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Oscillating facets

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Abstract. We study a singular one-dimensional parabolic problem with initial data in the BV space, i.e. the energy space, for various boundary data. We pay special attention to Dirichlet conditions, which need not be satisfied in a pointwise manner. The equation we study has two singular slopes, so that in principle solutions to the value problems may have an infinite number of oscillations, which seems surprising for a parabolic problem. We investigate this issue. We also study the facet creation process and the stopping of solutions caused by the evolution of facets. Our estimate of stopping time is based on the comparison principle for viscosity solutions. For this purpose we show that our solutions are viscosity solutions in the sense of [10].

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1. Introduction

We study here a singular diffusion problem

$$\frac{\partial u}{\partial t} = \big(\mathscr{L}(u_x)\big)_x \qquad \text{in } I_T := (a,b) \times (0,T), \tag{1.1}$$

where the nonlinearity is given by the following formula,

$$\mathscr{L}(p) = \operatorname{sgn}(p+1) + \operatorname{sgn}(p-1). \tag{1.2}$$

We consider various boundary conditions. We pay special attention to Dirichlet data, which need not be satisfied in a pointwise manner.

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Let us explain our motivation to study such problems. There is a sizable chunk of literature devoted to the role of curvature in the models of crystal growth, see [5] for a summary or [17] for further development. In a series of papers, [11], [12], [13], [14], [15], we studied the evolution of the so-called bent rectangles by the weighted mean curvature flow,

$$\beta V = \kappa_\gamma + \sigma. \tag{1.3}$$

The point is, the corners of these bent rectangles were formed by the facets meeting at the right angle. If we choose the local coordinate system in a proper way then, after simplifications preserving the main difficulties, system (1.3) looks like (1.1), this is presented in [15]. The main point is that nonlinearity (1.2) supports facets with different slopes.

Problem (1.1) is interesting even if $\mathscr{L}(p) = \operatorname{sgn}(p)$, see [3], [6], [8], [22], [23] and the references therein. Particularly interesting is the approach to crystal growth is presented by Spohn, [23], who discusses an equation like (1.1), but his choice of $\mathscr{L}$ involves also a degenerate term, which we drop for the sake of the simplicity of analysis.

The nonlinearity, we consider here, appears naturally, when we consider a corner formed by two evolving facets, (see below, cf. [15]). By a facet we mean a part of the graph of a solution to (1.1) with the slope corresponding to a jump in $\mathscr{L}$. In the present case, facets have slope ± 1. Facets will be defined rigorously in Subsection 4.1.

Our main objective is to study interactions of facets, especially in the case of oscillating data. In order to make equation (1.1) well-posed, we augment it with initial condition

$$u(x,0) = u_0(x) \qquad \text{for } x \in I := (a,b) \tag{1.4}$$

and either Dirichlet,

$$u(a) = A, \qquad u(b) = B, \tag{1.5}$$

periodic

$$u(a) = u(b) \tag{1.6}$$

or Neumann boundary data,

$$\mathscr{L}(u_x)|_{\partial I} = 0. \tag{1.7}$$

We put a cap on the oscillatory behavior of the data by requiring that $u_0 \in BV$.

One of the emerging problems is the meaning of (1.5). It turns out that our definition of solutions is too weak to guarantee that (1.5) is satisfied in a pointwise manner: the trace of solutions to (1.1), (1.4) and (1.5) need not be equal to the boundary data, but is suffices to prove existence and uniqueness. We elaborate on this in Definition 2.1 in Subsection 2.1 and §3, see also [1], [2], [20].

We notice that initial conditions from BV may have infinitely many facets of different slopes. We would like to determine if this is possible for any solution at $t > 0$. We shall see that most of the facet interactions are resolved instantly. Thus, at $t > 0$, we may have only a finite number of facets with non-zero curvature, see Theorem 4.5.

Our task involves re-examining the existence result of [21], because we consider less regular data than there. It is helpful to observe that (1.1) is formally a gradient flow of functional E on $L^2(I)$, defined by

$$E(u) = \int_I W(u_x)\, ds,$$

where $W(p) = |p+1| + |p-1|$. Obviously, E is well-defined iff $u \in BV(I)$. Thus, we will seek solutions with finite energy if $u_0 \in BV(I)$.

We also have to discuss the notion of a solution to (1.1) defined in [21], Theorem 1, because smooth solutions to the approximating system satisfying the Dirichlet data need not satisfy them in the limit if the convergence is too weak. In order to expose the issue of the Dirichlet boundary data, we will present explicit solutions in Proposition 4.13. We make additional comments when we characterize the steady states in Section 3.

We mention in passing that by a solution we mean a pair (u, Ω), where $\Omega(\cdot, t)$ is a selection of the subdifferential $\partial E\big(u(\cdot, t)\big)$. More details will be given in Section 2. It turns out that due to continuity of Ω studying its values gives a lot of information about solutions. In many instances, see Section 3, Subsection 4.1, this is our major tool.

Once the existence of solutions is established, we will characterize the steady states for all three boundary conditions. This is done in Section 3. In principle, they belong to BV. We will see that if A does not differ much from B, then the steady states are Lipschitz continuous functions satisfying (1.5) and such that

$$|u_x| \le 1. \tag{1.8}$$

In turns out however, that if the difference $B - A$ is big, then there are also discontinuous steady states belonging to BV. In other words the BV regularity of the steady states is optimal. On the other hand, we note that all steady states with the homogeneous Neumann data are Lipschitz functions satisfying (1.8).

Condition (1.8) permits seemingly unchecked oscillations. This seems surprising. We will present two justifications of this phenomenon. Namely, we notice at all times $t > 0$ there are only finitely many facets with non-zero curvature, see Theorem 4.5. The other explanation is that our solutions are viscosity solutions in the sense of [10]. We will see that in Section 5. In addition, the theory of viscosity solutions gives us a powerful tool like the Comparison Principle, see Theorem 5.3. It is used in the proof of the main result of Subsection 4.2, i.e. estimates on the stopping time of solutions, i.e. the time, after which the solution no longer changes. Results on the extinction times for solutions to the second-order total variation flow, the fourth-order total variation flow and a fourth-order surface diffusion law are estblished in [9]. The authors of [9] use quite a different approach, i.e. their main tools are energy estimates and Sobolev inequality.

In Section 4, we study the regularizing action of the flow when $u_0 \in BV$. We constructed solutions in the energy space, i.e., $u(t) \in BV$, by the way of examples, see Section 4.3, we shall see that discontinuities in u_0 persist. A more interesting observation is that $u_0 \in BV(I)$ implies that $u_t \in L^2(I_T)$ and this statement carries a lot of information about regularity and oscillatory behavior of solutions. Namely, for almost all $t > 0$ we have $u_t(\cdot, t) \in L^2(I)$. This implies that the number of facets with non-zero curvature is finite for almost all $t > 0$, see Theorem 4.5. The argument is based on the observation that $\Omega(\cdot, t) \in \partial E\big(u(\cdot, t)\big)$.

As we mentioned, $u(t)$ may have jumps as well as $u_x(t)$. We will see, see Theorem 4.6, that jumps of the derivative may not be arbitrary. This fact is well-known for the crystalline motion, see [7], [24] and the references therein. If at x_0 the interval with endpoints $u_x^+(x_0)$, $u_x^-(x_0)$ contains any of the singular slopes from $\{-1, 1\}$, then immediately the missing facet is created for $t > 0$. A similar statement holds if u has a jump discontinuity at x_0.

The conclusion that for almost all $t > 0$ solution $u(\cdot, t)$ has a finite number of non-zero curvature facets permits more detailed studies of the equations of the facet motion. In Section 4, we concentrate on estimates in terms of initial data. We see that facet interaction is the main mechanism for the stopping of solutions. Due to the fact that we have only a finite number of moving facets the task is easier. Our main tool is the comparison principle, Theorem 5.3, for viscosity solutions established in [10]. In particular, we give in Theorem 4.11 a simple (but not a closed formula) estimate for the stopping time in terms of the data.

In the last Section we show that the solutions we constructed are also viscosity solution in the sense of [10]. In the proof we consider all possible configurations of the facets, as a result the argument is not short, so the limitations of this method are clear.

2. Existence reexamined

We study (1.1) with either Dirichlet, periodic or Neumann boundary data. The first type of data requires a bit different treatment than the remaining ones. We introduce here the definition of solutions to (1.1) with various boundary data.

Once we cast our problem as a gradient flow

$$u_t \in -\partial E(u), \qquad u(0) = u_0. \tag{2.1}$$

for a proper convex and lower semicontinuous functional E, then we could invoke the general results on nonlinear semigroups, see [4], to conclude existence, uniqueness of solutions and their basic properties. This approach will work nicely if E is one of the following functionals on L^2,

$$\begin{aligned} E_1(u) &= \begin{cases} \int_I W(Du) & u \in BV(I), \\ +\infty & u \in L^2(I)\setminus BV(I), \end{cases} \\ E_2(u) &= \begin{cases} \int_{\mathbb{T}} W(Du) & u \in BV(\mathbb{T}), \\ +\infty & u \in L^2(\mathbb{T})\setminus BV(\mathbb{T}), \end{cases} \end{aligned} \tag{2.2}$$

where $\mathbb{T}$ is a flat one-dimensional torus identified with $[0, b)$, because E is proper convex and lower semicontinuous on L^2. These two functionals correspond to (1.1) with Neumann and periodic boundary data.

We stress that it is well-known, see [2] for the multidimensional case, that the case of Dirichlet is more difficult and the boundary need not be satisfied pointwise. This is so, because functional on $L^2(I)$, given by

$$E(u) = \begin{cases} \int_I |u_x - 1| + |u_x + 1| & \text{for } u \in BV(I),\ \gamma u(a) = A,\ \gamma u(b) = B, \\ +\infty & \text{else}, \end{cases}$$

is not lower semicontinuous on $L^2(I)$. In the formula above and throughout the paper we denote by γu the trace of u as a function from $BV(I)$, see [25].

In these circumstances we prefer to use the regularization technique to show existence of solutions to (1.1) with Dirichlet data. The advantage is that we see a justifications of the definition of the solution we adopt here.

Definition 2.1. We shall say that function $u \in L^2\big(0, T; L^2(I)\big)$ is a solution to (1.1) if $u \in L^\infty\big(0, T; BV(I)\big)$ and $u_t \in L^2\big(0, T; L^2(I)\big)$ and there is $\Omega \in L^2(0, T; W^{1,2})$, which satisfy the identity

$$\langle u_t, \varphi \rangle = -\int_I \Omega \varphi_x \, dx \tag{2.3}$$

for all test functions $\varphi \in C_0^\infty(I)$ and for almost every $t > 0$.

(A) We shall say that u, a solution to (1.1), satisfies the Neumann data (1.7), if

$$\Omega|_{\partial I} = 0 \qquad \text{for } a.e.\ t > 0.$$

(B) We shall say that u, a solution to (1.1), satisfies the Dirichlet data (1.5) at $x = a$ and $t > 0$ if

$$\gamma u(a) = A \qquad \text{or} \qquad \begin{cases} \text{if } \gamma u(a) > A, & \text{then } \gamma\Omega(a) = 2, \\ \text{if } \gamma u(a) < A, & \text{then } \gamma\Omega(a) = -2. \end{cases}$$

We shall say that u a solution to (1.1) satisfies the Dirichlet data (1.5) at $x = b$ and $t > 0$ if

$$\gamma u(b) = B \qquad \text{or} \qquad \begin{cases} \text{if } \gamma u(b) > B, & \text{then } \gamma\Omega(b) = -2, \\ \text{if } \gamma u(b) < B, & \text{then } \gamma\Omega(b) = 2. \end{cases}$$

We notice that the time regularity postulated in Definition 2.1 implies that solutions to (1.1) are in $C\big([0,T]; L^2(I)\big)$. Hence, we can impose initial conditions (1.4). At this point we stress that Ω is a selection of the composition of multi-valued operators $\mathscr{L} \circ u_x$.

Remark 2.2. We also notice that our definition of solutions to (1.1) with Dirichlet boundary data coincides with that used in [1], [2] or [20].

Here is ou existence result. We note that we consider less regular initial conditions than in [21].

Theorem 2.3. *Let us suppose that* $u_0 \in BV$, *then*

(1) *there exists a unique solution to* (1.1) *with boundary conditions* (1.5), *where* $A, B \in \mathbb{R}$;

(2) *there exists a unique solution to* (1.1) *with boundary conditions* (1.7);

(3) *there exists a unique solution to* (1.1) *with periodic boundary conditions* (1.6).

Moreover, for almost all $t > 0$

$$\int_I [W(u_x + h_x) - W(u_x)]\, dx \geq \int_I \Omega h_x \, dx, \tag{2.4}$$

where $h \in C_0^\infty(I)$.

Sketch of the proof. Parts (2) and (3) are conclusions from the classical semigroup theory, see [4], applied to E_1 and E_2 defined in (2.2). We can do this, because it is easy to check that E_1 and E_2 are proper, convex and lower semicontinuous. The choice of $\Omega(\cdot, t)$ is made implicitely in the following property of solution $\frac{d^+}{dt} u(t) = \partial E^o\big(u(t)\big)$, where $\partial E^o\big(u(t)\big)$ denotes the canonical selection of $\partial E\big(u(t)\big)$, hence (2.4) follows.

Below we present an argument for Dirichlet boundary data based on regularization. It is valid for other boundary data too. Due the regularity of such solutions and those obtained by the theory on nonlinear semigroups, it is easy to see that these types of solutions coincide.

