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Strongly anisotropic elliptic problems with regular and L^1 data

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Abstract. In this paper, we obtain the existence of weak solutions to a class of strongly anisotropic nonlinear elliptic boundary-value problems with nonlinear lower-order term with natural growth in an appropriate anisotropic function space. We investigate the cases where the right hand side term is regular or to be in L^1. A uniqueness result is also given in a particular case.

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1. Introduction

This work is devoted to the study of the anisotropic nonlinear elliptic problem

$$(P) \qquad \begin{cases} Au + g(x, u, Du) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ ($N \geq 3$), and A is a nonlinear operator acting from X into X^* (the space X as defined in Section 2) given by

$$Au = -\sum_{i=1}^{N} D_i(w_i|D_iu|^{q-2}D_iu) - \operatorname{div} a(x, u, Du)$$

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where D_i denotes the partial derivative $\dfrac{\partial}{\partial x_i}$, $q > 2$ and $w = \{w_i(x)\}_{i=1,\dots,N}$ is a vector of functions on Ω such that each w_i is a.e. positive in Ω and belong to $L^\infty(\Omega)$, while $a(x,s,\xi) = \{a_i(x,s,\xi)\}_{i=1,\dots,N} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-valued function, that is to say, measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω, and satisfies the following conditions.

(H1) There exist a constant $\beta > 0$ and a nonnegative function $k(x) \in L^1(\Omega)$ such that, for a.e. $x \in \Omega$, and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$

$$|a_i(x,s,\xi)| \le \beta\Big(k(x) + |s|^{\bar{p}} + \sum_{j=1}^N |\xi_j|^{p_j}\Big)^{1-1/p_i}, \qquad i = 1,\dots.N.$$

(The exponents p_i and $\bar{p}$ are defined below.)

(H2) For a.e. $x \in \Omega$, for every $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$

$$\big(a(x,s,\xi) - a(x,s,\xi')\big).(\xi - \xi') > 0.$$

(H3) There exists a constant $\alpha > 0$, such that for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$

$$a(x,s,\xi).\xi \ge \alpha \sum_{i=1}^N |\xi_i|^{p_i}.$$

As regard to the nonlinear lower-order term g, we assume that g has no growth conditions with respect to $|u|$, and satisfies the following classical sign condition and natural growth on $|Du|$:

(H4) For a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $g(x,s,\xi).s \ge 0$,

(H5) $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a Carathéodory function, such that for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$

$$|g(x,s,\xi)| \le h(|s|)\Big(c(x) + \sum_{j=1}^N |\xi_j|^{p_j}\Big), \qquad c \in L^1(\Omega),\ c \ge 0,$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function with finite values. Herein p_i are real positive numbers by $p_i > 2$, $i = 1,\dots,N$. We also assume $f \in L^m(\Omega)$, such that

$$m \ge \frac{N\bar{p}}{N\bar{p} - N + \bar{p}} \qquad \text{and} \qquad \bar{p} < N, \qquad \frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i},$$

or

$$\begin{cases} f \in L^1(\Omega), \\ \frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i < \frac{\bar{p}(N-1)}{N-\bar{p}} \quad \text{and} \quad \bar{p} < N, \\ |g(x,s,\xi)| \le h(|s|)\big(c(x) + \sum_{j=1}^N |\xi_j|^{\alpha_j}\big), \quad \alpha_j < p_j \ \forall j = 1,\dots,N. \end{cases} \tag{1}$$

The case of isotropic quasilinear elliptic equations ($p_i = p$, $i = 1,\dots,N$), where the principal part of the operator behaves like the Leray-Lions operator, has been the subject of numerous studies, we can cite, among others, the references [7], [9], [13], where the others obtained existence of solutions by considering lower order terms with quadratic growth or subquadratic growth with respect to the gradient. Let us also mention here the work of Bensoussan, Boccardo and Murat [5], where in particular, the authors proved the existence of solutions in the variational case by proceeding from different ideas which is based essentially on the strong convergence of the positive and negative parts of the approximate solution. In the anisotropic case, Li, Feng-Quan in [12] studied the problem (P) with $w_i = g = 0$ and a satisfying (H1)–(H3). The author proved the existence of a weak solution u in $\bigcap_{i=1}^N W_0^{1,(r_1,\dots,r_N)}(\Omega)$ with $r_i = \frac{p_i(\bar{p}-1)m^*}{\bar{p}}$ when $f \in L^m(\Omega)$ for $1 < m < \frac{N\bar{p}}{N\bar{p}-N+\bar{p}}$.

