Portugal. Math. (N.S.)
Vol. 72, Fasc. 4, 2015, 393–406
DOI 10.4171/PM/1972

Portugaliae Mathematica

On the group of germs of contact transformations

Ana Rita Martins and Orlando Neto*

Abstract. We show that the germ of a contact transformation Φ can be written in a unique way as a product $\Phi_1\Phi_2$, where Φ_1 only depends on the derivative of Φ and the derivative of Φ_2 is trivial. We show that each contact transformation with trivial derivative can be constructed solving a convenient Cauchy problem.

Mathematics Subject Classification (primary; secondary): 53D10; 37J55

Keywords: Contact geometry, Legendrian varieties

1. Introduction

When trying to obtain normal forms of Legendrian curves, to study its deformations or to construct its moduli spaces (see [1], [2] and [3]) it is essential to have a tool to construct germs of contact transformations of a contact manifold of dimension 3. The main purpose of this paper is to generalize Theorem 3.2 of [1] to a contact manifold of arbitrary dimension (see Theorem 4.1). This is an essential preliminary step to the study of the geometry of the germs of higher dimensional Legendrian varieties.

The most important tool in the proof of the main result is the Cauchy-Kowalevsky-Kashiwara theorem, a far reaching generalization of the classical Cauchy-Kowalevsky Theorem. This is probably the first non trivial down to earth application of that Theorem. Its cohomological formulation invites us to think that it only could be useful in more rarefied environments. This is not the case.

Corollaries 4.2 and 4.3 show how to construct two families of contact transformations particularly useful in the applications. Theorem 4.4 describes the group of germs of contact transformations.

*The authors would like to thank João Cabral for several useful conversations.

All the results of this paper are still true if we replace the complex analytic contact manifolds by real analytic contact manifolds.

2. Contact geometry

Let X be a germ at a point o of a complex manifold of dimension $2n+1$. Let $\mathcal{O}$ be the ring of holomorphic functions of X. Let m be the maximal ideal of $\mathcal{O}$. Let Ω^p be the module of germs of differential forms of degree p of X. A differential form $\omega \in \Omega^1$ is called a *contact form* if the wedge product $\omega(d\omega)^n$ does not vanish at o. If ω is a contact form it follows from the Darboux Theorem that there is a system of local coordinates

$$(x_1, \dots, x_n, y, p_1, \dots, p_n) \tag{1}$$

of X such that

$$\omega = dy - \sum_{i=1}^{n} p_i\, dx_i. \tag{2}$$

A submodule $\mathcal{L}$ of Ω^1 is called a *contact structure* on X if $\mathcal{L}$ is generated by a contact form. The pair $(X, \mathcal{L})$ is the germ of a contact manifold. A holomorphic map $\Phi : X \to X$ is called a *contact transformation* if for each generator ω of $\mathcal{L}$ there is $\varphi \in \mathcal{O}$ such that

$$\Phi^*\omega = \varphi\omega \quad \text{and} \quad \varphi(o) \neq 0.$$

Let M be a copy of $\mathbb{C}^{n+1}$ with coordinates $(x_1, \dots, x_{n+1})$. Let $\xi_1, \dots, \xi_{n+1}$ be holomorphic functions on the cotangent bundle T^*M of M such that

$$\theta = \sum_{i=1}^{n+1} \xi_i\, dx_i \tag{3}$$

is the canonical 1-form of T^*M.

Example 2.1. We call the projectivization $\mathbb{P}^*M$ of the vector bundle T^*M the *projective cotangent bundle* of M. Setting $p_i = -\xi_i/\xi_{n+1}$ and $y = x_{n+1}$, $i = 1, \dots, n$, (1) defines a system of local coordinates of $\mathbb{P}^*M$ at the point $(o, \langle dy \rangle)$ such that θ/ξ_{n+1} induces the contact form (2) on an open set of $\mathbb{P}^*M$. The germ of $\mathbb{P}^*M$ at $(o, \langle dy \rangle)$ and the contact structure generated by (2) define a germ of contact manifold.

Let $\pi : T^*M \to M$ be the cotangent bundle of a complex manifold M. A submanifold Λ of T^*M is called a *conic Lagrangean submanifold of* T^*M if Λ has the same dimension as M and $\theta|_\Lambda = 0$. Let N be a submanifold of M. There is one and only one conic Lagrangean submanifold Λ of T^*M such that $\pi(\Lambda) = N$. We denote Λ by $T^*_N M$ and call Λ the *conormal* of N. Given local coordinates $(x_1, \dots, x_{n+1})$ on an open subset V of M such that $N \cap V = \{x_1 = \cdots = x_k = 0\}$,

$$T^*_N M \cap \pi^{-1}(V) = \{x_1 = \cdots = x_k = \xi_{k+1} = \cdots = \xi_{n+1} = 0\},$$

where $\xi_1, \dots, \xi_{n+1}$ are holomorphic functions such that $\theta|_{\pi^{-1}(V)} = \sum_{i=1}^{n+1} \xi_i \, dx_i$.

