathbb{1}_{\{\sqrt{L}\le\sqrt{\lambda}\le\sqrt{2L}\}}(\lambda)\Bigg]|\lambda|^{-n}\,d\lambda \\
&\qquad + 2\int_{-\infty}^{+\infty}\Bigg[\frac{2|\lambda|}{\lambda_0}\|\phi\|_{C^1(\mathbb{R})}\mathbb{1}_{\{\sqrt{\lambda_0}\le|\lambda|\le\sqrt{2\lambda_0}\}}(\lambda) \\
&\qquad\qquad + \frac{2|\lambda|}{L}\|\phi\|_{C^1(\mathbb{R})}\mathbb{1}_{\{\sqrt{L}\le|\lambda|\le\sqrt{2L}\}}(\lambda)\Bigg]n|\lambda|^{-n-1}\,d\lambda \\
&\qquad + \int_{-\infty}^{+\infty}\mathbb{1}_{\{\sqrt{\lambda_0}\le|\lambda|\le\sqrt{2L}\}}(\lambda)n(n+1)|\lambda|^{-n-2}\,d\lambda.
\end{aligned}$$

By using that for $\alpha > 0$ we have

$$|\lambda|^k\mathbb{1}_{\{\sqrt{\alpha}\le|\lambda|\le\sqrt{2\alpha}\}}(\lambda) \le (2\alpha)^{k/2},$$

we obtain

$$\begin{aligned}
&\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee(\tau)\tau^2\|_\infty \\
&\quad\le \|\phi\|_{C^2(\mathbb{R})}\Big(\frac{10}{\lambda_0}\int_{\sqrt{\lambda_0}\le|\lambda|\le\sqrt{2\lambda_0}}|\lambda|^{-n}\,d\lambda + \frac{10}{L}\int_{\sqrt{L}\le|\lambda|\le\sqrt{2L}}|\lambda|^{-n}\,d\lambda\Big) \\
&\qquad + 2\|\phi\|_{C^1(\mathbb{R})}\Big(\frac{2\sqrt{2}}{\lambda_0^{1/2}}\int_{\sqrt{\lambda_0}\le|\lambda|\le\sqrt{2\lambda_0}}|\lambda|^{-n-1}\,d\lambda \\
&\qquad\qquad + \frac{2\sqrt{2}}{L^{1/2}}\int_{\sqrt{L}\le|\lambda|\le\sqrt{2L}}|\lambda|^{-n-1}\,d\lambda\Big) \\
&\qquad + n(n+1)\int_{\sqrt{\lambda_0}\le|\lambda|\le\sqrt{2L}}|\lambda|^{-n-2}\,d\lambda.
\end{aligned}$$

Calculating these integrals we find

$$
\begin{aligned}
&\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee(\tau)\tau^2\|_\infty \\
&\quad \le N_2\Biggl(\frac{20}{\lambda_0(n-1)}\bigl(\lambda_0^{(-n+1)/2}-(2\lambda_0)^{(-n+1)/2}\bigr) \\
&\qquad + \frac{20}{L(n-1)}\bigl(L^{(-n+1)/2}-(2L)^{(-n+1)/2}\bigr) \\
&\qquad + \frac{8\sqrt{2}}{n\sqrt{\lambda_0}}\bigl(\lambda_0^{-n/2}-(2\lambda_0)^{(-n+1)/2}\bigr) + \frac{8\sqrt{2}}{n\sqrt{L}}\bigl(L^{-n/2}-(2L)^{(-n+1)/2}\bigr) \\
&\qquad + n\bigl(\lambda_0^{(-n-1)/2}-(2L)^{(-n-1)/2}\bigr)\Biggr).
\end{aligned}
$$

This leads to the conclusion since this right-hand side is bounded by $16\sqrt{2}N_2\lambda_0^{(-n-1)/2}n$. □

5.6 Proposition. *For $n \ge 2$, $n \in \mathbb{N}$ we have*

$$\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee\|_\infty \le \frac{2\lambda_0^{(-n+1)/2}}{n-1}.$$