In the case of a datum f in $L^1(\Omega)$ or in $L^1 \log L^1(\Omega)$, with $w_i = g = 0$, and a does not depend on x and s, namely $a(x,s,\xi)$ is the vector field whose components are $|\xi_i|^{p_i-2}\xi_i$ ($i = 1,\dots,N$), then it has been proved in [6] that there exists a weak solution $u \in W_0^{1,(r_1,\dots,r_N)}(\Omega)$ for an anisotropic elliptic problem with Radon bounded measure data on Ω and $r_i \in \left[1, \frac{p_i(\bar{p}-1)N}{\bar{p}(N-1)}\right)$. If $f \in L^1(\Omega)$, $w_i = 0$, and g satisfies the following coercitivity condition, that is, there exists $\gamma > 0$ such that

$$|g(x,s,\xi)| \ge \gamma \sum_{i=1}^N |\xi_i|^{p_i}, \qquad \text{for } |s| \text{ sufficiently large},$$

it has been proved in [10] that there exists a weak solution $u \in W_0^{1,(p_1,\dots,p_N)}(\Omega)$ to the problem (P).

The purpose in this paper, is to follow the ideas of [5] to study such a problem in the anisotropic case. More precisely, under weak feeble restrictions on the functions a and g, namely, under the hypotheses (H1)–(H3) and g a nonlinear lower-order term, which depend on the solution and its gradient Du, having natural growth with respect to $|Du|$ with no growth restrictions on $|u|$ but a sign condition on g is assumed (see the conditions (H4) and (H5) above), we prove the existence result for the strongly anisotropic problem (P) when the datum f is assumed to be in $L^m(\Omega)$ with $m \ge \frac{N\bar{p}}{N\bar{p}-N+\bar{p}}$. In addition, if f belongs only in $L^1(\Omega)$, the existence of weak solutions is established. Also, in the case when f is in $L^1 \log L^1(\Omega)$, the existence result is, as well, deduced. We also prove the uniqueness of weak solutions to (P) in the case where g is a lower order term depending only on u. Our study is motivated by the use of a kind of anisotropic Sobolev inequality due to Troisi [22].

Let us point out that an interesting work in this direction can be found in [17] where the authors proved the existence of renormalized solutions for some anisotropic quasilinear elliptic equations. Finally, it would be interesting to mention that when g does not depend on Du, we refer the reader to the works [3] and [8], dealing with strongly nonlinear elliptic equations governed by a general class of anisotropic operators.

As a prototype example, we consider the model problem

$$-\sum_{i=1}^{N} D_i(w_i|D_iu|^{q-2}D_iu + |D_iu|^{p_i-2}D_iu) + g(x,u,Du) = f \qquad \text{in } \Omega,$$

where $g(x,s,\xi) = \frac{|s|\,\mathrm{sign}(s)}{\sqrt{1+|s|}}(\sum_{i=1}^{N}|\xi_i|^{\alpha_i})$, and the exponents $\alpha_i < p_i$ for $i = 1,\dots,N$.

2. Preliminaries

2.1. Anisotropic Sobolev spaces. We start by recalling that the notion of anisotropic Sobolev spaces were introduced and studied by Nikolskiĭ [15], Slobodeckiĭ [21], and Troisi [22], and later by Trudinger [23] in the framework of Orlicz spaces.

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 3)$ and let $p_1,\dots,p_N$ be N real numbers, with $1 < p_i < \infty$, $i = 1,\dots,N$. We denote by $W^{1,(p_1,\dots,p_N)}(\Omega)$, called anisotropic Sobolev space, the space of all real-valued functions $u \in L^{\bar{p}}(\Omega)$ such that the derivatives in the sense of distributions satisfy

$$D_iu \in L^{p_i}(\Omega) \qquad \text{for all } i = 1,\dots,N.$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p_1,\dots,p_N} = \left(\int_\Omega |u(x)|^{\bar{p}}\,dx\right)^{1/\bar{p}} + \sum_{i=1}^{N}\left(\int_\Omega |D_iu|^{p_i}\,dx\right)^{1/p_i}.$$