3. $\mathscr{D}$-modules

We shall recall here some results on $\mathscr{D}$-modules. References are made to [4] and [6] for details.

Let X be a complex analytic manifold. We denote by $\mathscr{O}_X$ the sheaf of holomorphic functions on X and by $\mathscr{D}_X$ the sheaf of holomorphic differential operators on X.

Recall that given local coordinates $(x_1, \dots, x_n)$ in an open subset U of X, every differential operator P in U can be written uniquely in the form

$$P = \sum_{|\alpha|=0}^{k} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{4}$$

where $k \in \mathbb{N}_0$ is the order of P, a_α is an holomorphic function in U, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},$$

for each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. Moreover, if one denotes by (x, ξ) the induced coordinates in $T^*U \simeq U \times \mathbb{C}^n$, we associate to P its *principal symbol*:

$$\sigma(P)(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha, \tag{5}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$.

Given a point o of X, let $\mathscr{O}_{X,o}$ (resp. $\mathscr{D}_{X,o}$) be the ring of germs of holomorphic functions (resp. differential operators) at the point o.

(AutoPDF V7 15/9/15 16:06) EMS (170×240mm) Tmath J-3037 PMS, 72:4 () PMU: (IDP) 27/8/2015 pp. 393–406 PMS_72-4_03 (p. 395)

A left $\mathscr{D}_{X,o}$-module is of finite type if it admits a presentation of the type

$$\sum_{j=1}^{s} P_{ij}u_j, \qquad i = 1, \dots, m_1. \tag{6}$$

If $\mathscr{M}$ is a finite type $\mathscr{D}_{X,o}$-module it admits a resolution of the type

$$\cdots \longrightarrow \mathscr{D}_{X,o}^{m_l} \xrightarrow{P_{l-1}} \cdots \xrightarrow{P_2} \mathscr{D}_{X,o}^{m_2} \xrightarrow{P_1} \mathscr{D}_{X,o}^{m_1} \xrightarrow{P} \mathscr{D}_{X,o}^{s} \longrightarrow \mathscr{M} \longrightarrow 0. \tag{7}$$

Applying the functor $\mathrm{Hom}_{\mathscr{D}_{X,o}}(\cdot, \mathscr{O}_{X,o})$ to the exact sequence above, after replacing $\mathscr{M}$ by 0, we obtain a complex

$$0 \longrightarrow \mathscr{O}_{X,o}^{s} \xrightarrow{P^t} \mathscr{O}_{X,o}^{m_1} \xrightarrow{P_1^t} \mathscr{O}_{X,o}^{m_2} \xrightarrow{P_2^t} \cdots \xrightarrow{P_{l-1}^t} \mathscr{O}_{X,o}^{m_l} \longrightarrow \cdots. \tag{8}$$

The k-th cohomology group of (8) is denoted by

$$\mathrm{Ext}^k_{\mathscr{D}_{X,o}}(\mathscr{M}, \mathscr{O}_{X,o}), \qquad k \geq 0.$$

The complex vector space $\mathrm{Hom}_{\mathscr{D}_{X,o}}(\mathscr{M}, \mathscr{O}_{X,o}) = \mathrm{Ext}^0_{\mathscr{D}_{X,o}}(\mathscr{M}, \mathscr{O}_{X,o})$ is the space of germs of holomorphic solutions of the homogeneous equation associated to (6). The other Ext^k's contain relevant information about the holomorphic solutions of the inhomogeneous equation associated to (6).

Let $\mathscr{M}$ be a $\mathscr{D}_{X,o}$-module of finite type. We associate to $\mathscr{M}$ a very important geometric invariant, its *characteristic variety* $\mathrm{Char}(\mathscr{M})$, an analytic subset of T_o^*X. Recall that if $\mathscr{M} = \mathscr{D}_{X,o}/\mathscr{I}$, where $\mathscr{I}$ is a left ideal of $\mathscr{D}_{X,o}$, one has:

$$\mathrm{Char}(\mathscr{M}) = \{\theta \in T_o^*X : \sigma(P)(\theta) = 0, \forall P \in \mathscr{I}\}.$$

Let Y be a submanifold of X. Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. The restriction functor $F \mapsto F|_Y$ that associates to a sheaf on X a sheaf on Y does not transform $\mathscr{M}$ into a $\mathscr{D}_Y$-module. In order to perform this operation we need to consider the *transfer module*

$$\mathscr{D}_{Y\to X} = \mathscr{O}_Y \underset{\mathscr{O}_X|_Y}{\otimes} \mathscr{D}_X|_Y,$$

which has a natural structure of $(\mathscr{D}_Y, \mathscr{D}_X|_Y)$-bimodule. We define the restriction of $\mathscr{M}$ to Y as the $\mathscr{D}_Y$-module

$$\mathscr{M}_Y = \mathscr{D}_{Y\to X} \underset{\mathscr{O}_X|_Y}{\otimes} \mathscr{M}|_Y.$$

This operation has an equivalent at the level of germs.