Proof. As

$$\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee\|_\infty \le \|\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}\|_1,$$

we conclude by simple calculations. □

5.7 Theorem. *For $n \ge 2$ and $N_2 = \|\phi\|_{C^2(\mathbb{R})}$, it holds*

$$\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee\|_1 \le \lambda_0^{-n/2}\biggl(\frac{4}{n-1} + 32\sqrt{2}N_2 n\biggr).$$

Proof. We split up the integral in $\mathbb{R}$ into an integral in $[-\lambda_0^{-1/2}, \lambda_0^{-1/2}]$ and outside, this yields

$$
\begin{aligned}
&\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee\|_1 \\
&\quad \le \|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee(\tau)\chi_{[-\lambda_0^{-1/2},\lambda_0^{-1/2}]}(\tau)\|_\infty \int_{-\lambda_0^{-1/2}}^{\lambda_0^{-1/2}} d\tau \\
&\qquad + \|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-n}]^\vee(\tau)\chi_{\mathbb{R}\setminus[-\lambda_0^{-1/2},\lambda_0^{-1/2}]}(\tau)\tau^2\|_\infty \int_{\mathbb{R}\setminus[-\lambda_0^{-1/2},\lambda_0^{-1/2}]} \frac{1}{\tau^2}\, d\tau.
\end{aligned}
$$

The conclusion then follows from Propositions 5.5 and 5.6. □

5.8 Proposition.

$$\|[\chi_{\lambda_0,L}(\lambda^2)]^\vee\|_1 \le \|[\phi(\lambda^2)]^\vee\|_1 + \|[\phi(\lambda^2)]^\vee\|_1^2.$$

Proof. By definition, we have

$$\begin{aligned}
\|[\chi_{\lambda_0,L}(\lambda^2)]^\vee\|_1 &= \left\|\left[\left(1-\phi\left(\frac{\lambda^2}{\lambda_0}\right)\right)\phi\left(\frac{\lambda^2}{L}\right)\right]^\vee\right\|_1 \\
&\le \left\|\left[\phi\left(\frac{\lambda^2}{L}\right)\right]^\vee\right\|_1 + \left\|\left[\phi\left(\frac{\lambda^2}{\lambda_0}\right)\right]^\vee \star \left[\phi\left(\frac{\lambda^2}{L}\right)\right]^\vee\right\|_1 \\
&\le \left\|\left[\phi\left(\frac{\lambda^2}{L}\right)\right]^\vee\right\|_1 + \left\|\left[\phi\left(\frac{\lambda^2}{\lambda_0}\right)\right]^\vee\right\|_1 \left\|\left[\phi\left(\frac{\lambda^2}{L}\right)\right]^\vee\right\|_1.
\end{aligned}$$

For $\alpha \ge 1$, the function $\lambda \mapsto \phi\left(\frac{\lambda^2}{\alpha}\right)$ is in $C^\infty(\mathbb{R})$ and has compact support. This justifes the above calculation. The right hand side of the last inequality is in fact independent of L and λ_0, as can be seen as follows: for $\alpha > 0$ we have

$$\left\|\left[\phi\left(\frac{\lambda^2}{\alpha}\right)\right]^\vee\right\|_1 = \sqrt{\alpha}\int_{-\infty}^{+\infty} |[\phi(\lambda^2)]^\vee(\sigma)|\,\frac{d\sigma}{\sqrt{\alpha}} = \|[\phi(\lambda^2)]^\vee\|_1. \qquad \square$$

5.9 Proposition.

$$\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-1}]^\vee\|_\infty \le \|[\phi(\lambda^2)]^\vee\|_1 + \|[\phi(\lambda^2)]^\vee\|_1^2.$$

Proof. We may write

$$\begin{aligned}
\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-1}]^\vee\|_\infty &= \|[\chi_{\lambda_0,L}(\lambda^2)]^\vee \star [\lambda^{-1}]^\vee\|_\infty \\
&\le \|[\chi_{\lambda_0,L}(\lambda^2)]^\vee\|_1 \|[\lambda^{-1}]^\vee\|_\infty.
\end{aligned}$$

due the fact that $\lambda \mapsto \chi_{\lambda_0,L}(\lambda^2)$ is a test function and $[\lambda^{-1}]^\vee(\tau) = -i\,\mathrm{sign}(\tau)$, $\tau \in \mathbb{R}$. The conclusion follows from Proposition 5.8. $\square$