Step 1. After regularizing $\mathscr{L}$ and u_0 we obtain a uniformly parabolic problem,

$$\begin{aligned} &\frac{\partial u^\epsilon}{\partial t} = \big(\mathscr{L}^\epsilon(u^\epsilon_x)\big)_x, && (x,t) \in I_T, \\ &u^\epsilon(x,0) = u^\epsilon_0(x), && x \in I, \\ &u^\epsilon(a,t) = A,\ u^\epsilon(b,t) = B, && t > 0, \end{aligned} \tag{2.5}$$

where ϵ is a regularizing parameter, $\mathscr{L}^\epsilon(p) = (\mathscr{L} * \eta_\epsilon)(p) + \epsilon p$ and $W^\epsilon(p) = (W * \eta_\epsilon)(p) + \frac{\epsilon}{2}p^2$. By the classical theory, see [18], we obtain existence and uniqueness of smooth solutions to (2.5).

From (2.5) we reach the following conclusion

$$\int_0^T \int_I (u^\epsilon_t)^2\, dx\, dt + \int_I W^\epsilon\big(u^\epsilon_x(x,T)\big)\, dx = \int_I W^\epsilon(u^\epsilon_{0,x})\, dx. \tag{2.6}$$

Now, we will pass to the limit. First of all, we notice that

$$\int_I W^\epsilon(u^\epsilon_{0,x}) \le 3|I| + 2 \sup_{\epsilon \in [0,1]} \int_I W(u^\epsilon_{0,x}) + \int_I \frac{\epsilon}{2} |u^\epsilon_{0,x}|^2\, dx =: M,$$

where we used that

$$\frac{\epsilon}{2} \int_I |u^\epsilon_{0,x}|^2\, dx \le C\|u_0\|^2_{BV}.$$

Since our bound on the right-hand-side (RHS) of (2.6) is independent of ϵ we conclude that

$$\int_0^T \int_I (u^\epsilon_t)^2 \le M \quad \text{and} \quad \int_I W^\epsilon\big(u^\epsilon_x(x,t)\big) \le M \quad \text{for all } t \in [0,T].$$

Thus, we can select a subsequence $\{u^\epsilon\}$ such that

$$u^\epsilon \rightharpoonup u \quad \text{in } L^2\big(0,T;L^2(I)\big) \qquad \text{and} \qquad u^\epsilon_t \rightharpoonup u_t \quad \text{in } L^2\big(0,T;L^2(I)\big).$$

Furthermore, by Aubin Lemma we deduce that u^ϵ converges to u in $L^p\big(0,T,L^q(I)\big)$, where p, q are arbitrary from the interval $(1,\infty)$. As a result, $\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^q} \to 0$, for almost all $t \in (0,T)$, when ϵ goes to 0. By the lower semicontinuity of the BV norm and by

$$\int_I W(Dv) = \int_I |D(v+x)| + |D(v-x)|, \tag{2.7}$$

we arrive at

$$\int_0^t \int_I u_t^2(x,s)\,dx\,ds + \int_I W(Du)(\cdot,t)\big)\,dx \le M \qquad \text{for almost all } t \in (0,T). \tag{2.8}$$

We note that (2.8) does not involve any statement on the boundary values of u.

Moreover, we have a bound on the BV norm of $u(\cdot,t)$,

$$\int_I |Du| \le \frac{1}{2}\int_I |D(u+x)| + \frac{1}{2}\int_I |D(u-x)| = \frac{1}{2}\int_I W(u_x)\,dx.$$

We also have to indicate a candidate for Ω as required by the definition of a solution. We set

$$\Omega^\epsilon(x,t) := \mathscr{L}^\epsilon\big(u^\epsilon(x,t)\big).$$

Since $u_t^\epsilon = \Omega_x^\epsilon$, then due to (2.6) we deduce that

$$\|\Omega^\epsilon\|_{L^2(0,T;H^1(I))} \le M_1 < +\infty. \tag{2.9}$$

Hence, we can select a subsequence,

$$\Omega^\epsilon \rightharpoonup \Omega \qquad \text{in } L^2\big(0,T;H^1(I)\big).$$

Moreover,

$$\int_0^T \int_I u_t\varphi\,dx\,dt = \int_0^T \int_I \Omega_x\varphi\,dx\,dt \qquad \text{for all } \varphi \in C_0^\infty\big((0,T)\times I\big).$$

At this point, we may apply [21], Lemma 2.1 to conclude that (2.3) holds. Moreover, [21], Lemma 2.2 implies (2.4).

Step 2. We have to show that u, the limit of solutions to the regularized problems, is a solution to (1.1) with boundary conditions, in the sense of Definition 2.1.

Let us suppose that t is such that $u(\cdot,t) \in BV(I)$ and $\Omega(\cdot,t) \in W^{1,2}(I)$. We consider first $x = a$ a boundary point of I. If $\gamma u(a,t) = A$, then u satisfies the Dirichlet boundary data. Let us suppose that $A + \delta := \gamma u(a,t) > A$. This means that for *any* sequence $\{x_n\}$, $a > x_n$, converging to a, we have $\lim_{n\to\infty} u(x_n,t) =$

$A+\delta$. Then, we select N such that for all $n > N$,

$$A + \frac{1}{2}\delta < u(x_n, t) = u(x_n, t) - u^\epsilon(x_n, t) + u^\epsilon(x_n, t).$$

Smooth solutions u^ϵ satisfy the boundary conditions in the above inequality, this implies that

$$\frac{1}{2}\delta < u(x_n, t) - u^\epsilon(x_n, t) + u_x^\epsilon(c_n, t)(x_n - a), \qquad c_n \in (a, x_n).$$

Since $u^\epsilon(\cdot, t)$ are commonly bounded in BV, then we deduce with the help of Helly's theorem that there is a subsequence of $u^\epsilon(\cdot, t)$, (we abstain from introducing a new notation), that converges to $u(\cdot, t)$ everywhere.

We fix n such that $\frac{\delta}{4(x_n - a)} > 1$. Next, we select $\epsilon > 0$ so that $u(x_n, t) - u^\epsilon(x_n, t) < \frac{1}{4}\delta$. Combining these facts, we reach $u_x^\epsilon(c_n, t) > 1$, hence $\Omega^\epsilon(c_n, t) = 2$. As a result, we conclude that

$$\gamma\Omega(a, t) = 2,$$

as desired. The analysis of the remaining cases is similar, we leave it to the interested reader. We conclude that u is indeed a solution to (1.1) satisfying (1.5).

Step 3. We shall establish uniqueness of solutions in the case Dirichlet boundary conditions. For the sake of simplicity of notation, we assume that $[a, b] = [0, b]$.

We notice that if u is a solution to (1.1), according to Definition 2.1, then $t \mapsto u(t) \in L^2(I)$ is continuous, in particular it makes sense to evaluate u at $t = 0$.

Let us suppose that u and v are solutions to (1.1) satisfying (1.5) with $u(0) = u_0 = v(0)$. For the sake of simplicity we assume that A and B in (1.5) are zero. We will extend u and v antisymetrcially to functions $\tilde{u}$, $\tilde{v}$, on $\tilde{I} = [-b, b]$. In the same way we extend $\Omega(u)$ and $\Omega(v)$, to $\tilde{I}$. We notice that $\tilde{u}$, $\tilde{v}$, $\tilde{\Omega}(\tilde{u})$ and $\tilde{\Omega}(\tilde{v})$ may be extended as periodic functions with period $2b$ to the line $\mathbb{R}$. We recall, see e.g. [19], that if $w \in H^1(I)$, $\varphi \in BV(I)$, then

$$\int_I w_x \varphi \, dx = -\int_I w D\varphi + \gamma(w\varphi)|_a^b.$$

We conclude with the help of this formula that

$$\frac{1}{2}\|\tilde{u} - \tilde{v}\|^2_{L^2(\tilde{I})}(T) = -\int_0^T \int_{\tilde{I}+\delta} \big(\tilde{\Omega}(\tilde{u}; x, t) - \tilde{\Omega}(\tilde{v}; x, t)\big)(\tilde{u}_x - \tilde{v}_x)\, dx\, dt,$$

$x = \delta$ is a point of continuity of $\tilde{u} - \tilde{v}$. On the other hand, monotonicity of $\mathscr{L}$ yields

$$\frac{1}{2}\|\tilde{u} - \tilde{v}\|^2_{L^2(I)}(\tau) \le 0. \tag{2.10}$$

We conclude that $u = v$, as desired. □

The same argument yields the contraction property for solutions, u, v, to (1.1) with data (1.5),

$$\|u - v\|_{L^2}(t_2) \le \|u - v\|_{L^2}(t_1), \qquad \|u(t_2)\|_{L^2} \le \|u(t_1)\|_{L^2}, \qquad t_2 > t_1. \tag{2.11}$$

These facts for solutions (1.1) with either Neumann or period data follow from the semigroup theory. We may summarize these observations in:

Corollary 2.4. *If u and v are two solutions to* (1.1) *satisfying Neumann, periodic or the same Dirichlet boundary conditions, then* (2.11) *hold.* □

It is also easy to establish:

Corollary 2.5. (a) *Let us suppose that u^n, Ω^n is a sequence of solutions to* (1.1) *such that $u^n \to u$ in $L^2(I_T)$ and $\Omega^n \rightharpoonup \Omega$ in $L^2(0, T; H^1(I))$, then u and Ω form a solution to* (1.1).

(b) *Let us suppose that $u_0^n \in C^\infty$, $u_0^n \to u_0$ in $L^2(I)$ and* $\sup \|u_0^n\|_{BV} < \infty$. *If $u_n \to u$ is a sequence of solutions to* (1.1) *with initial data u_0^n, then,*

$$u_n \to u \quad \text{in } L^2(I_T), \qquad \Omega_n \rightharpoonup \Omega \quad \text{in } L^2(0, T; H^1(I)).$$

Hence, u and Ω form a solution to (1.1) *with initial data u_0.* □

The following estimate for u_t is crucial for the rest of this paper.

Proposition 2.6. *Suppose that $u_0 \in BV(I)$ and u is the corresponding solution to* (1.1) *with either* (1.5), (1.6) *or* (1.7) *boundary conditions. Then, for a.e. $t \in (0, T)$,*

$$t\int_I u_t^2(t,x)\,dx \le \int_0^t\int_I u_t^2(s,x)\,dx\,ds \le M < \infty.$$

Proof. We proceed formally by differentiating equation (1.1) with respect to t and testing it with $u_t\varphi$, where φ is non-negative and it depends only on t and $\varphi(0) = 0$. We have

$$u_{tt}u_t\varphi = \mathscr{L}(u_x)_{xt}u_t\varphi.$$

Next, we integrate the above equation over I:

$$\frac{1}{2}\int_I \varphi \frac{d}{dt} u_t^2 = \int_I \mathscr{L}(u_x)_{xt} u_t \varphi.$$

We integrate by parts the right hand side of the above equations, then

$$\frac{1}{2}\int_I \varphi \frac{d}{dt} u_t^2 = -\int_I \mathscr{L}(u_x)_t u_{tx} \varphi.$$

Notice that, due to monotonicity of $\mathscr{L}$, we have $\mathscr{L}(u_x)_t u_{xt}\varphi = \mathscr{L}'(u_x) u_{xt}^2 \varphi \geq 0$. Hence,

$$0 \geq \int_I \varphi \frac{d}{dt} u_t^2 \, dx.$$

We integrate the above equation over $[0, t]$, then

$$0 \geq \int_0^t \int_I \varphi \frac{d}{dt} u_t^2 \, dx \, ds = -\int_0^t \int_I \varphi' u_t^2 \, dx \, ds + \int_I \varphi u_t^2(t) \, dx$$

If $\varphi(t) = t$ and $u_0 \in BV(I)$ then

$$t \int_I u_t^2(t) \, dx \leq \int_0^t \int_I u_t^2 \, dx \, ds \leq M < \infty. \tag{2.12}$$

A rigorous argument is based on approximation. □

Definition 2.7. We shall say that $t > 0$ is *typical* if

$$\int_I |u_t(x,t)|^2 \, dx < \infty \quad \text{and} \quad \int_I |\Omega_x(x,t)|^2 \, dx < \infty.$$

2.1. Discontinuous solutions. The type of initial conditions we consider permits discontinuous solutions. We make an observation about it.

Proposition 2.8. *Let us suppose that $u_0 \in BV(I)$ and u is the corresponding solution to* (1.1). *If $u(\cdot, t_0)$ has a jump discontinuity at x_0 and t_0 is typical, then* $|\Omega(x_0, t_0)| = 2$.

Proof. Let us consider the solutions u^ϵ to the regularized problem, approximating u. Since $u^\epsilon(\cdot, t)$ is a sequence of BV functions, then by Helly Theorem, we can select a subsequence (denoted by u^ϵ) converging to u everywhere. If Δ is the absolute value of the jump, then for a given ϵ we find δ_ϵ such that

$$\frac{1}{2}\Delta < |u^\epsilon(x_0+\delta_\epsilon, t_0) - u^\epsilon(x_0-\delta_\epsilon, t_0)| = 2|u^\epsilon_x(c_\epsilon, t_0)|\delta_\epsilon.$$

Thus, $|u^\epsilon_x(c_\epsilon, t_0)| > 1$, as a result $|\Omega^\epsilon(c_\epsilon, t_0)| = 2$. We can see that $|\Omega^\epsilon(c_\epsilon, t_0)| \to |\Omega(x_0, t_0)| = 2$, because $c_\epsilon \to x_0$. □

In the next section, we will discuss steady states of (1.1). We will see that jump discontinuities of the solution are allowed also in steady states.