The space $W_0^{1,(p_1,\dots,p_N)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$, the space of real indefinitely differentiable functions of compact support in Ω, with respect to the norm $\|.\|_{1,p_1,\dots,p_N}$. The theory of such anisotropic Sobolev spaces was developed in [16], [18], [19] and [22]. It was proved that $C_0^\infty(\Omega)$ is dense in $W_0^{1,(p_1,\dots,p_N)}(\Omega)$ and $(W_0^{1,(p_1,\dots,p_N)}(\Omega), \|.\|_{1,p_1,\dots,p_N})$ is a reflexive Banach space for any $p_i > 1$. We recall that

$$W_0^{1,(p_1,\dots,p_N)}(\Omega) = \bigcap_{i=1}^{N} W_0^{1,p_i}(\Omega),$$

where $W_0^{1,p_i}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{0,p_i} = \|u\|_{p_i} + \sum_{j=1}^{N} \|D_j u\|_{p_i}.$$

In the following, we assume

$$p_i > 2 \quad \text{for all } i = 1, \dots, N \quad \text{and} \quad \sum_{i=1}^{N} \frac{1}{p_i} > 1.$$

Let $q > 2$ and let w_i, $i = 1, \dots, N$ be measurable functions satisfying $w_i \in L^\infty(\Omega)$ and $w_i > 0$ for a.e. in Ω.

Consider the weighted Lebesgue space associated to w_i defined by

$$L^q_{w_i}(\Omega) = \{u = u(x) : w_i^{1/q} u \in L^q(\Omega)\}.$$

In this space we define the norm

$$\|u\|_{w_i} = \left(\int_\Omega w_i |u|^q \, dx\right)^{1/q}.$$

Let X be the Banach space, called also anisotropic Sobolev space, obtained as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_X = \|u\|_{\bar{p}} + \max_{i=1}^{N} \left[\|D_i u\|_{p_i} \vee \left(\int_\Omega w_i |D_i u|^q \, dx\right)^{1/q}\right]. \tag{2}$$

We denote by X^* the dual space of X.

Theorem 2.1 ([2]). *Let V, W be Banach spaces such that*

(i) *$V \cap W$ is dense in both V and W.*

(ii) *If $(u_h)_{h \in \mathbb{N}}$ is a sequence in $V \cap W$ such that*

$$\|u_h - u\|_V \to 0, \qquad \|u_h - v\|_W \to 0 \implies u = v \in V \cap W.$$

Then the map $V' + W'$ to $(V \cap W)'$ defined by

$$\langle z, u' + v' \rangle_{(V \cap W)'} := \langle z, u' \rangle_{V'} + \langle z, v' \rangle_{W'}, \qquad \forall z \in V \cap W$$

is an isometric isomorphism.

Proposition 2.2. *The function space X endowed with the norm* (2) *is a reflexive Banach space. Moreover, for any $\phi \in X^*$ there exist $F \in \prod_{i=1}^N L_{w_i}^{q'}(\Omega)$ and $G \in \prod_{i=1}^N L^{p_i'}(\Omega)$ such that*

$$\langle \phi, \varphi \rangle = \int_\Omega (\langle wD\varphi, F\rangle + \langle G, D\varphi\rangle), \qquad \forall \varphi \in X.$$

Proof. (The proof follows the lines to that of Oppezzi and Rossi [17] for the case $q = 2$. For the completeness and for the reader's convenience we present it here in detail).

We have X is isometrically isomorphic to a closed subspace of

$$L^{\bar{p}}(\Omega) \times \left(\prod_{i=1}^N \left(L^{p_i}(\Omega) \cap L_{w_i}^q(\Omega) \right) \right).$$

We prove for $i = 1, \ldots, N$, the reflexivity of the space $L^{p_i}(\Omega) \cap L_{w_i}^q(\Omega)$, endowed with the norm $\|.\|_{p_i} \vee \|.\|_{w_i}$. Let (u_k) be a bounded sequence in $L^{p_i}(\Omega) \cap L_{w_i}^q(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Then there exist $u \in L^{p_i}(\Omega)$ and $v \in L_{w_i}^q(\Omega)$ such that, by going to a subsequence if necessary

$$u_k \rightharpoonup u \text{ and } u_k \rightharpoonup v \text{ weakly in } L^{p_i}(\Omega) \text{ and } L_{w_i}^q(\Omega) \text{ respectively.}$$