Note that, in general, the restriction to Y of a $\mathscr{D}_{X,o}$-module of finite type need not to be of finite type over $\mathscr{D}_{Y,o}$. That property is only guaranteed if

$$\mathrm{Char}(\mathscr{M}) \cap T_Y^*X \subset T_X^*X. \tag{9}$$

When (9) holds, $\mathscr{M}$ is said to be *non-characteristic* relatively to Y.

We now recall the Cauchy-Kowalevsky-Kashiwara theorem:

Theorem 3.1. *Let Y be a submanifold of X and $o \in Y$. Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. Assume $\mathscr{M}_o$ is a $\mathscr{D}_{X,o}$-module of finite type. If Y is non characteristic for $\mathscr{M}$, $\mathscr{M}_{Y,0}$ is a $\mathscr{D}_{Y,o}$-module of finite type and the natural morphisms*

$$\mathrm{Ext}^k_{\mathscr{D}_{X,o}}(\mathscr{M}_o, \mathscr{O}_{X,o}) \to \mathrm{Ext}^k_{\mathscr{D}_{Y,o}}(\mathscr{M}_{Y,o}, \mathscr{O}_{Y,o}), \qquad k \geq 0 \tag{10}$$

are isomorphisms.

4. Contact transformations

Let $(X, \mathscr{L})$ be a germ of contact manifold of dimension $2n+1$. Let us fix a generator ω of $\mathscr{L}$ and system of local coordinates (1) such that (2) holds. Set $Y = \{p_1 = \cdots = p_n = 0\}$, $x = (x_1, \ldots, x_n)$ and $p = (p_1, \ldots, p_n)$. Denote by m the maximal ideal of $\mathbb{C}\{x, y, p\}$.

Theorem 4.1. *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}\{x, y, p\}, \beta_0 \in \mathbb{C}\{x, y\}$ be power series such that*

$$\frac{\partial \alpha_i}{\partial x_j}, \frac{\partial \beta_0}{\partial y} \in m, \qquad i, j = 1, \ldots, n. \tag{11}$$

There are $\beta, \gamma_1, \ldots, \gamma_n \in \mathbb{C}\{x, y, p\}$ such that $\beta - \beta_0 \in (p)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta, \gamma_1, \ldots, \gamma_n$ define an infinitesimal contact transformation Φ_{α,β_0} given by

$$(x, y, p) \mapsto (x_1 + \alpha_1, \ldots, x_n + \alpha_n, y + \beta, p_1 + \gamma_1, \ldots, p_n + \gamma_n). \tag{12}$$

The power series β and $\gamma_1, \ldots, \gamma_n$ are uniquely determined by these conditions.

If Φ is the contact transformation given by (12), $\Phi = \Phi_{\alpha,\beta_0}$, where $\beta_0 = \beta|_Y$.

Proof. The map (12) is a contact transformation if and only if there is $\varphi \in \mathbb{C}\{x_1, \ldots, x_n, y, p_1, \ldots, p_n\}$ such that $\varphi(0) \neq 0$ and

$$d(y + \beta) - \sum_{i=1}^n (p_i + \gamma_i) d(x_i + \alpha_i) = \varphi\Big(dy - \sum_{i=1}^n p_i \, dx_i\Big). \tag{13}$$

The condition (13) holds if and only if the conditions

$$\frac{\partial\beta}{\partial p_j} = \sum_{i=1}^{n} (p_i + \gamma_i) \frac{\partial\alpha_i}{\partial p_j}, \tag{14}$$

$$\varphi = 1 + \frac{\partial\beta}{\partial y} - \sum_{i=1}^{n} (p_i + \gamma_i) \frac{\partial\alpha_i}{\partial y}, \tag{15}$$

$$-p_j\varphi = \frac{\partial\beta}{\partial x_j} - \sum_{i=1}^{n} (p_i + \gamma_i) \left(\frac{\partial\alpha_i}{\partial x_j}\right) - (p_j + \gamma_j), \tag{16}$$

hold for $j = 1, \dots, n$. Set

$$D_i = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial y}, \quad i = 1, \dots, n, \quad A_{ij} = \delta_{ij} + D_i\alpha_j, \quad i, j = 1, \dots, n.$$

It follows by (15) and (16) that:

$$\sum_{j=1}^{n} A_{ij}(p_j + \gamma_j) = D_i\beta + p_i, \quad i = 1, \dots, n. \tag{17}$$