3. Steady states

We describe the multitude of the steady states and we will consider all three boundary conditions. We note that we frequently interchange symbols Ω and $\mathscr{L}(u_x) \in H^1(I)$. Here is our first observation.

Proposition 3.1. (a) *Let us suppose that a BV function u is a steady state solution to* (1.1), *i.e. there is* $\Omega \in H^1$, *understood as* $\mathscr{L}(u_x)$ *satisfying*

$$\big(\mathscr{L}(u_x)\big)_x = 0,$$

then $\mathscr{L}(u_x)$ *is a constant from the set* $\{\pm 2, \pm 1, 0\}$.

(b) *Let us suppose that* $u \in BV(I)$ *is as in* (a), *but it is not Lipschitz continuous, then* $\mathscr{L}(u_x)$ *is a constant from the set* $\{2, -2\}$.

Proof. We begin with part (a). Of course, $\mathscr{L}(u_x)$ is a constant from interval $[-2, 2]$. Let us suppose that $u_x > 0$ on a set E of a positive measure. Thus, on this set we have $\operatorname{sgn}(u_x + 1) = 1$, as a result $\mathscr{L}(u_x) \geq 0$ independently of the values of $\operatorname{sgn}(u_x - 1)$ on E. Let us suppose that $\mathscr{L}(u_x) \in (1, 2)$ on a set of positive measure. We know that since $\mathscr{L}(u_x)$ is in H^1, then it is a continuous function. Since

$$1 < \mathscr{L}(u_x) = \operatorname{sgn}(u_x + 1) + \operatorname{sgn}(u_x - 1),$$

and $\operatorname{sgn}(u_x + 1) = 1$ on E, then $0 < \operatorname{sgn}(u_x - 1) < 1$ and we deduce that $u_x = 1$. Due to the continuity of $\mathscr{L}(u_x)$ the set $\{x : \mathscr{L}(u_x) \in (1, 2)\} \subset E$ is open and we may consider one of its connected components, $\mathscr{C}$. Since $u_x = 1$ on $\mathscr{C}$, then $\mathscr{C}$ is a pre-image of a facet, as a result $\operatorname{sgn}(u_x - 1)$ may not be constant over $\mathscr{C}$. As a result $\mathscr{L}(u_x)$ cannot be equal to any number in the interval $(1, 2)$ on any open set.

Similarly, we deal with the case $\mathscr{L}(u_x) \in (-2, -1)$.

Let us now suppose that $\mathscr{L}(u_x) \in (0,1)$ on a set of positive measure. In this case we have

$$0 < \operatorname{sgn}(u_x + 1) + \operatorname{sgn}(u_x - 1) < 1.$$

But this implies an impossible situation of simultaneous $u_x + 1 = 0$ and $u_x - 1 = 0$ on the same set or $\operatorname{sgn}(u_x + 1) = 1$ and $\operatorname{sgn}(u_x - 1) < 0$, i.e. $u_x + 1 > 0$ and $u_x = 1$. The last situation occurs on a facet, where $\operatorname{sgn}(u_x - 1)$ may not be constant.

Similarly, we deal with the case $\mathscr{L}(u_x) \in (-1, 0)$.

Part (b) follows immediately from Proposition 2.8 and part (a). □

Our Proposition 3.1 states that the set of steady states may be very large. It should be stressed that it does not give a full description of this set, because the assumption is that $u \in BV$ conforms to Definition 2.1.

Proposition 3.2. *Let us suppose that $u \in BV$ is a solution to* (1.1) *in the sense of Definition* 2.1 *and it is time independent, i.e., function u satisfies*

$$\big(\mathscr{L}(u_x)\big)_x = 0 \qquad \text{in } (a,b).$$

(a) *If $\mathscr{L}(u_x) = 0$ at $x = a$ and $x = b$, then $|u_x| \le 1$. That is, u, a steady state of* (1.1) *with homogeneous Neumann boundary conditions* (1.7) *is a Lipschitz continuous function with the Lipschitz constant not exceeding* 1.

(b) *If u is a steady state of* (1.1) *with* (1.5) *and $A \le B$ (the case $B \le A$ is analogous), then:*
   (i) *if $(B - A)/(b - a) > 1$, then any increasing function satisfying the boundary data with $u_x \ge 1$ is a steady state.*
   (ii) *if $(B - A)/(b - a) = 1$, then $u(x) = x + A - a$ is the only steady state.*
   (iii) *if $(B - A)/(b - a) < 1$, then any Lipschitz continuous function with $|u_x| \le 1$ satisfying $u(a) = A$, $u(b) = B$ is a steady state of* (1.1) *with* (1.5).

(c) *If u satisfies the periodic boundary condition,* (1.6), *then u is a periodic Lipschitz continuous function with $|u_x| \le 1$.*

Proof. Part (a). Condition (1.7) and Proposition 3.1 jointly imply that $\Omega(x) \equiv 0$. This, in turn yields that

$$\operatorname{sgn}(u_x + 1) = -\operatorname{sgn}(u_x - 1) \neq 0.$$

We note that the case $\operatorname{sgn}(u_x + 1) = 0 = \operatorname{sgn}(u_x - 1)$ is impossible. Thus,

$$\operatorname{sgn}(u_x + 1) = 1 = -\operatorname{sgn}(u_x - 1).$$

which implies

$$u_x + 1 \geq 0 \quad \text{and} \quad u_x - 1 \leq 0$$

i.e. $|u_x| \leq 1$. In particular, u is Lipschitz continuous.

Part (b). Condition $A \leq B$ implies that u_x must be non-negative on a set of positive measure. If $\mathscr{L}(u_x) = 2$, then $\operatorname{sgn}(u_x + 1) = 1 = \operatorname{sgn}(u_x - 1)$, thus $u_x \geq 1$. This implies that any monotone increasing function with $u_x \geq 1$ and

$$\lim_{x \to a^+} u(x) \geq A, \qquad \lim_{x \to b^-} u(x) \leq B$$

is a steady state. In other words (i) holds.

If $\mathscr{L}(u_x) = 1$, then $\operatorname{sgn}(u_x + 1) = 1$ and $\operatorname{sgn}(u_x - 1) = 0$. This may occur only when $u_x = 1$, i.e. $u(x) = x + A - a$, thus $B = b - a + A$.

If $\mathscr{L}(u_x) = 0$, then $\operatorname{sgn}(u_x + 1) = 1$ and $\operatorname{sgn}(u_x - 1) = -1$. This means that $u_x + 1 \geq 0$ a.e. and $u_x - 1 \leq 0$ a.e. equivalently,

$$|u_x| \leq 1.$$

In particular $u(a) = A$ and $u(b) = B$, for otherwise Ω would be equal to 2.

Part (c). Of course, Ω may not be equal to ± 2, because this would imply that u is increasing (or decreasing) on I, which is not possible for a periodic function. The argument presented above applies for the cases $\Omega = \pm 1$, thus only $\Omega = 0$ is left. As a result, we have the same conclusion as in the case of Neumann data. □

Remark 3.3. 1) We shall see that if u a solution to (1.1), then this fact imposes restrictions of oscillations of u_x, cf. Theorem 4.5.

2) It seems that, in case (iii) an arbitrary number of oscillations is possible.

3) If $(B - A)/(b - a) \leq 1$, then all steady states are Lipschitz continuous. On the other hand, if $(B - A)/(b - a) > 1$, then all increasing functions u, not necessarily continuous, are steady states if $u_x \geq 1$ (see Proposition 2.8). Thus, we see that for $u_0 \in BV(I)$ the regularity $u \in L^\infty\big(0, T; BV(I)\big)$ is optimal.

4. Properties of solutions

We collect properties of solutions related to facets, defined in §4.1, and their evolution. In §4.2 we study stopping times.

4.1. Facets. In this section we will resolve whether solutions may have infinitely many facets. We will introduce necessary notions.

Definition 4.1. Let us set $\mathscr{P} = \{-1, 1\}$.

(a) We shall say that subset F of the graph of a solution to (1.1) is a *facet*, if

$$F \equiv F(\xi^-, \xi^+) = \{(x, u(x)) : u_x|_{[\xi^-, \xi^+]} \equiv p \in \mathscr{P}\}.$$

We write $u_x|_{[\xi^-, \xi^+]}$ with the understanding that the one-sided derivatives of u exist at ξ^+ and ξ^-.

Moreover, if $[\xi^-, \xi^+] \subset J$, J is an interval and $u_x|_J \equiv p \in \mathscr{P}$, then $[\xi^-, \xi^+] = J$. Interval $[\xi^-, \xi^+]$ is called the pre-image of facet F or a faceted region of u, (cf. Section 5).

(b) Facet $F(\xi^-, \xi^+)$ has *zero curvature*, if: (i) either Ω i.e. $\mathscr{L}(u_x)$ have the same value at ξ^- and ξ^+ or (ii) $\xi^- = a$ or $\xi^+ = b$ i.e. the facet hits the boundary (in the case of Dirichlet boundary conditions).

Remark 4.2. We notice that if for a facet $F = F(\xi^-, \xi^+)$ there is δ such that $u|_{(\xi^- - \delta, \xi^-)}$ and $u|_{(\xi^+, \xi^+ + \delta)}$ are both on different sides of line $l_p(x) = p(x - \xi^-) + u(\xi^-)$, then F has zero curvature. We will see this in the proof of Lemma 4.7.

According to this definition jumps of solutions are not facets, because they do not correspond to any jump singularity of $\mathscr{L}$.

We begin our analysis with the following observation.

Lemma 4.3. *Let us suppose that $t > 0$ is typical and $F(\xi_l, \xi_r)$ is a facet, then*

$$\lim_{x \to \xi_l} \Omega(x, t) = \omega^- \in \{-2, 0, 2\}, \qquad \lim_{x \to \xi_r} \Omega(x, t) = \omega^+ \in \{-2, 0, 2\}.$$

Proof. Step 1. We shall investigate the neighborhood of ξ_r assuming that $u(x, t)$ is absolutely continuous. Then, there are two possibilities, (1) for all sufficiently $\epsilon > 0$, the derivative u_x on $(\xi_r, \xi_r + \epsilon)$ assumes values from set $\{-1, 1\}$; (2) there is a sequence $\{x_n\}_{n=1}^{\infty}$ converging to ξ_r such that $\xi_r < x_n$ and $u_x(x_n, t)$ exists and $u_x(x_n, t) \neq \pm 1$.

If the first case occurs and there is $\delta > 0$ such that for all $x \in (\xi_r, \xi_r + \delta)$ we have $u_x(x, t) = -u_x|_{[\xi_l, \xi_r]}$, then one can see that

$$\lim_{x \to \xi_r} \Omega(x, t) = 0.$$

If instead there are two sequences x_n^+, x_n^- converging to ξ_r, such that $\xi_r < x_n^+$, x_n^- and $u_x(x_n^+, t) = 1$, $u_x(x_n^-, t) = -1$, then

$$\operatorname{sgn}\left(u_x(x_n^+, t) + 1\right) + \operatorname{sgn}\left(u_x(x_n^+, t) - 1\right) = 1 + \zeta_n^+,$$
$$\operatorname{sgn}\left(u_x(x_n^-, t) + 1\right) + \operatorname{sgn}\left(u_x(x_n^-, t) - 1\right) = \zeta_n^- - 1.$$

Since the right limit of Ω exists at ξ_r we deduce that $\zeta^\pm := \lim_{n\to\infty} \zeta_n^\pm$ satisfy $2 = \zeta^- - \zeta^+$. As a result, $\zeta^- = 1 = -\zeta^+$ and we conclude that $\Omega(x_n^+, t) = 0 = \Omega(x_n^-, t)$ and

$$\lim_{x\to\xi_r} \Omega(x,t) = 0.$$

Now, we consider the situation, when there is a sequence x_n converging to ξ_r, such that $\xi_r < x_n$ and $u_x(x_n,t)$ exists and $u_x(x_n,t) \neq \pm 1$. If this happens, then (a) $u_x(x_n,t) > 1$, (b) $u_x(x_n,t) \in (-1,1)$ or (c) $u_x(x_n,t) < -1$. Case (a) leads to the conclusion, that $\Omega(x_n,t) = 2$, hence

$$\lim_{x\to\xi_r} \Omega(x,t) = 2.$$

If (b) takes place, then $\Omega(x_n,t) = 0$ and $\lim_{x\to\xi_r} \Omega(x,t) = 0$.

Finally, (c) leads to $\Omega(x_n,t) = -2$. As a result, $\lim_{x\to\xi_r} \Omega(x,t) = -2$.

Step 2. Let us now suppose that the initial condition u_0 is in BV, we may no longer assume that $u(\cdot,t)$ is absolutely continuous for a.e. t. This solution may be approximated by solutions u^n for which conclusions of Step 1 are valid. Indeed, there is a sequence $u_0^n \in C^\infty$ and such that

$$\|u_0^n\|_{BV} \to \|u_0\|_{BV} \quad \text{and} \quad u_0^n \to u_0 \quad \text{in } L^2.$$

This assumptions guarantee that the corresponding solutions to (1.1) have the property

$$u_x^n \in L^\infty(0,T;BV),$$

i.e. $u^n(\cdot,t)$ is absolutely continuous for a.e. $t \in (0,T)$. For Dirichlet boundary data this is the content of [21], Theorem 1, for other boundary data this claim follows from the method of the proof of [21], Theorem 1.

Moreover, by Corollaries 2.4, 2.5 we know

$$u^n \to u \quad \text{in } L^2(I_T) \cap C([0,T];L^2), \qquad \Omega_n \rightharpoonup \Omega \quad \text{in } L^2(0,T;H^1).$$

We notice that for a.e. t sequence $\Omega^n(\cdot,t)$ converges uniformly, hence our claim follows for ξ_r. A similar conclusion may be drawn for ξ_l. □

We immediately deduce that: (see also [21]).