5.10 Proposition. *Let* $\lambda_0 \ge 1, L \ge 2\lambda_0$. *Then*

$$\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-1}]^\vee\|_1 \le \left(2(N_1 + N_1^2) + 32\sqrt{2}N_2\right)\sqrt{\lambda_0},$$

recalling that $N_1 = \|[\phi(\lambda^2)]^\vee\|_1$, $N_2 = \|\phi\|_{C^2(\mathbb{R})}$.

Proof. As before, we write

$$\|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-1}]^\vee\|_1 \le \|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-1}]^\vee\chi_{[-\lambda_0^{-1/2},\lambda_0^{-1/2}]}\|_\infty \int_{-\lambda_0^{-1/2}}^{\lambda_0^{-1/2}} d\tau$$
$$+ \|[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-1}]^\vee\chi_{\mathbb{R}\setminus[-\lambda_0^{-1/2},\lambda_0^{-1/2}]}(\tau)\tau^2\|_\infty \int_{\mathbb{R}\setminus[-\lambda_0^{-1/2},\lambda_0^{-1/2}]} \frac{1}{\tau^2}\,d\tau.$$

We finish the proof by using Propositions 5.5 and 5.9. □

Now we have all the ingredients to state and prove the L^∞-decay and the perturbation result.

5.11 Theorem. *Let $V, f, g \in L^1(\mathscr{R})$ be real valued, let V satisfy the conditions of Theorem* 1.1, $N \ge 2$, $0 < q_* < 1$, $\lambda_0 > \lambda_* = \frac{4(N-1)^2\|V\|_1^2}{N^2 q_*^2}$ *and* $L > 2\lambda_0$. *Then we have*

1.
$$|\langle e^{itH}\chi_{\lambda_0,L}(H)f, g\rangle|$$
$$\le \left(\sum_{k=0}^\infty \left(\frac{2(N-1)}{N}\right)^k \|V\|_1^k \|\mathscr{F}^{-1}[\chi_{\lambda_0,L}(\lambda^2)|\lambda|^{-k}]\|_1\right)\|f\|_1\|g\|_1|t|^{-1/2},$$
for $t \neq 0$.

2. $\|e^{itH}\chi_{\lambda_0}(H)\|_{1,\infty} \le 4\left(A + B\frac{\|V\|_1}{\sqrt{\lambda_0}}\right)|t|^{-1/2}$, $t \neq 0$, *where* $A = N_1 + N_1^2$ *with* $N_1 := \|\mathscr{F}^{-1}[\phi(\lambda^2)]\|_1$, $N_2 := \|\phi\|_2$ *and* $B = M\frac{(N-1)}{N}\frac{1}{(1-q_*)^2}$ *with* $M := 32\sqrt{2}\max\{N_1 + N_1^2; N_2\}$,
3. $\|e^{itH}\chi_{\lambda_0}(H) - e^{itH_0}\chi_{\lambda_0}(H_0)\|_{1,\infty} \le 4B\frac{\|V\|_1}{\sqrt{\lambda_0}}|t|^{-1/2}$, $t \neq 0$, *with* B *as in* 2. *In particular we have*

$$e^{itH}\chi_{\lambda_0}(H)f \to e^{itH_0}\chi_{\lambda_0}(H_0)f \qquad \text{for } \lambda_0 \to \infty$$

uniformly on $\mathscr{R}$ for every fixed $t > 0$ or also uniformly on $\mathscr{R} \times [\epsilon, \infty)$ with respect to the weight $|t|^2$ on the time axis for any positive ϵ.