Since $\varphi w_i \in L^{p_i'}(\Omega)$, we see that

$$\int_\Omega u_k \varphi w_i \, dx \to \int_\Omega u \varphi w_i \, dx \quad \text{and} \quad \int_\Omega u_k \varphi w_i \, dx \to \int_\Omega v \varphi w_i \, dx,$$

so that $u = v$ by arbitrariness of φ and assumptions on w_i. Now by Theorem 2.1, thanks to the density of $L^{p_i}(\Omega) \cap L_{w_i}^q(\Omega)$ in both $L^{p_i}(\Omega)$ and $L_{w_i}^q(\Omega)$, it results that $\left(L^{p_i}(\Omega) \cap L_{w_i}^q(\Omega)\right)'$ is isomorphic to $L^{p_i'}(\Omega) + L_{w_i}^{q'}(\Omega)$ according to the definition

$$\langle z, u' + v' \rangle_{(L^{p_i}(\Omega) \cap L_{w_i}^q(\Omega))'} = \langle z, u' \rangle_{(L^{p_i}(\Omega))'} + \langle z, v' \rangle_{(L_{w_i}^q(\Omega))'}$$
$$\forall z \in L^{p_i}(\Omega) \cap L_{w_i}^q(\Omega).$$

On the other hand, by density of $C_0^\infty(\Omega)$ in both $\left(L^{p_i}(\Omega)\right)'$ and $\left(L_{w_i}^q(\Omega)\right)'$ it results

$$\langle u_k, u' \rangle_{(L^{p_i}(\Omega))'} \to \langle u, u' \rangle_{(L^{p_i}(\Omega))'} \quad \text{and} \quad \langle u_k, v' \rangle_{(L_{w_i}^q(\Omega))'} \to \langle u, v' \rangle_{(L_{w_i}^q(\Omega))'},$$

for all $u' \in \left(L^{p_i}(\Omega)\right)'$ and $v' \in \left(L_{w_i}^q(\Omega)\right)'$. Hence $u_k \rightharpoonup u$ in X and the reflexivity is proved.

Let $P : X \to \prod_{i=1}^N \big(L^{p_i}(\Omega) \cap L^q_{w_i}(\Omega)\big)$ such that $P(u) = Du$. If $\phi \in X^*$ is given, we define

$$\begin{aligned} \phi' : P(X) &\to \mathbb{R} \\ u &\mapsto \phi'\big(P(u)\big) = \phi(u) \end{aligned}.$$

Clearly $\phi' \in P(X)'$, so, by Hahn-Banach theorem there exists a norm preserving extension $\tilde{\phi} \in \big(\prod_{i=1}^N \big(L^{p_i}(\Omega) \cap L^q_{w_i}(\Omega)\big)\big)'$ of ϕ'. Therefore, by the isomorphism between $\big(L^{p_i}(\Omega) \cap L^q_{w_i}(\Omega)\big)'$ and $L^{p_i'}(\Omega) + \big(L^q_{w_i}(\Omega)\big)'$, there exist $F = (F_1, \dots, F_N)$ and $G = (G_1, \dots, G_N)$ such that

$$G_i + F_i \in L^{p_i'}(\Omega) + \big(L^q_{w_i}(\Omega)\big)', \qquad i = 1, \dots, N$$

and

$$\phi(\zeta) = \sum_{i=1}^N \langle \zeta_i, F_i \rangle_{(L^q_{w_i}(\Omega))'} + \langle \zeta_i, G_i \rangle_{L^{p_i'}(\Omega)} \qquad \text{for each } \zeta \in \prod_{i=1}^N \big(L^{p_i}(\Omega) \cap L^q_{w_i}(\Omega)\big).$$

Now, if $\varphi \in X$ we have

$$\tilde{\phi}(\varphi) = \phi'(D\varphi) = \sum_{i=1}^N (\langle D_i\varphi, F_i \rangle_{(L^q_{w_i}(\Omega))'} + \langle D_i\varphi, G_i \rangle_{L^{p_i'}(\Omega)}).$$

This concludes the Proof of Proposition 2.1. □

3. Main result

If $\xi : \Omega \to \mathbb{R}^N$ we introduce the notation $w\xi = (w_1\xi_1, \dots, w_N\xi_N)$, so that the operator $A : X \to X^*$ defined by

$$Au = -\sum_{i=1}^N D_i\big(w_i|D_iu|^{q-2}D_iu + a_i(x, u, Du)\big)$$

is well defined.

In this section we formulate and prove the main results of this paper. Our first main result is the following.