Proposition 4.4. *For a typical $t > 0$, $u(t)$ does not contain any degenerate $(\xi^- = \xi^+)$ facet with non-zero curvature.*

Proof. Continuity of Ω implies that $\Omega(\xi^-, t) = \Omega(\xi^+, t)$. Hence $F(\xi)$ has zero curvature. □

We present our main structural theorem.

Theorem 4.5. *If* $u_0 \in BV(I)$ *and* u *is the corresponding solution to* (1.1) *with either Dirichlet* (1.5)*, periodic* (1.6) *or Neumann* (1.7) *boundary conditions, then for almost all* $t > 0$ *the number of facets with non-zero curvature is finite.*

Proof. It is sufficient to consider a typical $t > 0$. The argument is based on the following estimate for the velocity of facet $F(\xi^-, \xi^+)$. Once we set $\omega^\pm := \Omega(\xi^\pm, t)$, then we notice,

$$\Omega_x(x, t) = \frac{\omega^+ - \omega^-}{\xi^+ - \xi^-}, \qquad \text{for } x \in F(\xi^-, \xi^+). \tag{4.1}$$

First we use the fact that Ω is the minimal section of $\partial E(u)$, established for solutions satisfying Neumann or periodic boundary conditions. Thus, Ω minimizes the functional

$$\int_{\xi^-}^{\xi^+} |\zeta_x|^2 \, dx$$

among H^1 functions with $\zeta(\xi^\pm) = \Omega(\xi^\pm, t)$. As a result, Ω is a linear function. Thus (4.1) holds.

Now, we study eq. (1.1) with Dirichlet data and let us suppose that u is a solution to this problem. By reflection we may extend u to a periodic function. Thus, we may apply results already established for periodic solutions, i.e. if $F(\xi^-, \xi^+)$ is a facet, then Ω restricted to $[\xi^-, \xi^+]$ is an affine function.

By Proposition 2.6 we know that for almost all $t > 0$ we have $\displaystyle\int_I u_t^2 \, dx \le \frac{M}{t}$, where M depends on data only. Thus, we square the RHS of (1.1) and integrate $u_t^2 = |\Omega_x|^2$ over I. We notice that

$$\int_I u_t^2 \, dx = \int_I |\Omega_x|^2 \, dx \ge \sum_{F(I_\iota)} \int_{I_\iota} |\Omega_x|^2 \, dx = \sum_{F(I_\iota)} \frac{(\omega^+ - \omega^-)^2}{\xi_i^+ - \xi_i^-}.$$

Here, $\{F(I_\iota) : \iota \in J\}$ is the collection of all non-zero curvature facets. We immediately conclude that the number of facets with non-zero curvature is finite. □

After these preparations we are going to present the basic facts about the facet creation process. Our main tool is the analysis of the continuity of $\Omega(\cdot, t)$. The following theorem tells us that a solution u does not miss any of the preferred directions $-1, 1$, even if the datum does.

Theorem 4.6. *Let us suppose that u is a solution to* (1.1), $t > 0$ *is a typical time instance, i.e.* $u_t(\cdot, t), \Omega_x(\cdot, t) \in L^2(I)$. *If* $x_0 \in I$ *and* $u_x^-(x_0, t) < u_x^+(x_0, t)$, (*resp.* $u_x^+(x_0, t) < u_x^-(x_0, t)$), *then*

$$\big(u_x^-(x_0, t), u_x^+(x_0, t)\big) \cap \{-1, +1\} = \emptyset,$$
$$(resp.\ \big(u_x^+(x_0, t), u_x^-(x_0, t)\big) \cap \{-1, +1\} = \emptyset).$$

Proof. First, we consider the case of $u(\cdot, t)$ being absolutely continuous.

Let us assume that the opposite happens, i.e. there is p from $\{-1, 1\}$ such that

$$p \in \big(u_x^-(x_0, t), u_x^+(x_0, t)\big).$$

For the sake of definiteness, we assume that $p = 1$. In other words,

$$u_x^-(x_0, t) < 1 < u_x^+(x_0, t).$$

Thus, for all $x > x_0$ sufficiently close to x_0 we have

$$1 < \frac{u(x, t) - u(x_0, t)}{x - x_0}.$$

We conclude that there exists sequence x_n^+, converging to x_0 such that $x_0 < x_n^+$ and $1 < u_x(x_n^+, t)$. Hence,

$$\Omega(x_n^+, t) = \operatorname{sgn}\big(u_x(x_n^+, t) + 1\big) + \operatorname{sgn}\big(u_x(x_n^+, t) - 1\big) = 2.$$

Continuity of Ω at x_0 implies that $\Omega(x_0, t) := \lim_{x_n^+ \to x_0^+} \Omega(x_n^+, t) = 2$. On the other hand, $u_x^-(x_0, t) < 1$ and we see that

$$\frac{u(x, t) - u(x_0, t)}{x - x_0} < 1,$$

i.e. there exists a sequence $x_n^- < x_0$ converging to x_0 such that $u_x(x_n^-, t) < 1$. As a result

$$\Omega(x_n^-, t) = \operatorname{sgn}\big(u_x(x_n^-, t) + 1\big) + \operatorname{sgn}\big(u_x(x_n^-, t) - 1\big) = \zeta_n - 1.$$

We consider three cases depending on the behavior of $u_x(x_n^-, t)$. We set, $\Omega(x_0, t) = \lim_{x_n^- \to x_0} \Omega(x_n^-, t)$. Now, if $u_x(x_n^-) < -1$, then $\zeta_n = -1$ and $\Omega(x_0, t) = -2$. If $u_x(x_n^-) > -1$, then $\zeta_n = 1$ and $\Omega(x_0, t) = 0$. Furthermore, if $u_x(x_n^-) = -1$, then $\zeta_n \leq 1$ and then $\Omega(x_0, t) \leq 0$.

Let us consider the possibility that $u_x^+ = 1$. In this case, $u_x^- < 1$, but $\frac{u(x)-u(x_0)}{x-x_0} > 1$, thus $\Omega(x_0,t) = 1 + \zeta^+ = 2$ and $\Omega(x_0,t) = -1 + \zeta^- \leq 0$.

Let us consider a general datum, i.e. $u(\cdot,t) \in BV(I)$. We may assume that $u(\cdot,t)$ has a jump discontinuity at x_0. It follows from Proposition 2.8 that $|\Omega(x_0,t)| = 2$. In order to fix our attention we assume that $\Omega(x_0,t) = 2$. By the continuity of Ω, there is neighborhood U of x_0 such that $\Omega(x,t) > 2 - \varepsilon$ for $x \in U$. Hence, there does not exist any $x \in U$ such that $u_x(x,t) < 1$. Therefore, $u_x(x,t) \geq 1$ for all $x \in U$. Thus, the open interval with endpoints u_x^-, u_x^+ does not contain $+1$ nor -1. A similar result holds for $\Omega(x_0,t) = -2$. □

We introduce a piece of convenient notation. Let us suppose that $t > 0$ is typical and $u(\cdot,t) \in BV(J)$, $J = [\alpha,\beta]$, has a facet $F(\xi^-,\xi^+)$, we assume that $\alpha < \xi^- \leq \xi^+ < \beta$. We introduce the *transition numbers* $\chi_s = \chi_s(u,x)$, $s = l,r$, by the following formulas,

$$\chi_l = \begin{cases} +1 \text{ if } u \geq \ell_p \text{ in } \{x \in J : x \leq x_0\}, \\ -1 \text{ if } u \leq \ell_p \text{ in } \{x \in J : x \leq x_0\}, \end{cases}$$
$$\chi_r = \begin{cases} +1 \text{ if } u \geq \ell_p \text{ in } \{x \in J : x \geq x_0\}, \\ -1 \text{ if } u \leq \ell_p \text{ in } \{x \in J : x \geq x_0\}, \end{cases} \tag{4.2}$$

where ℓ_p is the line with slope p containing facet $F(\xi^-,\xi^+)$.

We notice that we can improve formula (4.1).

Lemma 4.7. *If u is a solution to* (1.1) *and $F(\xi^-,\xi^+)$ is one of the facets, $\xi^- \neq a$ and $\xi^+ - \neq b$, then for a typical $t > 0$,*

$$\Omega_x = \frac{\chi_l + \chi_r}{\xi^+ - \xi^-}. \tag{4.3}$$

Proof. We are going to find values of Ω at ξ^-, ξ^+. We treat ξ^+ first. Initially, we assume that $u(\cdot,t) \in AC(I)$ and $t > 0$ is a typical time instance. Moreover, at $t > 0$ $u(\cdot,t)$ does not have any degenerate facets, as guaranteed by Proposition 4.4. Thus, for facet $F(\xi^-,\xi^+)$ there is such $\epsilon > 0$ that $u|_{(\xi^+,\xi^++\epsilon)}$ is either above l_p, i.e. the line containing $F(\xi^-,\xi^+)$, or below it.

If u is above l_p, then

$$u(x,t) - u(\xi^+,t) > p(x - \xi^+) \qquad \text{for all } x \in (\xi^+, \xi^+ + \epsilon),$$

where p is the slope of l_p. This implies that there exists sequence $x_n \in (\xi^+,\xi^+ + \epsilon)$, converging to ξ^+ such that $u_x(x_n) > p$. We notice that if $p = 1$, then

$$\Omega(x_n,t) = \operatorname{sgn}\big(u_x(x_n,t) + 1\big) + \operatorname{sgn}\big(u_x(x_n,t) - 1\big) = 1 + 1 = p + \chi_r.$$

If $p = -1$, then we know from Theorem 4.6 that $u_x(x_n, t) \in (-1, 1)$. Thus we see that

$$\Omega(x_n, t) = \operatorname{sgn}\big(u_x(x_n, t) + 1\big) + \operatorname{sgn}\big(u_x(x_n, t) - 1\big) = 1 - 1 = p + \chi_r.$$

Since $\Omega(\cdot, t)$ is continuous, we conclude that

$$\Omega(\xi^+, t) = p + \chi_r. \tag{4.4}$$

A similar reasoning performed for interval $(\xi^- - \epsilon, \xi^-)$ yields

$$\Omega(\xi^-, t) = p - \chi_r, \tag{4.5}$$

provided that $u(\cdot, t)$ is continuous. Thus, (4.1) implies (4.3).

Let us suppose now that $u(\cdot, t)$ is no longer absolutely continuous in any neighborhood of x_0. What may happen is:

(a) $u(\cdot, t)$ has a jump discontinuity at x_0, thus by Proposition 2.8 we know that $\Omega(x_0, t) = 2$.

(b) $u(\cdot, t)$ is continuous at x_0. We consider

$$u(x_n, t) - u(x_0, t) = p(1 + \delta_n)(x_n - x_0) > p(x_n - x_0).$$

If u^ϵ is the regularized solution, then we can find sequence ϵ_n converging to 0 such that

$$u^{\epsilon_n}(x_n, t) - u^{\epsilon_n}(x_0, t) \geq p\left(1 + \frac{1}{2}\delta_n\right)(x_n - x_0) > p(x_n - x_0).$$

Thus, we may use the argument from the first part. However, even if $u^k(\cdot, t)$ converges to $u(\cdot, t)$ in L^2, then $\Omega^k(\cdot, t)$ converges uniformly to $\Omega(\cdot, t)$. Thus, (4.4) and (4.5) remain valid for solutions with $u_0 \in BV$. Thus, in all cases, mentioned above, we obtain that (4.3) holds. □

The two previous results are concerned with typical time instances. In particular, they do not preclude the possibility of shrinking a non-zero curvature facet to a point at an exceptional time. Now, we present an improvement of Theorem 4.5. The Proposition below is not a direct consequence of regularity.

We used ξ^-, ξ^+ to denote the endpoints of the facet pre-image. Now, it is advantageous to show the dependence of ξ^-, ξ^+ on h the 'distance' of the facet from the x_1-axis. If $t_0 > 0$ is a typical time instance, then we set $u(\cdot) = u(\cdot, t_0)$ and $h_0 = u(x_0, t_0)$. We have

$$\xi^- = \inf\{x : u(x, t) = p(x - x_0) + h\}, \tag{4.6}$$

$$\xi^+ = \sup\{x : u(x, t) = p(x - x_0) + h\}. \tag{4.7}$$

We notice that $\xi^{\pm}$ are well-defined. Moreover, due to the argument in the above proof,

$$\begin{aligned} &\frac{d\xi^+}{dh} \geq 0 \quad \text{and} \quad \frac{d\xi^-}{dh} \leq 0 \quad \text{if } \chi_l + \chi_r > 0, \\ &\frac{d\xi^+}{dh} \leq 0 \quad \text{and} \quad \frac{d\xi^-}{dh} \geq 0 \quad \text{if } \chi_l + \chi_r < 0. \end{aligned} \tag{4.8}$$

Proposition 4.8. *No facet with non-zero curvature may shrink to a point at any* $t > 0$.

Proof. Step 1. We begin with one facet $F(\xi^-, \xi^+)$ with non-zero curvature. We assume that $F(\xi^-, \xi^+)$ does not intersect other non-zero curvature facets. Let

$$L = \xi^+ - \xi^-.$$

After taking the time derivative, we have

$$\frac{d}{dt}(\xi^+ - \xi^-) = \left(\frac{d\xi^+}{dh} - \frac{d\xi^-}{dh}\right)\frac{dh}{dt}.$$

We notice that due to (4.8) the RHS is always non-negative.