Proof.
<u>1.:</u>
At first we consider $f \in L^1(\mathscr{R}) \cap L^2(\mathscr{R})$, the estimates then extend to $f \in L^1(\mathscr{R})$. From Stone's formula, the fact that the spectrum of H is absolutely continuous on $[0, \infty)$ (Remark 4.16) and the limiting absorption principle proved in Theorem 4.14 we deduce

$$\langle e^{itH}\chi_{\lambda_0,L}(H)f, g\rangle = \frac{1}{2i\pi}\int_0^\infty e^{it\lambda}\chi_{\lambda_0,L}(\lambda)\langle (R_V(\lambda + i0) - R_V(\lambda - i0))f, g\rangle\,d\lambda.$$

As V, f and g are real valued, we obtain

$$\langle e^{itH}\chi_{\lambda_0,L}(H)f,g\rangle = \frac{1}{\pi}\int_0^\infty e^{it\lambda}\chi_{\lambda_0,L}(\lambda)\Im\langle R_V(\lambda+i0)f,g\rangle\,d\lambda.$$

Using Proposition 5.1 part 3. and the change of variables $\lambda = \mu^2$ we find

$$\begin{aligned}&\langle e^{itH}\chi_{\lambda_0,L}(H)f,g\rangle\\ &\quad = \frac{2}{\pi}\int_0^\infty e^{it\mu^2}\chi_{\lambda_0,L}(\mu^2)\sum_{k=0}^{\infty}\Im\big\langle R_0(\mu^2+i0)\big(-VR_0(\mu^2+i0)\big)^k f,g\big\rangle\mu\,d\mu.\end{aligned}$$

Fubini's Theorem, whose hypotheses are fulfilled thanks to the inequality in the proof of Proposition 5.1 part 3., leads to

$$\begin{aligned}&\langle e^{itH}\chi_{\lambda_0,L}(H)f,g\rangle\\ &\quad = \frac{2}{\pi}\sum_{k=0}^{\infty}\int_{\mathscr{R}}\int_{\mathscr{R}^k}\prod_{j=1}^{k}V(x_j)\int_{\mathscr{R}}\\ &\qquad\Big(\int_0^\infty e^{it\mu^2}\chi_{\lambda_0,L}(\mu^2)N(x,x_1,\dots,x_k,y,\mu)\mu\,d\mu\Big)f(y)\,dy\,dx_1\dots dx_k g(x)\,dx,\end{aligned}$$

where $N(x,x_1,\dots,x_k,y,\mu)$ is defined by

$$\begin{aligned}&N(x,x_1,\dots,x_k,y,\mu)\\ &\quad = (-1)^k\Im\Big(K_0(x,x_1,\mu^2+i0)\prod_{j=1}^{k-1}K_0(x_j,x_{j+1},\mu^2+i0)K_0(x_k,y,\mu^2+i0)\Big).\end{aligned}$$

Using again Proposition 5.1, after some elementary calculations, we find that

$$\begin{aligned}&N(x,x_1,\dots,x_k,y,\mu)\\ &\quad = -\frac{1}{\mu^{k+1}N^{k+1}}e^{ik\pi/2}\sum_{n=1}^{2^k}\Big(1-\frac{N}{2}\Big)^{\alpha_n}\Big(\frac{N}{2}\Big)^{\beta_n}\big(e^{id_n\mu}+(-1)^k e^{-id_n\mu}\big),\end{aligned}$$

with $\alpha_n,\delta_n\in\mathbb{N}$ such that $\alpha_n+\delta_n = k+1$ and d_n are real numbers that depend on x, y and x_j, $j=1,\dots,k$. Using this expression in the previous one, we obtain

$$\begin{aligned}&\langle e^{itH}\chi_{\lambda_0,L}(H)f,g\rangle\\ &\quad = -\frac{1}{\pi}\sum_{k=0}^{\infty}N^{-(k+1)}e^{ik\pi/2}\int_{\mathscr{R}}\int_{\mathscr{R}^k}\prod_{j=1}^{k}V(x_j)\int_{\mathscr{R}}\sum_{n=1}^{2^k}\Big(1-\frac{N}{2}\Big)^{\alpha_n}\Big(\frac{N}{2}\Big)^{\beta_n}\\ &\qquad\Big(\int_{-\infty}^{\infty}e^{it\mu^2}\chi_{\lambda_0,L}(\mu^2)e^{id_n\mu}\mu^{-k}\,d\mu\Big)f(y)\,dy\,dx_1\dots dx_k g(x)\,dx.\end{aligned}$$