Theorem 3.1. *Let*

$$q > 2,\ \bar{p} < N, \quad \text{and} \quad m \geq \frac{N\bar{p}}{N\bar{p} - N + \bar{p}}, \qquad \frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}. \tag{3}$$

with $p_i > 2$, $i = 1, \dots, N$. *Assume* (H1)–(H5) *hold, and* $f \in L^m(\Omega)$. *Then the problem* (P) *has at least one solution* $u \in X$ *such that*

$$\begin{cases} g(x,u,Du) \in L^1(\Omega) \quad \text{and} \quad g(x,u,Du)u \in L^1(\Omega), \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x,u,Du)v\,dx = \langle f, v\rangle, \quad \forall v \in X. \end{cases}$$

In order to prove Theorem 3.1, we need the following anisotropic Sobolev inequality.

Lemma 3.2 ([22]). *Let* $p_i > 1$, $i = 1, \dots, N$ *and* $u \in W_0^{1,(p_1,p_2,\dots,p_N)}(\Omega)$. *Then*

$$\|u\|_s \le C \prod_{i=1}^N \|D_i u\|_{p_i}^{1/N}$$

where $s = \bar{p}^* = \dfrac{N\bar{p}}{N - \bar{p}}$ *if* $\bar{p} < N$ *with* $\bar{p}$ *given by* $\dfrac{1}{\bar{p}} = \dfrac{1}{N}\sum_{i=1}^N \dfrac{1}{p_i}$. *The constant* C *depends on* p_i *and* N. *Furthermore, if* $\bar{p} \ge N$, *this inequality is true for all* $s \ge 1$ *and* C *depends also on* s *and* $|\Omega|$.

Remark 3.3. If we assume that the exponents p_i satisfy $\bar{p} < N$. Then $\bar{p}^*$ is given by

$$\bar{p}^* = \frac{N}{\displaystyle\sum_{i=1}^N \frac{1}{p_i} - 1},$$

and there are continuous embeddings $W_0^{1,(p_1,p_2,\dots,p_N)}(\Omega) \hookrightarrow L^s(\Omega)$ for all $s \le \bar{p}^*$ which turn out to be compact only when $s < \bar{p}^*$.

Proof of Theorem 3.1. The proof of this theorem needs several steps.

`Step 1: Existence of the approximate problem.` Set

$$g_\varepsilon(x,u,Du) = \frac{g(x,u,Du)}{1 + \varepsilon|g(x,u,Du)|} \qquad \text{for } \varepsilon > 0,$$

and consider

$$b_\varepsilon(u,v) = \int_\Omega g_\varepsilon(x,u,Du)v\,dx,$$

for all $u, v \in X$.

Observe that $b_\varepsilon(u,v)$ is well defined since $g_\varepsilon(x,u,Du)$ is bounded with compact support. Define the following operator

$$\begin{aligned} G_\varepsilon u : X &\to \mathbb{R} \\ v &\to \int_\Omega g_\varepsilon(x,u,Du)v\,dx. \end{aligned}$$

Definition 3.4. The operator B from X to its dual X^* is called of the calculus of variations type, if B is bounded and is of the form

$$B(u) = B(u,u) \tag{4}$$

where $(u,v) \to B(u,v)$ is an operator from $X \times X$ into X^* satisfying the following properties:

$$\begin{gathered} \forall u \in X,\ v \to B(u,v) \text{ is bounded hemicontinuous from } X \text{ into } X^* \\ \text{and } \big(B(u,u) - B(u,v), u - v\big) \geq 0, \end{gathered} \tag{5}$$

$$\forall v \in X,\ u \to B(u,v) \text{ is bounded hemicontinuous from } X \text{ into } X^* \tag{6}$$

$$\begin{gathered} \text{if } u_n \rightharpoonup u \text{ weakly in } X \text{ and if } \big(B(u_n,u_n) - B(u_n,u), u_n - u\big) \to 0 \\ \text{then, } \forall v \in X,\ B(u_n,v) \rightharpoonup B(u,v) \text{ weakly in } X^*, \end{gathered} \tag{7}$$

$$\begin{gathered} \text{if } u_n \rightharpoonup u \text{ weakly in } X \text{ and if } B(u_n,v) \rightharpoonup \psi \text{ weakly in } X^*, \\ \text{then, } \big(B(u_n,v), u_n\big) \to (\psi, u). \end{gathered} \tag{8}$$