Step 2. Now we consider two intersecting facets $F(\xi^-, \xi) = F_1$, $F(\xi, \xi^+) = F_2$ with non-zero curvature (see Figure 1), i.e $\big(\xi, u(\xi)\big)$ is the intersection point.

For facet F_1 it is advantageous to use ξ^- defined by (4.6) and for F_2 to use ξ^+ defined by (4.7). Point ξ is the intersection of lines containing F_1 and F_2, i.e.,

$$\xi = \frac{x_1 + x_2}{2} + \frac{h_2 - h_1}{2p},$$

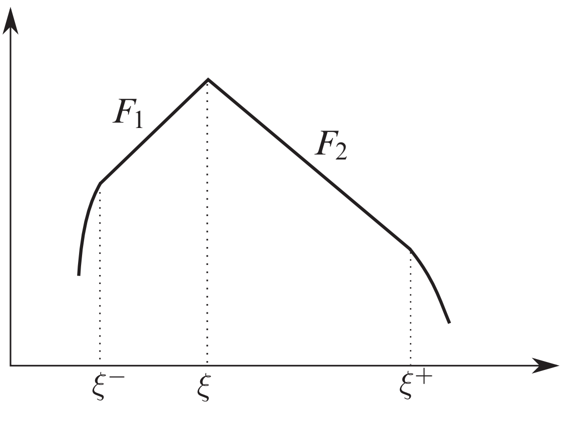

Figure 1. Two interacting facets.

where $(x_i, h_i) \in F_i$, $i = 1, 2$ are fixed and p is the slope of F_1. If we set $L_1 = \xi - \xi^-$, $L_2 = \xi^+ - \xi$, then we see that

$$\frac{d}{dt}L_1 = \frac{d\xi}{dt} - \frac{d\xi^-}{dh_1}\frac{dh_1}{dt} = \frac{1}{2p}\left(\frac{dh_2}{dt} - \frac{dh_1}{dt}\right) - \frac{d\xi^-}{dh_1}\frac{dh_1}{dt},$$
$$\frac{d}{dt}L_2 = \frac{1}{2p}\left(\frac{dh_1}{dt} - \frac{dh_2}{dt}\right) + \frac{d\xi^+}{dh_2}\frac{dh_2}{dt}.$$

We notice that the last terms are always non-negative. Moreover from the formula for vertical velocity, (4.3), we calculate $\frac{dh_1}{dt}$, $\frac{dh_2}{dt}$ and we have

$$\frac{d}{dt}L_1 = \frac{\chi_l + \chi_r}{2p}\left(\frac{L_1 - L_2}{L_1 L_2}\right) - \frac{d\xi^-}{dh_1}\frac{dh_1}{dt}, \qquad \frac{d}{dt}L_2 = \frac{\chi_l + \chi_r}{2p}\left(\frac{L_2 - L_1}{L_1 L_2}\right) + \frac{d\xi^+}{dh_2}\frac{dh_2}{dt}.$$

We notice that if $L_1 \to 0$ and $L_2 \geq \delta > 0$, then we see that $\frac{dL_1}{dt} > 0$, also if $L_2 \to 0$ and $L_1 \geq \delta > 0$, then $\frac{dL_2}{dt} > 0$, but this is impossible. Finally, we notice that $L_1 + L_2 \to 0$ is impossible too, because

$$\frac{d}{dt}(L_1 + L_2) = \frac{d\xi^+}{dt} - \frac{d\xi^-}{dt} \geq 0. \qquad \square$$

We see that only zero curvature facets may shrink to a point. On the other hand zero curvature facets are created during collisions.

4.2. Stopping time. We notice that the diffusion is so strong that for all initial data u_0 the solution stops evolving in finite time. The examples from Section 4.3 give explicit bounds in the case of initial data $u_0 \in BV(I)$. The explicit bounds we give in Theorem 4.11 depend in a crucial way on the Comparison Principle, Theorem 5.3, which is valid for viscosity solutions. Thus, we have to check that our solution is indeed a solution in the viscosity sense. This is done in next Section.

Definition 4.9. If u is a solution to (1.1), then a number $T_{\text{ext}} > 0$ is the stopping time for u iff $u_t \equiv 0$ for all $t > T_{\text{ext}}$ and for all $\epsilon > 0$ we have $u_t \not\equiv 0$ on $(T_{\text{ext}} - \epsilon, T_{\text{ext}})$.

First, we establish the following proposition, which will be helpful in the next theorem.

Proposition 4.10. *Suppose that v and u^k, $k \in \mathbb{N}$ are solutions to* (1.1), *T^k_{ext} is the stopping time of u^k. If $u^k \to v$ in $L^2(I_T)$, $\Omega^k \rightharpoonup \Omega$ and T_{ext} is the stopping time* *of v, then*

$$T_{\text{ext}} \le \limsup_{k\to\infty} T^k_{\text{ext}}.$$

Proof. Set $\overline{T} := \limsup_{k\to\infty} T^k_{\text{ext}}$. It is sufficient to show that if $\delta > 0$ and $h \in \mathbb{R}$ are such that $|h| < \delta$, then

$$v(\overline{T} + \delta + h) - v(\overline{T} + \delta) = 0.$$

Indeed, since $u^k \in C\big([0, \overline{T} + \delta + h]; L^2(I)\big)$ and u^k converges uniformly to v in $C\big([0, \overline{T} + \delta + h]; L^2(I)\big)$, then we have

$$v(\overline{T} + \delta + h) - v(\overline{T} + \delta) = \lim_{k\to\infty} [u^k(\overline{T} + \delta + h) - u^k(\overline{T} + \delta)] = 0. \tag{4.9}$$

The limit is in $L^2(I)$. This is so, because for sufficiently large $k \in \mathbb{N}$, we have $T^k_{\text{ext}} < \min\{\overline{T} + \delta + h, \overline{T} + \delta\}$. This implies that $u^k(\overline{T} + \delta + h) = u^k(\overline{T} + \delta) = u^k(T^k_{\text{ext}})$. Hence, (4.9) follows. □

Theorem 4.11. *Let us suppose that $\delta > 0$ is any typical time. We set $u_0 := u(\delta)$, where u is a solution to* (1.1). *We assume that u_0 is differentiable away from the endpoint of the facets. Let us suppose that x_1 and x_2 are such points that*

(P) $(u_0)_x(x_1) = (u_0)_x(x_2)$, $\big(x_i, u_0(x_i)\big) \in F_i$, $i = 1, 2$, facets F_1, F_2 are different, there is no $x_0 \in [x_1, x_2]$ such that $(u_0)_x(x_0) = -(u_0)_x(x_1)$

We denote the time after which F_1 and F_2 collide by $T(x_1, x_2)$. Then, $T(x_1, x_2)$ is finite and the stopping time T_{ext} of u_0 can be estimated as

$$T_{\text{ext}} \le \max\{T(x_1, x_2) : x_1, x_2 \textit{ satisfy } (P)\}.$$

Proof. By assuming that $\delta > 0$ is typical, in virtue of Proposition 4.4 and Proposition 4.8 we consider situation, when all facets have been already created and their lengths are positive. It follows from Theorem 4.5 that for $t > \delta > 0$ we have finitely many facets with non-zero curvature. Let

$$\mathscr{F}^+ = \{F(\xi^-, \xi^+) : u_x|_{[\xi^-,\xi^+]} = 1, \chi_l(\xi^-) + \chi_r(\xi^+) \neq 0\},$$
$$\mathscr{F}^- = \{F(\xi^-, \xi^+) : u_x|_{[\xi^-,\xi^+]} = -1, \chi_l(\xi^-) + \chi_r(\xi^+) \neq 0\}.$$

Since the number of elements in $\mathscr{F}^+ \cup \mathscr{F}^-$ is finite, we can order them,

$$F_1(\xi_1^-, \xi_1^+), \ldots, F_N(\xi_N^-, \xi_N^+),$$

where

$$a \le \xi_1^- < \xi_1^+ \le \xi_2^- < \xi_2^+ \le \cdots \xi_i^- < \xi_i^+ \le \cdots \xi_N^- < \xi_N^+ \le b.$$

Facets occurring in this sequence can be grouped in the following way

$$F_i, \dots, F_{i+l},$$

where $F_j \in \mathscr{F}^+$, $j = i, \dots, i+l$ (or respectively, they belong to $\mathscr{F}^-$) and $F_{i-1}, F_{i+l+1} \in \mathscr{F}^-$ (respectively, they are in $\mathscr{F}^+$) as far as $i > 1$, $i + l < N$.

Let us consider a typical group $F_i, \dots, F_{i+l}$, for the sake of definiteness, we assume that $F_j \in \mathscr{F}^+$, $j = i, \dots, i+l$.

We prove our theorem by induction with respect to l. We notice that l is always odd. Let $l = 1$, We will estimate the stopping time using the following procedure. We take

$$x_1 \in [\xi_i^-, \xi_i^+], \qquad x_2 \in [\xi_{i+1}^-, \xi_{i+1}^+],$$

such that $u_x(x_1) = u_x(x_2) = 1$ and we set

$$x_0 = \max\{y : u_x(y) = -1 \wedge y < x_1\}, \qquad x_3 = \min\{y : u_x(y) = -1 \wedge x_2 < y\}.$$

It follows from the definition of x_0, x_3 that the graph of u_0 restricted to interval $[x_0, x_3]$ is contained in the strip limited by the tangents at the points $x_0, \dots, x_3$ (see Figure 2). We denote these tangent lines by $\ell_0, \dots, \ell_3$, i.e. $(x_i, u_0(x_i)) \in \ell_i$, $i = 0, \dots, 3$.

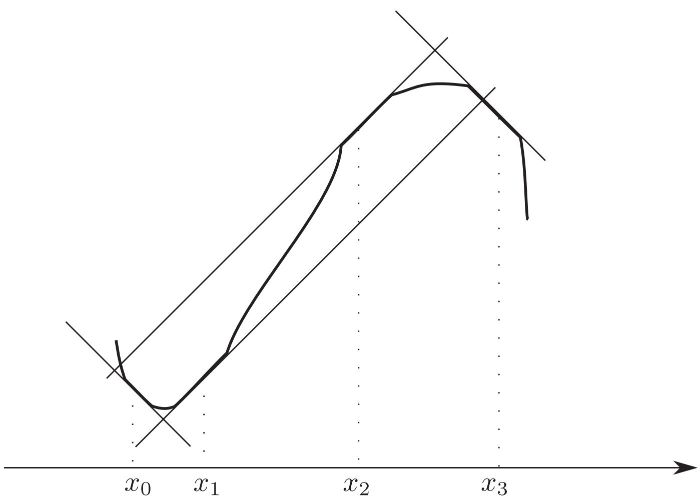

Figure 2. Case $l = 1$.

First, we calculate the upper and lower estimates of function u. We define $w \geq u \geq v$ in the following way:

$$w(x,\delta) = \begin{cases} \min\{\ell_2,\ell_3\}, & x \in [x_2,x_3], \\ u(x,\delta), & x \notin [x_2,x_3], \end{cases} \qquad v(x,\delta) = \begin{cases} \max\{\ell_0,\ell_1\}, & x \in [x_0,x_1], \\ u(x,\delta), & x \notin [x_0,x_1]. \end{cases}$$

Consider solutions w, v of (1.1), $t \geq \delta$ with initial conditions $w_0 = w(x,\delta)$, $v_0 = v(x,\delta)$, respectively. By the comparison principle, we have

$$v(x,t) \leq u(x,t) \leq w(x,t) \qquad \text{for } t \geq \delta,\ x \in I.$$

We denote by $p_1(t)$, (resp. $p_2(t)$) the line parallel to ℓ_1, (resp. ℓ_2) and passing through the point $\big(x_1, v(x_1,t)\big)$, (resp. $\big(x_1, w(x_1,t)\big)$. We can estimate the time t_1 such that $p_1(t_1) = p_2(t_2)$. Indeed, if $\frac{dh_1}{dt}$, $\frac{dh_2}{dt}$ are vertical velocities of p_1, p_2, respectively, then

$$\int_\delta^{t_1} \left(\frac{dh_1}{dt} - \frac{dh_2}{dt}\right) = d, \tag{4.10}$$

where $d = \ell_2(x_1) - \ell_1(x_1)$. We recall formula (4.1),

$$\frac{dh_1}{dt} = \frac{2}{L_1}, \qquad \frac{dh_2}{dt} = -\frac{2}{L_2},$$

where L_1 is the length of facet at $\big(x_1, v(x_1,t)\big)$, L_2 is the length of facet at $\big(x_2, w(x_2,t)\big)$.

Note that $L_1 + L_2 \leq \lambda$, where λ is the distance between projections of points $\ell_0 \cap \ell_1$ and $\ell_2 \cap \ell_3$ on the x_1-axis. It follows from (4.10) that

$$d = \int_\delta^{t_1} \left(\frac{dh_1}{dt} - \frac{dh_2}{dt}\right) = \int_\delta^{t_1} \left(\frac{2}{L_1} + \frac{2}{L_2}\right) \geq \int_\delta^{t_1} \frac{4}{\lambda}.$$

As a result,

$$T(x_1,x_2) \leq (t_1 - \delta) \leq \frac{d\lambda}{4} = \frac{\lambda}{4}\big(\ell_2(x_1) - \ell_1(x_1)\big).$$

Next, we proceed inductively. Suppose that $l > 1$. We denote by ℓ_j lines containing facets F_j, $j = i, \ldots, i+l$. We choose k and m such that

$$|\ell_k(x) - \ell_m(x)| = \max\{|\ell_r(x) - \ell_s(x)| : r,s \in \{i,\ldots,i+l\}\}.$$

This means that line ℓ_j containing F_j lies within the strip bounded by ℓ_k, ℓ_m. There are two cases to consider:

(i) there exists a facet F_j contained within the strip bounded by ℓ_k, ℓ_m;

(ii) all other facets are contained in the ℓ_k or ℓ_m. In such a case let m be the largest and k the smallest index that fulfills this assumption.