Noting that $\left|\left(1-\frac{N}{2}\right)^{\alpha_n}\left(\frac{N}{2}\right)^{\beta_n}\right| = \left(\frac{N}{2}-1\right)^{\alpha_n}\left(\frac{N}{2}\right)^{\beta_n} \leq (N-1)^{\alpha_n}(N-1)^{\beta_n} = (N-1)^{k+1}$, we find

$$\begin{aligned}
&|\langle e^{itH}\chi_{\lambda_0,L}(H)f,g\rangle| \\
&\quad \leq \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(N-1)^{k+1}}{N^{k+1}}\int_{\mathscr{R}}\int_{\mathscr{R}^k}\prod_{j=1}^{k}|V(x_j)| \\
&\qquad \int_{\mathscr{R}}\sum_{n=1}^{2^k}\left|\int_{-\infty}^{\infty}e^{it\mu^2}\chi_{\lambda_0,L}(\mu^2)e^{id_n\mu}\mu^{-k}\,d\mu\right||f(y)|\,dy\,dx_1\ldots dx_k|g(x)|\,dx.
\end{aligned}$$

Setting

$$S_k = \sup_{d\in\mathbb{R}}\left|\int_0^{\infty}e^{i(t\mu^2+d\mu)}\chi_{\lambda_0,L}(\mu^2)\mu^{-k}\,d\mu\right|,$$

we deduce that

$$|\langle e^{itH}\chi_{\lambda_0,L}(H)f,g\rangle| \leq \frac{1}{\pi}\|f\|_1\|g\|_1\sum_{k=0}^{\infty}\frac{2^{k+2}(N-1)^{k+1}}{N^{k+1}}\|V\|_1^k S_k.$$

We observe that

$$S_k = \sup_{a\in\mathbb{R}}\left|\int_{-\infty}^{+\infty}e^{i(t\lambda^2+a\lambda)}\chi_{\lambda_0,L}(\lambda^2)\lambda^{-k}\,d\lambda\right| \leq \|\mathscr{F}^{-1}[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-k}]\|_1|t|^{-1/2}, \qquad t\neq 0,$$

since the quantity inside the absolute value is the solution of the free Schrödinger operator on $\mathbb{R}$ at time t and position a for the initial condition $\mathscr{F}^{-1}[\chi_{\lambda_0,L}(\lambda^2)\lambda^{-k}]$, see for example [23], p. 60 Theorem IX.30. The convergence of the series will follow from the proof of 2.

<u>2.:</u>

First let $f,g\in L^1(\mathscr{R})$ be real valued. With $\tilde{q}(\lambda) := \frac{2(N-1)\|V\|_1}{N\sqrt{\lambda}}$ and the assumptions $0<q_*<1$ and $\lambda_*\leq\lambda_0$ it follows $0<\tilde{q}(\lambda_0)<\tilde{q}(\lambda_*)=q_*<1$. Therefore

$$\sum_{k=1}^{\infty}\tilde{q}(\lambda_0)^k k = \frac{1}{\left(1-\tilde{q}(\lambda_0)\right)^2}$$

converges. Thus we can apply Theorem 5.3 and obtain together with 1.

$$\begin{aligned}
|\langle e^{itH}\chi_{\lambda_0}(H)f,g\rangle| &\leq \left(\sum_{k=0}^{\infty}\left(\frac{2(N-1)}{N}\right)^k\|V\|_1^k c(k)\lambda_0^{-k/2}\right)\|f\|_1\|g\|_1|t|^{-1/2} \\
&= \left(\sum_{k=0}^{\infty}\tilde{q}(\lambda_0)^k c(k)\right)\|f\|_1\|g\|_1|t|^{-1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \Big(c(0) + \sum_{k=1}^{\infty} \tilde{q}(\lambda_0)^k c(k) \Big) \|f\|_1 \|g\|_1 |t|^{-1/2} \\
&\le \Big(c(0) + M \sum_{k=1}^{\infty} \tilde{q}(\lambda_0)^k k \Big) \|f\|_1 \|g\|_1 |t|^{-1/2} \\
&= \left(c(0) + M \frac{1}{\big(1 - \tilde{q}(\lambda_0)\big)^2} \right) \|f\|_1 \|g\|_1 |t|^{-1/2} \\
&\le \left(c(0) + M \frac{1}{(1 - q_*)^2} \right) \|f\|_1 \|g\|_1 |t|^{-1/2}
\end{aligned}
$$