Setting $B_i = D_i\beta + p_i$ and $X_i = p_i + \gamma_i$, $i = 1, \dots, n$, one can write the equations (17) as the matrix identity

$$AX = B,$$

where A (resp. B and C) is the square matrix (resp. column matrices) (A_{ij}) (resp. (B_i) and (X_i)). Hence, by Cramer's rule, there are matrices A_i, $i = 1, \dots, n$, such that

$$p_i + \gamma_i = \frac{|A_i|}{|A|}. \tag{18}$$

Replacing in (14), one gets

$$|A| \frac{\partial\beta}{\partial p_j} = \sum_{i=1}^{n} |A_i| \frac{\partial\alpha_i}{\partial p_j}. \tag{19}$$

On the other hand, developing the determinants along the i-th column, there are $c_{ik} \in m$ such that

$$|A_i| = \sum_{k=1}^{n} (\delta_{ik} + c_{ik})(p_k + D_k\beta), \quad i = 1, \dots, n, \tag{20}$$

(AutoPDF V7 15/9/15 16:06) EMS (170×240mm) Tmath J-3037 PMS, 72:4 () PMU: (IDP) 27/8/2015 pp. 393–406 PMS_72-4_03 (p. 398)

and

$$|A| = \sum_{k=1}^{n} (\delta_{ik} + c_{ik}) A_{ki}, \qquad i = 1, \dots, n. \tag{21}$$

More precisely, c_{ik} belongs to the ideal generated by $(D_j \alpha_l)_{j,l=1,\dots,n}$.
Setting

$$a_{jk} = \sum_{i=1}^{n} (\delta_{ik} + c_{ik}) \frac{\partial \alpha_i}{\partial p_j}, \qquad j, k = 1, \dots, n,$$

(19) can be written as

$$|A| \frac{\partial \beta}{\partial p_j} - \sum_{k=1}^{n} a_{jk} D_k \beta = \sum_{k=1}^{n} a_{jk} p_k. \tag{22}$$

Therefore if β exists, β is the solution of the Cauchy problem

$$P_j \beta = g_j, \qquad j = 1, \dots, n, \qquad \beta - \beta_0 \in (p), \tag{23}$$

where

$$P_j = |A| \frac{\partial}{\partial p_j} - \sum_{k=1}^{n} a_{jk} D_k,$$

and

$$g_j = \sum_{k=1}^{n} a_{jk} p_k, \qquad j = 1, \dots, n.$$

Consider the $\mathscr{D}_X$-module

$$\mathscr{M} = \mathscr{D}_X \Big/ \Big(\sum_{j=1}^{n} \mathscr{D}_X P_j \Big).$$

Since $|A| \neq 0$ near the origin and

$$T^*_Y X = \{(x, y, p, \xi, \eta, \rho) \in T^* X : p = \xi = \eta = 0\},$$

$\mathscr{M}$ is non characteristic relatively to Y in a neighborhood of the origin. Moreover, it follows by the division theorems for $\mathscr{D}$-modules that $\mathscr{M}_{Y,0} \simeq \mathscr{D}_{Y,0}$.

(AutoPDF V7 15/9/15 16:06) EMS (170×240mm) Tmath J-3037 PMS, 72:4 () PMU: (IDP) 27/8/2015 pp. 393–406 PMS_72-4_03 (p. 399)

Therefore, by the Cauchy-Kowalevsky-Kashiwara Theorem

$$\mathrm{Ext}^1_{\mathscr{D}_{X,0}}(\mathscr{M}_0, \mathscr{O}_{X,0}) \simeq \mathrm{Ext}^1_{\mathscr{D}_{Y,0}}(\mathscr{M}_{Y,0}, \mathscr{O}_{Y,0}) = 0. \tag{24}$$

Let us consider differential operators $R_1, \dots, R_n$ such that $\sum_{i=1}^n R_i P_i = 0$. If

$$P_i \beta = g_i, \qquad i = 1, \dots, n, \tag{25}$$

$\sum_{i=1}^n R_i g_i = \sum_{i=1}^n R_i P_i \beta = 0$. Hence, the compatibility condition

$$\sum_{i=1}^n R_i g_i = 0 \tag{26}$$

is a necessary condition for the existence of a solution of (25). Let

$$\mathscr{D}_{X,0}^N \xrightarrow{Q} \mathscr{D}_{X,0}^n \xrightarrow{P} \mathscr{D}_{X,0} \to 0,$$

be a resolution of $\mathscr{M}_0$, where $N \in \mathbb{N}$, $Q \in M_{N \times n}(\mathscr{D}_{X,0})$ and $P = [P_1 \dots P_n]^t \in M_{n \times 1}(\mathscr{D}_{X,0})$.