We first consider case (i). Since $F_j = F(\xi_j^-, \xi_j^+)$ and for some $\epsilon > 0$, $u|_{[\xi_j^- - \epsilon, \xi_j^+ + \epsilon]}$ is in a half-plane with boundary ℓ_j, it follows that in the group $F_i, \dots, F_{i+l}$, except for F_j, F_k, F_m, there is one additional facet $F_{j+\epsilon}$ adjacent to F_j, where $\epsilon = 1$ or $\epsilon = -1$. We are looking for a point of intersection of ℓ_j and the graph of u such that

$$\bar{x} = \max\{\tilde{x} < \xi_j^- : \ell_j(\tilde{x}) = u(\tilde{x}, \delta)\} \quad \text{if } \epsilon = -1$$

or

$$\bar{x} = \min\{\tilde{x} > \xi_j^+ : \ell_j(\tilde{x}) = u(\tilde{x}, \delta)\} \quad \text{if } \epsilon = 1.$$

We consider

$$v(x, \delta) = \begin{cases} u(x, \delta), & x \notin [\bar{x}, \xi_j^-], \\ \ell_j(x), & x \in [\bar{x}, \xi_j^-]. \end{cases}$$

Then v has $l - 2$ facets with non-zero curvature. Function $w(x, \delta)$ is defined analogously, i.e. we set, if $\epsilon = -1$

$$\tilde{x} = \min\{y > \xi_{j-1}^+ : \ell_{j-1}(x) = u(x, \delta)$$

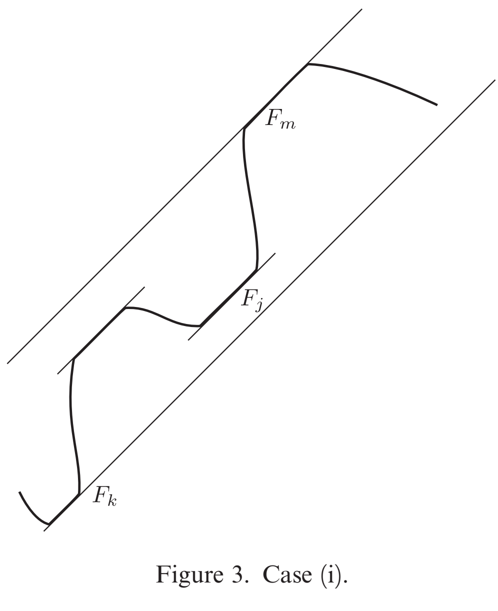

Figure 3. Case (i).

or, if $\epsilon = 1$,

$$\tilde{x} = \max\{y > \xi^+_{j+1} : \ell_{j+1}(x) = u(x,\delta).$$

For $\epsilon = -1$, we define,

$$w(x,\delta) = \begin{cases} u(x,\delta), & x \notin [\xi^+_{j-1}, \tilde{x}], \\ \ell_j(x), & x \in [\xi^+_{j-1}, \tilde{x}]. \end{cases}$$

A similar definition is for $\epsilon = 1$. We have

$$w(x,\delta) \geq u(x,\delta) \geq v(x,\delta).$$

In case (ii) we proceed analogously, except that we now only have lines ℓ_k and ℓ_m. (To define v we combine the facet, which creates ℓ_k, with the nearest facet with the same curvature). We see that w and v have $l-2$ facets with non-zero curvature. We shall use the induction hypothesis that we have the estimate of stopping time for function u in case $l-2$. Thus, we arrive at an estimate for $T(x_k, x_m)$,

$$T(x_k, x_m) \leq \frac{\lambda}{4}\big(\ell_m(x_k) - \ell_k(x_k)\big),$$

where λ is the distance from the x_1-coordinate of $\ell_0 \cap \ell_m$ to $\ell_m \cap \ell_{m+1}$, where l_0 is the line passing through F_{i-1} and ℓ_{m+1} is the line including F_{i+l+1}.

First, we will estimate the time $T(x_k, x_m)$ after which v and w collide in the strip bounded by lines ℓ_k and ℓ_m. For this purpose we take

$$\underline{v} \leq v \leq \underline{w}, \qquad \bar{v} \leq w \leq \overline{w}.$$

We notice that our estimates on $T(x_k, x_m)$, we are developing here, depend only on the parameters of the strip determined by the lines ℓ_k, ℓ_m bounding a part of the graph of solution u. By construction, the same bounding box is for u and v, w yielding the same estimate for collision time $T(x_k, x_m)$. The estimate is made on the premise that lines ℓ_k, ℓ_m sweep the strip.

We note that in the case of Dirichlet boundary conditions if facet F touches the boundary, then F has zero curvature. We proceed as earlier with one difference. In equation (4.10) we have only one non-zero vertical velocity. But this in some cases gives us even better stopping time. In the case of Neumann data we extend u_0 by odd reflection. This reflection does not lengthen the stopping time. Hence the claim. □

Remark 4.12. The inspection of the proof shows that it does not require any differentiability of a solution. In our calculations we depended on the fact that for typical $t > 0$, all facets are created and there are no missing directions (see Theorem 4.6 and Proposition 4.8). Thus, the argument remains valid also when initial condition u_0 is in BV and we consider the solution at a typical time $\delta > 0$.

4.3. Examples. Here, we present two examples highlighting the main issues addressed in this paper.

We study solutions when the initial data are discontinuous at $x = a$ or $x = b$. We see that the solution does not satisfy the boundary condition in a pointwise manner for $t > 0$.

Proposition 4.13. *Let us suppose that $I = (-1, 1)$ and $u(t, -1) = u(t, 1) = 0$.*

(a) *If $u_0(x) = -|x| + d$, where $d > 1$, then*

$$u(x,t) = -|x| + d - 2\min\left\{t, \frac{d-1}{2}\right\}$$

and $\Omega(u; x, t) = -2x$. In particular, $T_{\text{ext}} = \frac{d-1}{2}$ and $u(x, T_{\text{ext}}) = 1 - |x|$.

(b) *If $u_0(x) = -|x| + e$, where $e < 1$, then*

$$u(x,t) = \max\left\{u_0(x), |x| - 2 + e + 2\min\left\{\sqrt{2t}, \frac{(1-e)}{2}\right\}\right\} \tag{4.11}$$

and $T_{\text{ext}} = \frac{(1-e)^2}{8}$. In particular,

$$u(x, T_{\text{ext}}) = \begin{cases} -x - 1, & x \in \left[-1, -\frac{(1+e)}{2}\right], \\ -|x| + e, & x \in \left[-\frac{(1+e)}{2}, \frac{(1+e)}{2}\right], \\ x - 1, & x \in \left[\frac{(1+e)}{2}, 1\right]. \end{cases}$$

Proof. We conduct calculations similar to that in the proof of Proposition 4.8. (a) We are interested in how long it takes for the facets to reach the stopping time. We have $h(0) = d$, $h(T_{\text{ext}}) = 1$, the facets have constant length, $L(h) = 1$. We notice that $\Omega(x, t) = -2|x|$, hence

$$\frac{dh}{dt}(t) = \frac{-2}{L(h)} \qquad i.e. \qquad \frac{dh}{dt}(t)L(h) = -2.$$

We integrate the above equation over $[0, T_{\text{ext}}]$

$$\int_0^{T_{\text{ext}}} \frac{dh}{dt}(t)L(h)\,dt = -2T_{\text{ext}}.$$

We see that $T_{\text{ext}} = \frac{d-1}{2}$.

(b) We see that due to Theorem 4.6 at points $x = -1$, and $x = 1$ two symmetric facets are created, $F(-1, -\eta)$, $F(\eta, 1)$. We will follow $F(\eta, 1)$, hence we have $h(0) = -1 + e$, $h(T_{\text{ext}}) = 0$. The length of the facet is $L(h) = \frac{1}{2}(1 + h - e)$. At the same time $L = 1 - \eta(t)$, as a result,

$$\Omega(x,t) = \begin{cases} \frac{2x - 2\eta(t)}{1-\eta(t)}, & x \in [\eta(t), 1], \\ 0, & x \in [-\eta(t), \eta(t)], \\ \frac{2x + 2\eta(t)}{1-\eta(t)}, & x \in [-1, -\eta(t)]. \end{cases}$$

Using this information we write equation for h,

$$\frac{dh}{dt}(t) = \frac{2}{L(h)} \qquad i.e. \qquad \frac{dh}{dt}(t) L(h) = 2.$$

We integrate the above equation to get

$$h(t) = 2\sqrt{2t} - 1 + e \qquad \text{and} \qquad \eta(t) = 1 - \sqrt{2t}.$$

Thus, we see that $T_{\text{ext}} = \frac{(1-e)^2}{8}$ and (4.11) holds. □

Example 1. We look at a solution with oscillating initial data. Theorem 4.11 implies that facets close to zero get killed first, so that the stopping time is estimated by using the parameters corresponding to the biggest humps in the data. Let us consider $u_0(x) = x^2 \sin(x^{-1}) \in BV(I)$, where $I = (-1, 1)$. We see that for any $t > 0$ most of the facet interactions are over, only a finite number of facets with non-zero curvature are left. We approximate u_0 with

$$u_0^n(x) = \begin{cases} 0, & x \in \left[-\frac{1}{n\pi}, \frac{1}{n\pi}\right], \\ x^2 \sin(x^{-1}), & x \in [-1, 1] \setminus \left[-\frac{1}{n\pi}, \frac{1}{n\pi}\right]. \end{cases}$$

Due to Proposition 4.10 and Theorem 4.11 we have an estimate on the stopping time for the evolution with initial condition u_0. However, we provide no closed formula for it.

5. Viscosity solutions

There are two main reasons for introducing the theory of viscosity solutions in this paper. Firstly, we would like to check if the oscillatory behavior of solutions is 'correct'. At the same time the theory of viscosity solutions will give us an additional tool like the comparison principle, see Theorem 5.3, which is used in the proof of Theorem 4.11. This is the second reason for dealing with the theory of viscosity solutions.

Our exposition is based on [10], adapted to the setting of (1.1). It is clear that we have to give meaning to $\big(\mathscr{L}(\varphi_x)\big)_x$ for a proper choice of test functions φ. Our experience with the theory of nonlinear semigroups suggests that it is advantageous to work with $\big(W_p(\varphi_x)\big)_x$, when $W(p)$ is a convex function.

We are going to present the necessary notions. A function $f \in C(I)$ is called *faceted* at x_0 with *slope* $p \in \{-1, 1\} =: \mathscr{P}$ on I (or *p-faceted* at x_0) if there is a closed nontrivial finite interval $\tilde{I} \subset I$ containing x_0 such that f coincides with an affine function

$$\ell_p(x) = p(x - x_0) + f(x_0) \qquad \text{in } \tilde{I}$$

and $f(x) \neq \ell_p(x)$ for all $x \in J \backslash \tilde{I}$, where J is a neighborhood of $\tilde{I}$ in I. Interval $\tilde{I}$ is denoted by $R(f, x_0)$.

We will denote by $C^2_P(I)$ the set of $f \in C^2(I)$ such that f is p-faceted at x_0 whenever $f'(x_0) \in \{-1, 1\}$. Let $A_P(I_T)$ be the set of all admissible functions ψ on I_T i.e. ψ is of the form

$$\psi(x, t) = f(x) + g(t), \qquad f \in C^2_P(I),\ g \in C^1(0, T).$$

The definition of $\big(W_p(\varphi_x)\big)_x$ is non-local for p-faceted $\varphi \in C^2_P(I)$. It involves a solution of an obstacle problem, which we will describe momentarily.

We assume that $\Delta > 0$, $\chi_l, \chi_r \in \{1, -1\}$ and $J = [\alpha, \beta] \subset I$ are given. We set

$$K^Z_{\chi_l\chi_r}(J) = \Big\{ \zeta \in H^1(J) : |Z(x) - \zeta(x)| \leq \frac{\Delta}{2}, x \in J,$$
$$Z(\alpha) - \chi_l \frac{\Delta}{2} = \zeta(\alpha), Z(\beta) + \chi_r \frac{\Delta}{2} = \zeta(\beta) \Big\}.$$

We also introduce

$$\mathscr{J}^Z_{\chi_l\chi_r}(\zeta, J) = \begin{cases} \int_J |\zeta'(x)|^2\, dx & \text{if } \zeta \in H^1(J), \\ +\infty & \text{if } \zeta \in L^2(J) \backslash H^1(J). \end{cases}$$

Let us call by $\xi^{Z,J}_{\chi_l\chi_r}$ the unique solution to the obstacle problem

$$\min\{\mathscr{J}^Z_{\chi_l\chi_r}(\zeta, J) : \zeta \in K^Z_{\chi_l\chi_r}(J)\}. \tag{5.1}$$

It is easy to see that for any affine function Z, the minimizer is an affine function too. This is the case considered in this paper. But in general, even if $Z \in C^2$, then it is well-known that the unique solution $\xi^{Z,J}_{\chi_l\chi_r}$ belongs to $C^{1,1}(J)$, see [16].