for all $t \neq 0$. Since spectral measures are finite and since $\lim_{L\to\infty} \chi_{\lambda_0, L} = \chi_{\lambda_0}$ pointwise, we can replace $\chi_{\lambda_0, L}$ by χ_{λ_0} in the last inequality using dominated convergence. By linearity a factor 4 appears for complex valued f and g. This ends the proof of 2.

3.:

Stone's formula applied to H_0 yields

$$\langle e^{itH_0} \chi_{\lambda_0, L}(H_0) f, g \rangle = \frac{1}{\pi} \int_0^{\infty} e^{it\lambda} \chi_{\lambda_0, L}(\lambda) \Im \langle R_0(\lambda + i0) f, g \rangle \, d\lambda,$$

which is the first term in the expansion for $\langle e^{itH} \chi_{\lambda_0, L}(H) f, g \rangle$. Therefore

$$
\begin{aligned}
&\big\langle \big(e^{itH} \chi_{\lambda_0, L}(H) - e^{itH_0} \chi_{\lambda_0, L}(H_0) \big) f, g \big\rangle \\
&\quad = \frac{2}{\pi} \int_0^{\infty} e^{it\mu^2} \chi_{\lambda_0, L}(\mu^2) \sum_{k=1}^{\infty} \Im \big\langle R_0(\mu^2 + i0) \big(-V R_0(\mu^2 + i0) \big)^k f, g \big\rangle \mu \, d\mu.
\end{aligned}
$$

Now the same proof as in 2. but without the first term yields the assertion. □

5.2. Low energy estimate. In this section we consider the case of initial conditions with compact energy band with respect to the spectral representation. Again we adapt the reasoning of [16] to the transmission situation.

For any smooth and compactly supported cut-off function χ in $\mathbb{R}$, by Theorem 4.15 we have for any $x, x' \in R$

$$2i\pi \int_0^{+\infty} e^{it\lambda} \chi(\lambda) E_{ac}(d\lambda)(x, x') = -2i\pi \int_0^{+\infty} e^{it\lambda} \chi(\lambda) \Im K(x, x', \lambda) \, d\lambda,$$

and by the change of variables $\lambda = \mu^2$, we get

$$2i\pi \int_0^{+\infty} e^{it\lambda} \chi(\lambda) E_{ac}(d\lambda)(x, x') = -4i\pi \int_0^{+\infty} e^{it\mu^2} \chi(\mu^2) \Im K(x, x', \mu^2) \mu \, d\mu.$$

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Now recalling the definition of K, we again distinguish between the following three cases:

1. If $x, x' \in R_j$ with $x' > x$, then

$$K(x, x', \mu^2) = \frac{1}{W_{j,-}(\mu)} f_{j,-}(\mu, x) f_{j,+}(\mu, x') + \frac{c_{j,-,2}(\mu)}{i\mu f_{j+1,+}(\mu, 0) s(\mu)} f_{j,+}(\mu, x) f_{j,+}(\mu, x').$$

As $\overline{f_{j,\pm}(\mu, x)} = f_{j,\pm}(-\mu, x)$, we deduce

$$\begin{aligned} 2i\pi \int_0^{+\infty} e^{it\lambda}\chi(\lambda) E_{ac}(d\lambda)(x, x') &= -2i\pi \int_{-\infty}^{+\infty} e^{it\mu^2}\chi(\mu^2)\mu \frac{1}{W_{j,-}(\mu)} f_{j,-}(\mu, x) f_{j,+}(\mu, x')\, d\mu \\ &\quad - 2\pi \int_{-\infty}^{+\infty} e^{it\mu^2}\chi(\mu^2) \frac{c_{j,-,2}(\mu)}{f_{j+1,+}(\mu, 0) s(\mu)} f_{j,+}(\mu, x) f_{j,+}(\mu, x')\, d\mu. \end{aligned}$$