Since

$$\ker(\mathscr{O}_{X,0}^n \xrightarrow{Q} \mathscr{O}_{X,0}^N)/P\mathscr{O}_{X,0} \simeq \mathrm{Ext}^1_{\mathscr{D}_{X,0}}(\mathscr{M}_0, \mathscr{O}_{X,0}) = 0,$$

the compatibility condition (26) is also sufficient to ensure the existence of a solution of (25), which implies the existence of a solution of the Cauchy problem (23).

Now, if $\sum_{i=1}^n R_i P_i = 0$,

$$\sum_{i=1}^n R_i g_i = \sum_{i=1}^n R_i \Big(\sum_{k=1}^n a_{ik} p_k \Big) = \sum_{i=1}^n R_i \Big(\sum_{k=1}^n a_{ik} D_k y \Big) = \sum_{i=1}^n R_i (P_i y) = 0.$$

Hence, the compatibility condition is verified.

If we combine (18), (20) and (21), it follows that

$$\gamma_i = \frac{\sum_{k=1}^n (\delta_{ik} + c_{ik})(p_k + D_k \beta)}{\sum_{k=1}^n (\delta_{ik} + c_{ik}) A_{ki}} - p_i = \frac{\sum_{k=1}^n (\delta_{ik} + c_{ik})(p_k + D_k \beta - A_{ki} p_i)}{|A|},$$

for $i = 1, \dots, n$. Hence

$$\gamma_i = \frac{\sum_{k=1, k \neq i}^n c_{ik}(p_k + D_k \beta - p_i D_i \alpha_k) + (1 + c_{ii})(D_i \beta - p_i D_i \alpha_i)}{|A|}, \tag{27}$$

for $i = 1, \dots, n$, which is enough to conclude that the γ_i's are also determined by the α_i's and β_0. □

Let $a \in \mathbb{N}^n$. If $a_1 + \cdots + a_n \geq 2$, $\det(a_i - \delta_{ij}) \neq 0$. Hence, the linear system

$$\sum_{k=1}^{n} (a_k - \delta_{kj}) c_k = 1, \qquad j = 1, \ldots, n \tag{28}$$

determines $c_1, \ldots, c_n \in \mathbb{Q}^n$.

Corollary 4.2. *Let $a \in \mathbb{N}^n$ and $\lambda \in \mathbb{C}^*$. Assume that $a_1 + \cdots + a_n \geq 2$. Set*

$$\beta = \lambda p^a, \qquad \alpha_i = c_i \frac{\partial \beta}{\partial p_i}, \qquad \gamma_i = 0, \qquad i = 1, \ldots, n.$$

Then $\alpha_1, \ldots, \alpha_n, \beta, \gamma_1, \ldots, \gamma_n$ define a contact transformation of type (12).

Proof. Set $\beta_0 = 0$. Following the Proof of Theorem 4.1

$$A_{ij} = \delta_{ij}, \qquad B_i = p_i, \qquad |A_i| = p_i, \qquad i, j = 1, \ldots, n.$$

By (18), $\gamma_i = 0$, $i = 1, \ldots, n$. By (20), $c_{ik} = 0$, $i, k = 1, \ldots, n$. Moreover,

$$a_{ji} = \frac{\partial \alpha_i}{\partial p_j}, \qquad g_j = \sum_{k=1}^{n} \frac{\partial \alpha_k}{\partial p_j} p_k, \qquad i, j = 1, \ldots, n,$$

and β is a solution of the Cauchy problem

$$\frac{\partial \beta}{\partial p_j} - \sum_{k=1}^{n} \frac{\partial \alpha_k}{\partial p_j} D_k \beta = \sum_{k=1}^{n} \frac{\partial \alpha_k}{\partial p_j} p_k, \qquad j = 1, \ldots, n, \ \beta \in (p). \tag{29}$$

The initial condition is satisfied, $D_i \beta = 0$, for $i = 1, \ldots, n$, and

$$\sum_{k=1}^{n} \frac{\partial \alpha_k}{\partial p_j} p_k = \sum_{k=1}^{n} p_k \frac{\partial}{\partial p_j} \left(c_k \frac{\partial \beta}{\partial p_k} \right) = \sum_{k=1}^{n} (a_k - \delta_{kj}) c_k \frac{\partial \beta}{\partial p_j} = \frac{\partial \beta}{\partial p_j}. \tag{30}$$

for $j = 1, \ldots, n$. □

Corollary 4.3. *Let $a, b \in \mathbb{N}^n$ and $\lambda \in \mathbb{C}^*$. Assume that $a_1 + \cdots + a_n \geq 2$. Set*

$$\beta_* = \lambda p^a x^b, \qquad \beta_0 = 0, \qquad \alpha_i = c_i \frac{\partial \beta_*}{\partial p_i}, \qquad i = 1, \ldots, n.$$