Of course, Z is defined up to an additive constant, which we have to choose properly. If $F = F(\xi^-, \xi^+)$ is a facet, and $u_x|_{(\xi^-, \xi^+)} = p \in \mathscr{P}$, then we set $Z = px$ and $\Delta = W_p(p^+) - W_p(p^-) = 2$.

We define $\Lambda_W^Z(\varphi)$ as follows. We stress that in [10] we denoted the same object by $\Lambda_W^{Z'}(\varphi)$, but here we opt for a simpler notation.

If $\varphi \in C^2$ and $\varphi_x \notin \mathscr{P}$, then we set

$$\Lambda_W^Z(\varphi)(x) := \big(W_p(\varphi_x)\big)_x.$$

If $\varphi \in C_P^2$ is p-faceted at x_0, then we denote its faceted region $R(\varphi, x_0)$ by J. We take $Z := p \in \mathscr{P}$, then we set

$$\Lambda_W^Z(\varphi)(x) := \frac{d}{dx}\xi_{\chi_l\chi_r}^{Z,J}(x).$$

Transition numbers χ_l, χ_r are defined by (4.2).

It turns out that the non-local definition of $\Lambda_W^Z(\varphi)$ has the desired property.

Proposition 5.1 ([10, Theorem 2.4]). *Assume that I_1 and I_2 are bounded open intervals and $\xi_{\chi_l\chi_r}^{Z,I_i}$, $i = 1,2$ is the solution to* (5.1). *We write $\Lambda_{\chi_l\chi_r}(x,J)$ for $\frac{d}{dx}\xi_{\chi_l\chi_r}^{Z,J}(x)$.*

(i) *If $I_2 \subset I_1$, then*

$$\Lambda_{--}(x, I_2) \le \Lambda_{\pm\pm}(x, I_1) \le \Lambda_{++}(x, I_2) \quad \text{for } x \in I_2.$$

(ii) *If $a \le c < b \le d$ for $I_1 = (a,b)$, $I_2 = (c,d)$, then for $x \in (c,b)$*

$$\Lambda_{\pm-}(x, I_1) \le \Lambda_{+\pm}(x, I_2), \qquad \Lambda_{-\pm}(x, I_2) \le \Lambda_{\pm+}(x, I_1).$$

After these preparations we may define the test functions and viscosity solutions to (1.1).

Definition 5.2. A real-valued function u on I_T is a (viscosity) *subsolution* of (1.1) in I_T if its upper-semicontinuous envelope u^* is finite in $\bar{I}_T$ and

$$\psi_t(\hat{t}, \hat{x}) - \Lambda_W^{Z(\hat{t},\cdot)}\big(\psi(\hat{t})\big)(\hat{x}) \le 0 \tag{5.2}$$

whenever $\big(\psi, (\hat{t}, \hat{x})\big) \in A_P(I_T) \times I_T$ fulfills

$$\max_{I_T}(u^* - \psi) = (u^* - \psi)(\hat{t}, \hat{x}). \tag{5.3}$$

Here, $\psi(\hat{t})$ is a function on Ω defined by $\psi(\hat{t}) = \psi(\hat{t}, \cdot)$ and u^* is defined by

$$u^*(t,x) = \lim_{\varepsilon \downarrow 0} \sup\{u(s,y) : |s - t| < \varepsilon, |x - y| < \varepsilon, (s,y) \in I_T\} \qquad \text{for } (t,x) \in \overline{I_T}.$$

We also set $u_* = (-u^*)$.

A (viscosity) *supersolution* is defined by replacing $u^*(<\infty)$ by the lower-semicontinuous envelope $u_*(>-\infty)$, max by min in (5.3) and the inequality (5.2) by the opposite one. If u is both a sub- and supersolution, it is called a *viscosity solution*. Hereafter, we avoid using the word viscosity, if there is no ambiguity. Function ψ satisfying (5.3) is called a test function of u at $(\hat{t}, \hat{x})$.

The main tool we acquire from the theory of viscosity solutions is the Comparison Principle.

Theorem 5.3 ([10, Theorem 4.1]). *Let u and v be respectively sub- and supersolutions of* (1.1) *in $I_T = I \times (0, T)$, where I is a bounded open interval. If $u^* \le v_*$ on the parabolic boundary $\partial_p I_T \big(= [0, T) \times \partial I \cup \{0\} \times \bar{I}\big)$ of I_T, then $u^* \le v_*$ in I_T.*

In order to use Theorem 5.3, we have to check that the solutions we constructed in Theorem 2.3, are actually viscosity solutions.

Theorem 5.4. *If $u_0 \in BV$ and u is the corresponding solution to* (1.1) *with either boundary condition* (1.5), (1.6) *or* (1.7), *then u is a viscosity solution to* (1.1).

Before we engage into the proof, we make observations facilitating the argument. We set,

$$\Xi(u) = \{[\xi^-, \xi^+] \subset [a, b] : [\xi^-, \xi^+] \text{ is the pre-image of facet } F(\xi^-, \xi^+) \text{ of } u\}.$$

Lemma 5.5. *Let us suppose that $u(\cdot, t_0)$ is continuous at x_0 and u_t exists at (x_0, t_0), where $t_0 > 0$. Then one of the following possibilities holds:*

(a) *x_0 is in the complement of the sum of all pre-images of facets, i.e. $x_0 \in I \setminus \bigcup \Xi\big(u(\cdot, t)\big)$. Hence, $u_t(x_0, t_0) = 0$.*

(b) *x_0 is in the interior of the pre-image of a facet, $x_0 \in \big(\xi^-(t_0), \xi^+(t_0)\big)$ and either* (i) *the length of the interval $\xi^+(t) - \xi^-(t)$ as a function of time is continuous at t_0 and $\chi_r + \chi_l$ is a non-zero constant for all t from a neighborhood of t_0 or* (ii) *$\chi_r + \chi_l = 0$ for all t from a neighborhood of t_0.*

(c) *$x_0 \in \{\xi^-(t_0), \xi^+(t_0)\}$ and either* (i) *facet $F\big(\xi^-(t_0), \xi^+(t_0)\big)$ has zero curvature or* (ii) *$\chi_r + \chi_l \neq 0$ and $\big(x_0, u_0(x_0, t_0)\big) \in F\big(\xi_1^-(t_0), \xi_1^+(t_0)\big) \cap F\big(\xi_2^-(t_0), \xi_2^+(t_0)\big)$ and functions $\xi_1^+(\cdot) - \xi_1^-(\cdot)$, $\xi_2^+(\cdot) - \xi_2^-(\cdot)$ are continuous at $t = t_0$ and equal at $t = t_0$. We note that* (i) *includes the case of a facet passing through the boundary data.*

Proof. If $u_t(x_0, t_0)$ exists, then this means that $u_t^+(x_0, t_0) = u_t^-(x_0, t_0)$. We know how to calculate $u_t^+(x_0, t_0)$. First, we consider the case of x_0 belonging to $I \setminus \bigcup \Xi\big(u(\cdot, t_0)\big)$. Since facets with a non-zero curvature are expanding, then

$x_0 \in I \setminus \bigcup \Xi(u(\cdot, t))$ for all $t < t_0$. Since $u_t(x_0, t_0)$ exists, we deduce that it must be zero. Thus, (a) holds.

Let us now assume that x_0 belongs to the interior of the pre-image of a facet, i.e. $x_0 \in (\xi^-(t_0), \xi^+(t_0))$. Then always,

$$u_t^+(x_0, t_0) = \frac{\chi_r + \chi_l}{\xi^+(t_0) - \xi^-(t_0)}. \tag{5.4}$$

Existence of derivative u_t at t_0 implies that (b) holds.

Let us suppose that not only $x_0 \in \{\xi^-, \xi^+\}$ but also $x_0 \in \partial(\bigcup \Xi(u(\cdot, t_0)))$. If $F(\xi^-(t), \xi^+(t))$ have zero curvature for all $t < t_0$ sufficiently close to t_0, then $u_t^-(x_0, t_0) = 0$ and (5.4) imply that $F(\xi^-(t_0), \xi^+(t_0))$ must have zero curvature.

If $\chi_l(\xi^-(t)) + \chi_r(\xi^+(t)) \neq 0$ for $t < t_0$ sufficiently close to t_0, then $x_0 \in I \setminus \bigcup \Xi(u(\cdot, t))$. In this case, (5.4) and $u_t^-(x_0, t_0) = 0$ imply again that $\chi_r + \chi_l = 0$, i.e. facet $F(\xi^-(t_0), \xi^+)(t_0)$ has zero curvature. This includes the case of a facet satisfying the boundary conditions. Hence, $u_t(x_0, t_0) = 0$.

If $(x_0, u(x_0, t_0)) \in F(\xi_1^-(t_0), \xi_1^+(t_0)) \cap F(\xi_2^-(t_0), \xi_2^+(t_0))$, then $u_t^+(x_0, t_0)$ does not exist unless $\xi_1^+(\cdot) - \xi_1^-(\cdot) = \xi_2^+(\cdot) - \xi_2^-(\cdot)$ and the transition numbers of $F(\xi_1^-(t_0), \xi_1^+(t_0))$ and $F(\xi_2^-(t_0), \xi_2^+(t_0))$ are the same. Existence of $u_t(x_0, t_0)$ and (5.4) imply the continuity of functions $\xi_1^+(\cdot) - \xi_1^-(\cdot)$, $\xi_2^+(\cdot) - \xi_2^-(\cdot)$ at $t = t_0$. Thus, we showed that (c) takes place. □

We notice that fulfilling any of the conditions (a) to (c) is also sufficient for the existence of $u_t(x_0, t_0)$.

As important as Lemma 5.5 is understanding the behavior of u near (x_0, t_0), when $u(x_0, \cdot)$ is not differentiable with respect to t at t_0.

Lemma 5.6. *Let us suppose that $u(\cdot, t_0)$ is continuous at x_0 and u_t does not exist at (x_0, t_0). Then one of the following possibilities holds:*

(a) *x_0 is in the interior of a pre-image of a facet, $x_0 \in (\xi^-(t_0), \xi^+(t_0))$ and $\chi_l(t_0) + \chi_r(t_0) = 0$, while $\chi_l(t) + \chi_r(t) \neq 0$ for $t < t_0$ close to t_0. This case includes the situation when a facet hits the boundary of I at time t_0.*

(b) *x_0 is in the interior of a pre-image of a facet, $x_0 \in (\xi^-(t_0), \xi^+(t_0))$ and $\chi_l(t) + \chi_r(t) = \mathrm{const} \neq 0$, for $t \leq t_0$ close to t_0 and $\xi^+(t) - \xi^-(t)$ has a jump at $t = t_0$.*

(c) *x_0 belongs to the boundary of $\bigcup \Xi(u(\cdot, t))$, while for all $t < t_0$ sufficiently close to t_0, point x_0 belongs to $I \setminus \bigcup \Xi(u(\cdot, t))$. In this case $u_t^+(x_0, t_0)$ is given by (5.4) and $u_t^-(x_0, t_0) = 0$.*

(d) *$x_0 \in F(\xi_1^-(t_0), \xi_1^+(t_0)) \cap F(\xi_2^-(t_0), \xi_2^+(t_0))$ and functions $\xi_1^+(t_0) - \xi_1^-(t_0) \neq \xi_2^+(t_0) - \xi_2^-(t_0)$.*

Proof. Here, we listed cases complementary to those enumerated in Lemma 5.5, thus no further argument is necessary. □

We notice that in case (d) function $x \mapsto u_t^+(x, t_0)$ is discontinuous at $x = x_0$.

We are now ready for the *proof of Theorem* 5.4. We will show that u is a subsolution. The argument that u is also a supersolution is similar and it will be omitted. Let us take $(x_0, t_0) \in I_T$ and u^* the upper semicontinuous envelope of u. There are the following four cases to consider:

1) x_0 is a point of continuity of $u(\cdot, t)$, i.e. $u^*(x_0, t_0) = u(x_0, t_0)$;
2) x_0 is a point of discontinuity of $u(\cdot, t)$. We note that jumps are the only possible discontinuities of BV functions.

In each of the above situations either:

a) x_0 belongs to a pre-image of facet F, i.e. $x_0 \in [\xi^-, \xi^+]$ or
b) the converse holds.

Step 1. We begin with case 1 a). Let us take a test function $\psi(x, t) = f(x) + g(t)$ such that $\psi(x_0, t_0) = u(x_0, t_0)$ and

$$\max(u - \psi) = u(x_0, t_0) - \psi(x_0, t_0). \tag{5.5}$$

We have to proceed according to the properties of the test function. Let us assume first that $u_t(x_0, t_0)$ exists. Then $g'(t_0) = u_t(x_0, t_0)$ and we need to consider the following cases.

If F has zero curvature with slope $p = 1$ (similarly if $p = -1$), then we have the following cases:

(i) $F\big(R(f, x_0)\big)$ has a non-zero curvature and $\xi^- \le x^- < \xi^+ \le x^+$ (or $x^- \le \xi^- < x^+ \le \xi^+$), where $[x^-, x^+] = R(f, x_0)$. Then Proposition 5.1 implies $\Lambda_{+-}(x, [\xi^-, \xi^+]) \le \Lambda_{++}\big(x, R(f, x_0)\big)$ (or $\Lambda_{-+}(x, [\xi^-, \xi^+]) \le \Lambda_{++}\big(x, R(f, x_0)\big)$). Hence, (5.2) holds.

(ii) $F\big(R(f, x_0)\big)$ has zero curvature and $\xi^- \le x^- < \xi^+ \le x^+$ (or $x^- \le \xi^- < x^+ \le \xi^+$). Then Proposition 5.1 implies $\Lambda_{+-}(x, [\xi^-, \xi^+]) \le \Lambda_{+-}\big(x, R(f, x_0)\big)$ (or $\Lambda_{-+}(x, [\xi^-, \xi^+]) \le \Lambda_{++}\big(x, R(f, x_0)\big)$). Hence, (5.2) holds.