The first term of this right hand side was estimated in Lemma 4 of [16], hence it remains to estimate the second term. For that purpose, we set

$$\begin{aligned} T_2(t, x, x') &:= \int_{-\infty}^{+\infty} e^{it\mu^2}\chi(\mu^2) \frac{c_{j,-,2}(\mu)}{f_{j+1,+}(\mu, 0) s(\mu)} f_{j,+}(\mu, x) f_{j,+}(\mu, x')\, d\mu \\ &= \int_{-\infty}^{+\infty} e^{it\mu^2}\chi(\mu^2) e^{i\mu(x+x')} \frac{c_{j,-,2}(\mu)}{f_{j+1,+}(\mu, 0) s(\mu)} m_{j,+}(\mu, x) m_{j,+}(\mu, x')\, d\mu. \end{aligned}$$

Hence denoting by

$$p(\mu) := \frac{c_{j,-,2}(\mu)}{f_{j+1,+}(\mu, 0) s(\mu)},$$

we have shown in Corollary 4.10 that this function belongs to $H^1(-R, R)$, for all $R > 0$. Since the mapping

$$q : \mu \to \chi(\mu^2) m_{j,+}(\mu, x) m_{j,+}(\mu, x'),$$

has compact support and is in $C^1(\mathbb{R})$ with the property

$$|q(\mu)| + |\dot{q}(\mu)| \le C,$$

for some $C > 0$ independent of x and x' due to (4.28) and (4.44), we deduce that the product pq belongs to $H^1(\mathbb{R})$. By Plancherel theorem (see for instance

[23], p. 60), we deduce that

$$T_2(t,x,x') = t^{-1/2}\int_{-\infty}^{+\infty} \mathscr{F}^{-1}(pq)(\xi + x + x')e^{-i\xi^2/t}\,d\xi.$$

and consequently

$$\begin{aligned}|T_2(t,x,x')| &\le |t|^{-1/2}\int_{-\infty}^{+\infty} |\mathscr{F}^{-1}(pq)(\xi + x + x')|\,d\xi \\ &\le |t|^{-1/2}\int_{-\infty}^{+\infty} |\mathscr{F}^{-1}(pq)(\xi)|\,d\xi \\ &\le C|t|^{-1/2}\|pq\|_{H^1(\mathbb{R})},\end{aligned}$$

for some $C > 0$.

2. If $x, x' \in R_j$ with $x' < x$, then

$$\begin{aligned}K(x,x',\mu^2) = {}&\frac{1}{W_{j,-}(\mu)} f_{j,-}(\mu,x')f_{j,+}(\mu,x) \\ &+ \frac{c_{j,-,2}(\mu)}{i\mu f_{j+1,+}(\mu,0)s(\mu)} f_{j,+}(\mu,x)f_{j,+}(\mu,x').\end{aligned}$$

In that case the first term was treated in Lemma 4 of [16], while the second term is the same as before.

3. If $x \in R_j$ and $x' \in R_k$ with $k \ne j$, then

$$K(x,x',\mu^2) = \frac{1}{i\mu f_{j+1,+}(\mu,0)s(\mu)} f_{j,+}(\mu,x)f_{k,+}(\mu,x').$$

Therefore in that case we have

$$\begin{aligned}&2i\pi\int_0^{+\infty} e^{it\lambda}\chi(\lambda)E_{ac}(d\lambda)(x,x') \\ &\quad = 2\pi\int_{-\infty}^{+\infty} e^{it\mu^2}\chi(\mu^2)\frac{1}{f_{j+1,+}(\mu,0)s(\mu)} f_{j,+}(\mu,x)f_{j,+}(\mu,x')\,d\mu.\end{aligned}$$

Since $\frac{1}{f_{j+1,+}(\mu,0)}$ is in $C^1(\mathbb{R})$, by Corollary 4.9, the function

$$\mu \to \frac{1}{f_{j+1,+}(\mu,0)s(\mu)}$$

belongs to $H^1(-R,R)$, for all $R > 0$ and we conclude as for $T_2(t,x,x')$. □

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Received March 29, 2013; revised July 1, 2014

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