(AutoPDF V7 15/9/15 16:06) EMS (170×240mm) Tmath J-3037 PMS, 72:4 () PMU: (IDP) 27/8/2015 pp. 393–406 PMS_72-4_03 (p. 401)

Then the associated contact transformation $\Phi_{\alpha,0}$ *is such that*

$$\beta = \beta_* + \varepsilon, \qquad \textit{where } \varepsilon \in (p)^{|a|+1} x^b \tag{31}$$

and

$$\gamma_i = \frac{\sum_{k=1,k\neq i}^{n} (-1)^{i+k} p_k D_i \alpha_k + D_i \beta + \varepsilon_i}{|A|}, \qquad \varepsilon_i \in (p)^{|a|+1} \left(\frac{\partial x^b}{\partial x_l}\right)_{l=1,\dots,n}, \tag{32}$$

$i = 1, \dots, n.$

Proof. It follows from (21) that

$$|A| = \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(\delta_{ik} + c_{ik})}{n} A_{ki}. \tag{33}$$

Hence, (22) can be written as:

$$\sum_{i,k=1}^{n} (\delta_{ik} + c_{ik}) \left(\frac{A_{ki}}{n} \frac{\partial \beta}{\partial p_j} - \frac{\partial \alpha_i}{\partial p_j} D_k \beta - p_k \frac{\partial \alpha_i}{\partial p_j} \right) = 0.$$

Since $A_{ki} = \delta_{ki} + D_k \alpha_i$, β is the solution of the Cauchy problem

$$(1 + W) \frac{\partial \beta}{\partial p_j} = U_j + V_j, \qquad j = 1, \dots, n, \ \beta \in (p),$$

where

$$W = \sum_{i=1}^{n} \left(\frac{c_{ii} + D_i \alpha_i}{n} + \sum_{k=1}^{n} \frac{c_{ik} D_k \alpha_i}{n} \right), \qquad V_j = \sum_{i=1}^{n} \left(p_i \frac{\partial \alpha_i}{\partial p_j} + \sum_{k=1}^{n} c_{ik} p_k \frac{\partial \alpha_i}{\partial p_j} \right),$$

$$U_j = \sum_{i=1}^{n} \left(\frac{\partial \alpha_i}{\partial p_j} D_i \beta + \sum_{k=1}^{n} c_{ik} \frac{\partial \alpha_i}{\partial p_j} D_k \beta \right), \qquad j = 1, \dots, n.$$

Since $U_j \in (p)^{|a|-2} x^b$, $j = 1, \dots, n$ and $V_j \in (p)^{|a|-1} x^b$,

$$\frac{\partial \beta}{\partial p_j} \in (p)^k x^b, \qquad j = 1, \dots, n, \tag{34}$$

with $k = |a| - 2$. Since $\beta \in (p)$, $\beta, U_j \in (p)^{|a|-1} x^b$, $j = 1, \dots, n$. Therefore, (34) holds with $k = |a| - 1$ and $\beta, U_j \in (p)^{|a|} x^b$, $j = 1, \dots, n$. Since $W, c_{ik} \in (p)$,

(AutoPDF V7 15/9/15 16:06) EMS (170×240mm) Tmath J-3037 PMS, 72:4 () PMU: (IDP) 27/8/2015 pp. 393–406 PMS_72-4_03 (p. 402)

$i, k = 1, \ldots, n$

$$\frac{\partial \beta}{\partial p_j} \equiv \sum_{i=1}^{n} p_i \frac{\partial \alpha_i}{\partial p_j} \mod (p)^{|a|} x^b, \qquad j = 1, \ldots, n. \tag{35}$$

Repeating the argument of (30),

$$\frac{\partial \beta_*}{\partial p_j} = \sum_{i=1}^{n} p_i \frac{\partial \alpha_i}{\partial p_j}, \qquad j = 1, \ldots, n. \tag{36}$$

By (35) and (36),

$$\frac{\partial (\beta - \beta_*)}{\partial p_j} \in (p)^{|a|} x^b, \qquad j = 1, \ldots, n.$$

Since $\beta - \beta_* \in (p)$, (31) follows.

Let now $\gamma_1, \ldots, \gamma_n$ as in Theorem 4.1.

If we denote by I the ideal generated by $(D_\ell \alpha_r)_{\ell, r = 1, \ldots, n}$, it is easy to prove that, if $n \geq 2$ and $i, k = 1, \ldots, n$, $i \neq k$, one has $c_{ik} = (-1)^{i+k} D_k \alpha_i + d_{ik}$, for some $d_{ik} \in I^2$, which entails (32). □

Let $\mathscr{C}$ be the group of germs of contact transformations of $(X, \mathscr{L})$. Set

$$\mu = \{u \in T_o X : \alpha(o)(u) = 0 \text{ for each } \alpha \in \mathscr{L}\}.$$

Since

$$(d\omega)(o) = \sum_{i=1}^{n} dx_i \, dp_i$$

and $\mu = T_o\{y = 0\}$,

$$\sigma = (d\omega)(o)|_\mu$$

is a linear symplectic form on μ. Given $\varphi \in \mathscr{O}$ such that $\varphi(o) \neq 0$,

$$d(\varphi\omega)(o)|_\mu = \varphi(o)(d\omega)(o)|_\mu + (d\varphi)(o)\omega(o)|_\mu = \varphi(o)\sigma.$$

Hence σ is canonically defined modulo the product by a non vanishing scalar.