(iii) $F\big(R(f, x_0)\big)$ has a non-zero curvature and $R(f, x_0) \subset [\xi^-, \xi^+]$. Then Proposition 5.1 implies $\Lambda_{+-}(x, [\xi^-, \xi^+]) \le \Lambda_{++}\big(x, R(f, x_0)\big)$. Hence, (5.2) holds.

Let us consider $x_0 \in [\xi^-, \xi^+]$ and facet $F(\xi^-, \xi^+)$ has $\chi_l + \chi_r = -2$. In this case $\Omega_x|_{(\xi^-, \xi^+)} = -2/(\xi^+ - \xi^-)$ and $\Omega_x = \Lambda_{--}\big(x, (\xi^-, \xi^+)\big)$. If the graphs of u and f are below the line l_p passing through $F(\xi^-, \xi^+)$, then the faceted region $R(f, x_0)$ contains $[\xi^-, \xi^+]$. Then we consider $\Lambda_{--}\big(x, R(f, x_0)\big)$. We also notice that

$$\Lambda_{--}\big(x, R(f, x_0)\big) \geq \Lambda_{--}\big(x, (\xi^-, \xi^+)\big).$$

Hence, (5.2) holds.

The other possible tangency configurations of u and f are analyzed as in (i)–(iii) above. The details are left to the interested reader.

The case $x_0 \in [\xi^-, \xi^+]$ when $\chi_l + \chi_r = 2$ is simpler, because there is just one way how f may touch u. Namely, we have $\chi_l = \chi_r = 1$ and $R(f, x_0) \subset [\xi^-, \xi^+]$. Thus, $\Lambda_{++}\big(x, R(f, x_0)\big) \geq \Lambda_{++}\big(x, (\xi^-, \xi^+)\big)$ and (5.2) follows.

Step 2. Now, we work assuming that $u_t(x_0, t_0)$ does not exist. We have two major subcases:

$$x_0 \in (\xi^-, \xi^+), \tag{5.6}$$

$$x_0 \in \{\xi^-, \xi^+\}. \tag{5.7}$$

Our analysis is based on Lemma 5.6. If (5.6) and the case Lemma 5.6 (a) hold, then $F(\xi^-, \xi^+)$ has zero curvature and there exist facets $F\big(\zeta^-(t), \zeta^+(t)\big)$, such that

$$\lim_{t\to t_0^-} \zeta^\pm(t) =: \zeta^\pm \in [\xi^-, \xi^+]. \tag{5.8}$$

First, we consider

$$x_0 \in (\xi^-, \xi^+)\backslash\{\zeta^-, \zeta^+\}.$$

We will separately study $x_0 \in \{\zeta^-, \zeta^+\}$. We have two obvious possibilities for $F\big(\zeta^-(t), \zeta^+(t)\big)$, either $\chi_l(t) + \chi_r(t) < 0$ or $\chi_l(t) + \chi_r(t) > 0$, here we use the short-hands, $\chi_l(t) \equiv \chi_l\big(\zeta^-(t)\big)$, $\chi_r(t) \equiv \chi_l\big(\zeta^+(t)\big)$.

If $\chi_l(t) + \chi_r(t) < 0$ for $t < t_0$ close to t_0, then there is no $g \in C^1(0, T)$ such that

$$\begin{aligned} &f(x_0) + g(t_0) = u(x_0, t_0), \\ &u(x, t) \leq g(t) + f(x) \quad \text{in a neighbourhood of } (x_0, t_0). \end{aligned} \tag{5.9}$$

On the other hand, if $\chi_l(t) + \chi_r(t) > 0$ for $t < t_0$ close to t_0, then there exists g satisfying (5.9) with $g'(t_0) \in [0, A]$.

If $x_0 \in [\zeta^-, \zeta^+]$, when $\zeta^\pm$ are defined by (5.8), then by Lemma 5.6, we deduce that $A = \frac{2}{\zeta^+ - \zeta^-}$. We have further subcases to consider:

(α) $\zeta^+ = \xi^+$, $\zeta^- = \xi^-$, i.e. facet $F(\xi^-, \xi^+)$ is a result of a collision of a facet moving upward with the boundary of I.

(β) facet $F(\xi^-, \xi^+)$ is a result of a collision of a facet moving upward with a facet passing through the boundary data.

(γ) facet $F(\xi^-,\xi^+)$ is a result of a collision of a facet moving upward with another facet moving downward.

Let us consider the resulting limitations on f and $R(f,x_0)$. Case (α) does not bring any. If (β) occurs, then (5.9) implies that either

$$R(f,x_0) \subset [\zeta^-,\zeta^+] \tag{5.10}$$

or

$$R(f,x_0) \not\subset [\zeta^-,\zeta^+] \tag{5.11}$$

but $g'(t_0)=0$. Finally, (γ) and (5.9) imply that $R(f,x_0) \subset [\zeta^-,\zeta^+]$, because the situation is similar to that studied in the lines above formula (5.9).

Let us check that u is a subsolution in these cases. We notice that

$$\Omega_x(x_0,t_0) = 0 = u_t^+(x_0,t_0). \tag{5.12}$$

It will be easier if we start with (γ) first. In this case we have $R(f,x_0) \subset [\zeta^-,\zeta^+]$ and $\Lambda_W^Z(f,x) = \Lambda_{++}\big(x, R(f,x_0)\big)$. Since $g'(t_0) \le \frac{2}{\zeta^+-\zeta^-} = \Lambda_{++}(x,[\zeta^-,\zeta^+])$, then we infer from Proposition 5.1 that $\Lambda_{++}\big(x,R(f,x_0)\big) \ge \Lambda_{++}(x,[\zeta^-,\zeta^+])$, thus (5.2) holds.

In case (α) there is no apparent restriction on $R(f,x_0)$ but it is not clear, which minimization problem is the correct one if $R(f,x_0)$ intersects the boundary of I. Since we developed ideas for the Dirichlet boundary condition through the periodic boundary data, we first extend u antisymetrcially to get a periodic function. We see that (α) corresponds to (γ) considered above. Thus, we immediately conclude that if (α) holds, then (5.2) is satisfied as well. We also check it directly. Since we extended u antisymetrcially, then (α) corresponds to the collision of two facets, one is moving downward the other is moving upward. In this case, $\Lambda_W^Z(f,x) = \Lambda_{++}\big(x,R(f,x_0)\big)$. Since we have

$$\Omega_x(x,t_0) = \Lambda_W^Z(u,x) = 0,$$

and $g'(t_0) \le \frac{2}{\zeta^+-\zeta^-} = \Lambda_{++}(x,[\zeta^-,\zeta^+])$, then we infer from Proposition 5.1 that

$$\Lambda_{++}\big(x,R(f,x_0)\big) \ge \Lambda_{++}(x,[\zeta^-,\zeta^+]),$$

thus (5.2) indeed holds.

Now, we consider (β). If the subcase (5.10) holds, then $R(f,x_0) \subset [\zeta^-,\zeta^+]$ and $\Lambda_W^Z(f,x) = \Lambda_{++}\big(x,R(f,x_0)\big)$. Since $g'(t_0) \le \frac{2}{\zeta^+-\zeta^-} = \Lambda_{++}(x,[\zeta^-,\zeta^+])$, then we infer from Proposition 5.1 that

$$\Lambda_{++}\big(x,R(f,x_0)\big) \ge \Lambda_{++}(x,[\zeta^-,\zeta^+]),$$

thus (5.2) holds.

In subcase (5.11), $R(f, x_0) \subset [\xi^-, \xi^+]$ and if $R(f, x_0)$ does not intersect ∂I, then $\Lambda_W^Z(f) = \Lambda_{++}\big(x, R(f, x_0)\big) \geq \Lambda_W^Z(u)$. On the other hand, if $R(f, x_0)$ intersects ∂I, then we proceed as above in case (α) and take $\Lambda_{++}\big(x, R(f, x_0)\big)$ for $\Lambda_W^Z(f)$. Hence, $\Lambda_W^Z(f) \geq \Lambda_W^Z(u)$. As a result, in both cases (5.2) holds.

Now, we come back to the left out case, i.e. $x_0 \in \{\zeta^-, \zeta^+\}$. If $F(\xi^-, \xi^+)$ does not touch the boundary, then for $t < t_0$ close to t_0, x_0 does not belong to any preimage of any facet. This is so because $\zeta^{\pm}(t)$ are not constant, see (4.8), unless $\zeta^{\pm}(t)$ are points of discontinuity of u. Hence, there is no test function satisfying (5.5).

The other case is that $F(\zeta^-, \zeta^+)$ intersects the boundary. As usually, we have two possibilities for this facet, either $\chi_l(t) + \chi_r(t) < 0$ or $\chi_l(t) + \chi_r(t) > 0$. If $\chi_l(t) + \chi_r(t) < 0$, then there is no $g \in C^1(0, T)$ such that (5.5) holds. If $\chi_l(t) + \chi_r(t) > 0$, then we proceed as in previous cases.

Step 3. We consider the situation when (5.6) and the case Lemma 5.6 (b) hold. Thus, a moving facet collides with a zero curvature facet. We have the situation similar to that in Step 1. Thus, we may rule out the case of $(\chi_l + \chi_r)(t^-) = -2$ as impossible to satisfy (5.9).

If $(\chi_l + \chi_r)(t^+) = 2$, we conclude that the only possibility for $R(f, x_0)$ is that $R(f, x_0) \subset [\zeta^-, \zeta^+]$ and we have $\Lambda_W^Z(f) = \Lambda_{++}\big(x, R(f, x_0)\big)$. Arguing as before we see that $g'(t) \in [0, 2/(\xi^+ - \xi^-)]$ and $2/(\xi^+ - \xi^-) = \Lambda_{++}(x, [\xi^-, \xi^+])$, but $\Omega_x = 0$. As a result, (5.2) holds.

Step 4. Let us assume that (5.7) and case (c) of Lemma 5.6 hold. But there is no test function $\psi(x, t) = f(x) + g(t)$ such that (5.9) holds.

Step 5. Let us assume that (5.7) and case (d) of Lemma 5.6 hold. If $F(\xi^-, \xi^+)$ has positive curvature, i.e. $\chi_l + \chi_r > 0$, then Lemma 5.6 (d) and (5.9) imply that there is no test function. On the other hand, i.e. if $\chi_l + \chi_r < 0$, then there are test functions. In this configuration $u_t^+(x_0, t_0) < 0$ and $u_t^-(x_0, t_0) = 0$. We may assume that $\big(\xi, u(\xi)\big)$ is a common point of two facets $F[\xi^-, \xi]$ and $F[\xi, \xi^+]$. Without the loss of generality, we may assume that $F(\xi^-, \xi)$ has zero curvature while $F(\xi, \xi^+)$ has negative curvature, i.e. $\chi_l + \chi_r = -2$.

We deduce, that $g'(t_0) \in [A, 0]$, where $A = \frac{-2}{\xi^+ - \xi}$. Moreover, f may be faceted with facets $p = \pm 1$ as well as $|f'(x_0)| < 1$. Then, it is easy to check that (5.2) holds.

Due to Lemma 5.5 and 5.6 all cases corresponding to 1a) are exhausted.

Step 6. Let us now consider situation corresponding to 2a) and its consequences. If this occurs, then x_0 belongs to facet $F = F(\xi^-, \xi^+)$ and u is discontinuous at x_0. This discontinuity implies that x_0 as an endpoint of facet $F\big(\xi^-(t), \xi^+(t)\big)$ does not move for t in a neighborhood of t_0.

Furthermore, it may happen that $u_t(x_0, t_0)$ exists. Then our argument is similar to that used in Step 1, while taking into account that $|\Omega(x_0, t_0)| = 2$ and $u^*(x_0, t_0) = u(x_0, t_0)$ or $u^*(x_0, t_0) \neq u(x_0, t_0)$. The details are left to the interested reader except for a new situation arising when the test function has a slope different from ± 1. For the sake of definiteness we assume that the slope of F is 1. If $u_t(\cdot, t)$ is additionally continuous at x_0, then $u_t(x_0, t_0) = 0$. If $u_t(\cdot, t)$ is discontinuous at x_0, then $u_t(x_0, t_0) = -2/(\xi^+ - \xi^-)$. Any non-faceted test function $\psi(x,t) = f(x) + g(t)$ must be such that $f'(x_0) > 1$. If this happens, then $\Lambda_W^Z = \big(W_p\big(f'(x)\big)\big)_x\big|_{x=x_0} = 0$. Hence, (5.2) holds.

The case when F has slope -1 is handled in the same way.

If $u_t(x_0, t_0)$ does not exist, then we have several sub-cases:

1^* facet is an effect of the collision of $F(x_0, \xi^+)$ with $F(\zeta^-, \zeta^+)$. Furthermore, $F(\zeta^-, \zeta^+)$ may have positive or zero curvature;

2^* $F(a, x_0)$ is an effect of a collision of $F(\zeta^-, x_0)$, $(p = -1)$, with the boundary;

3^* $F(a, x_0)$ is an effect of a collision of $F(\zeta^-, x_0)$, $(p = -1)$, with a facet touching the boundary.

Those situations are analogous to that considered in Step 2, where we have a discontinuity of u at t_0. We also note that this discontinuity of u may lead to non-faceted test functions as in the previous paragraph. The details, however, are left to the interested reader.

The cases 1b) and 2b) are now easy and they are left to the reader. □

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