If $\Phi \in \mathscr{C}$,

$$d\Phi^* y = \Phi^* \, dy = (\Phi^* \omega)(o) = \varphi(o)\omega(o) = \varphi(o) \, dy.$$

(AutoPDF V7 15/9/15 16:06) EMS (170×240mm) Tmath J-3037 PMS, 72:4 () PMU: (IDP) 27/8/2015 pp. 393–406 PMS_72-4_03 (p. 403)

Hence

$$\Phi^* y = \lambda y + \varepsilon, \tag{37}$$

where $\lambda = \varphi(o) \in \mathbb{C}^*$ and $\varepsilon \in m^2$. Moreover, $\Phi^*(dy)(o) = \lambda\, dy(o)$. Therefore

$$D\Phi(o)(\mu) = \mu.$$

Let $\mathscr{G}$ be the group of $\Phi \in \mathscr{C}$ such that for each $\alpha \in \mathscr{L}$, there is $\varphi \in \mathscr{O}$ such that

$$\Phi^*\alpha = \varphi\alpha, \quad \text{and} \quad \varphi(o) = 1.$$

Let $\Phi \in \mathscr{G}$. Since

$$(\Phi^*\, d\omega)(o) = (d\omega)(o),$$

then

$$D\Phi(o)^*\sigma = \sigma.$$

Hence each $\Phi \in \mathscr{G}$ induces a linear symplectic transformation of μ. The symplectic group $Sp(\mu, \sigma)$ is composed of the linear transformations

$$p \mapsto Ap + Bx, \qquad x \mapsto Cp + Dx, \tag{38}$$

such that $A, B, C, D \in M_n(\mathbb{C})$ and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

There is a morphism of groups from $Sp(\mu, \sigma)$ into $\mathscr{G}$ that associates to (38) the *paraboloidal contact transformation* (see [5]) given by (38) and

$$y \mapsto y + \frac{1}{2} p^t C^t A p + p^t C^t B x + \frac{1}{2} x^t D^t B x. \tag{39}$$

We will denote by $\mathscr{P}$ the group of paraboloidal contact transformations.

Let $\mathscr{S}$ be the group of scalar contact transformations σ_λ, $\lambda \in \mathbb{C}^*$, such that

$$\sigma_\lambda(x, y, p) = (\lambda x, \lambda y, p). \tag{40}$$

Let $\mathscr{Q}$ be the group of contact transformations Φ such that $D\Phi(o)$ equals the identity.

Theorem 4.4. *Given $\Phi \in \mathscr{C}$ there are σ_λ, $\Pi \in \mathscr{P}$ and $\Psi \in \mathscr{Q}$ such that*

$$\Phi = \sigma_\lambda \Pi \Psi. \tag{41}$$

Moreover, σ_λ, Π and Ψ are determined by Φ.

The group $\mathscr{Q}$ is the group of contact transformations Φ_{α,β_0} such that

$$\alpha_i, \qquad \frac{\partial \beta_0}{\partial x_j} \in m^2, \qquad \frac{\partial \beta_0}{\partial y} \in m, \qquad i, j = 1, \dots, n. \tag{42}$$

Proof. There is a morphism $\Phi \mapsto \sigma_\lambda$ from $\mathscr{C}$ onto $\mathscr{S}$, where λ is given by (37). Its kernel equals $\mathscr{G}$.

There is a morphism from $\mathscr{G}$ into $\mathscr{P}$, that associates to $\Phi \in \mathscr{G}$ the paraboloidal transformation associated to $D\Phi(o)|_\mu$. Since $\mathscr{P} \subset \mathscr{G}$, this morphism is onto. It follows from (37) that if $D\Phi(o)|_\mu$ equals the identity of μ, $D\Phi(o)$ equals the identity of $T_o X$.

Since $\mathscr{S} \subset \mathscr{C}$ and $\mathscr{P} \subset \mathscr{G}$, the exact sequences

$$1 \to \mathscr{G} \to \mathscr{C} \to \mathscr{S} \to 1 \quad \text{and} \quad 1 \to \mathscr{Q} \to \mathscr{G} \to \mathscr{P} \to 1$$

split. Hence, (41) holds.

Let $\alpha_1, \dots, \alpha_n, \beta, \beta_0, \gamma_1, \dots, \gamma_n$ in the situation of Theorem 4.1. We start by noticing that the condition $\alpha_i \in m^2$, for $i = 1, \dots, n$, entails $c_{ik} \in m$, for $i, k = 1, \dots, n$.

Assume that (42) holds. By (14),

$$\frac{\partial \beta}{\partial p_j} \in m, \qquad j = 1, \dots, n.$$

Hence, it follows by (27) that $\gamma_i \in m^2$. Therefore, $\Phi_{\alpha,\beta_0} \in \mathscr{Q}$.

Conversely, if $\Psi \in \mathscr{Q}$, there are $\alpha_1, \dots, \alpha_n, \beta, \gamma_1, \dots, \gamma_n \in m^2$ such that Ψ equals to (12). Therefore, $\Psi = \Phi_{\alpha,\beta_0}$ and by (27), $D_i\beta \in m^2$, $i = 1, \dots, n$, which in turn implies (42). □

Following the notation of Example 2.1, let N be a smooth hypersurface of M. Let $q \in N$. Assume that f define N in a neighbourhood of q and $df(o) \neq 0$. Consider X the germ of $\mathbb{P}^*M$ at $o = (q, \langle df \rangle)$. Set $\Lambda = T_o T_N^* M$. Given local coordinates $(x_1, \dots, x_n, y)$ on a neighborhood of q such that y defines N near o, the germ of $T_N^* M$ at o equals

$$\{y = p_1 = \cdots = p_n = 0\}.$$

Set

$$\mathscr{C}_N = \{\Phi \in \mathscr{C} : D\Phi(\sigma)(\Lambda) = \Lambda\}, \qquad \mathscr{P}_N = \mathscr{C}_N \cap \mathscr{P}.$$

(AutoPDF V7 15/9/15 16:06) EMS (170×240mm) Tmath J-3037 PMS, 72:4 () PMU: (IDP) 27/8/2015 pp. 393–406 PMS_72-4_03 (p. 405)

Note that $\mathscr{P}_N$ is the group of paraboloidal contact transformations defined by A, B, C, D such that $D = 0$.

Corollary 4.5. *Given* $\Phi \in \mathscr{C}_N$*, there are* $\sigma_\lambda \in \mathscr{S}$, $\Pi \in \mathscr{P}_N$ *and* $\Psi \in \mathscr{Q}$ *such that* (41) *holds. Moreover,* σ_λ, Π *and* Ψ *are determined by* Φ.

Remark 4.6. We can call the linear map $D\varphi(o)$ the infinitesimal model of the holomorphic map φ. In the say same spirit, it makes sense to look at a symplectic map $D\Psi(o)$ as the infinitesimal model of a symplectic map Ψ. Can we find an "infinitesimal model" for a contact transformation? Indeed, there is no "contact linear algebra". Nevertheless, if $\Phi \in \mathscr{G}$, $D\Phi(o)|_\mu : \mu \to \mu$ is a symplectic linear map. The paraboloidal contact transformation Π behaves as an infinitesimal model for Φ. If $\Phi \in \mathscr{C}$, $\sigma_\lambda \Pi$ behaves as an infinitesimal model of Φ.

Notice that groups $\mathscr{S}$ and $\mathscr{P}$ depend on the choice of the system of local coordinates (1).

References

[1] A. Araújo and O. Neto, *Moduli of Germs of Legendrian Curves*, Ann. Fac. Sci. Toulouse Math., Vol. XVIII, 4, (2009), 645–657.

[2] A. Martins, M. Mendes and O. Neto, *Deformations of Legendrian curves*, in preparation.

[3] J. Cabral and O. Neto, *Microlocal versal deformations of the plane curves* $y^k = x^n$, C. R. Acad. Sci. Paris, Ser. I 347 (2009), 1409–1414.

[4] M. Kashiwara, *$\mathscr{D}$-modules and microlocal calculus*, Translations of Mathematical Monographs, **217** American Math. Soc. (2003).

[5] M. Kashiwara, *Systems of microdifferential equations*. Progress in Mathematics, 34. Birkhauser, Notes by Teresa Monteiro Fernandes.

[6] P. Schapira, *Microdifferential systems in the complex domain*. Springer-Verlag, (1985).

Received April 3, 2014; revised October 22, 2014

A. R. Martins, Department of Mathematics of the Faculty of Sciences, University of Lisbon, Campo Grande, Lisboa, Portugal
E-mail: ardmmc@gmail.com

O. Neto, CMAF and Department of Mathematics of the Faculty of Sciences, University of Lisbon, Campo Grande, Lisboa, Portugal
E-mail: orlando60@gmail.com

(AutoPDF V7 15/9/15 16:06) EMS (170×240mm) Tmath J-3037 PMS, 72:4 () PMU: (IDP) 27/8/2015 pp. 393–406 PMS_72-4_03 (p. 406)