Proposition 3.5. *Under the assumptions* (H1), (H2) *and* (H3), *the operator* $B_\varepsilon = A + G_\varepsilon$ *is of the calculus of variations type. Moreover,* B_ε *is coercive, in the sense that:*

$$\lim_{\|u\|_X \to +\infty} \frac{\langle B_\varepsilon u, u\rangle}{\|u\|_X} = +\infty,$$

Proof. Put for $u, v, \psi \in X$

$$b_1(u,v,\psi) = \sum_{i=1}^N \int_\Omega \big(w_i |D_i v|^{q-2} D_i v D_i \psi + a_i(x,u,Dv) D_i \psi\big)\,dx$$

and

$$b_\varepsilon(u,\psi) = \int_\Omega g_\varepsilon(x,u,Du)\psi\,dx.$$

Then the mapping $\psi \mapsto b_1(u, v, \psi) + b_\varepsilon(u, \psi)$ is continuous in X and

$$b_1(u, v, \psi) + b_\varepsilon(u, \psi) = \langle B_\varepsilon(u, v), \psi \rangle, \qquad B_\varepsilon(u, u) = B_\varepsilon u.$$

The conditions (5) and (6) follows easily from (H1). Indeed for the boundedness we have

$$\begin{aligned}
&|\langle (A + G_\varepsilon)u, v \rangle| \\
&\quad = |\langle Au, v \rangle| + \left| \int_\Omega g_\varepsilon(x, u, Du) v \, dx \right| \\
&\quad \le \left(\left(\int_\Omega w_i |D_i u|^q \, dx \right)^{1/q} \right)^{q/q'} \left(\int_\Omega w_i |D_i v|^q \, dx \right)^{1/q} \\
&\qquad + \beta \sum_{i=1}^N \left[\left(\int_\Omega \left(k(x) + |u|^{\bar{p}} + \sum_{j=1}^N |D_j u|^{p_j} \, dx \right) \right)^{1/p_i'} \left(\int_\Omega |D_i v|^{p_i} \, dx \right)^{1/p_i} \right] \\
&\qquad + \left| \int_\Omega g_\varepsilon(x, u, Du) v \, dx \right|,
\end{aligned}$$

which gives using the fact that $|g_\varepsilon(x, u, Du)| \le \frac{1}{\varepsilon}$,

$$\begin{aligned}
|\langle (A + G_\varepsilon)u, v \rangle| &\le \|u\|_X^{q/q'} \|v\|_X + c_1 \|v\|_X (c_2 + \|u\|_X)^\gamma + c(\varepsilon) \|v\|_X \\
&\le \|v\|_X \left(\|u\|_X^{q/q'} + c_1 (c_2 + \|u\|_X)^\gamma + c(\varepsilon) \right),
\end{aligned}$$

where c_1, c_2 and $c(\varepsilon)$ are positive constants and γ is a positive real number. This implies the boundedness of $A + G_\varepsilon$.

Now, to show that $B_\varepsilon = A + G_\varepsilon$ is hemicontinuous, let $\lambda \to \lambda_0$ and prove that

$$\langle B_\varepsilon(u + \lambda v), \psi \rangle \to \langle B_\varepsilon(u + \lambda_0 v), \psi \rangle$$

for all $u, v, \psi \in X$. Since for a.e. $x \in \Omega$

$$\begin{aligned}
&w_i |D_i(u + \lambda v)|^{q-2} D_i(u + \lambda v) + a_i\big(x, u + \lambda v, \nabla(u + \lambda v)\big) \\
&\qquad \to w_i |D_i(u + \lambda_0 v)|^{q-2} D_i(u + \lambda_0 v) + a_i\big(x, u + \lambda_0 v, \nabla(u + \lambda_0 v)\big)
\end{aligned}$$

as $\lambda \to \lambda_0$, thanks to the growth condition (H1), we have

$$w_i |D_i(u + \lambda v)|^{q-2} D_i(u + \lambda v) \to w_i |D_i(u + \lambda_0 v)|^{q-2} D_i(u + \lambda_0 v)$$

weakly in $\big(L^q_{w_i}(\Omega)\big)'$ and

$$a_i\big(x, u + \lambda v, \nabla(u + \lambda v)\big) \to a_i\big(x, u + \lambda_0 v, \nabla(u + \lambda_0 v)\big)$$

weakly in $L^{p_i'}(\Omega)$ as $\lambda \to \lambda_0$.

Therefore

$$\langle b_1(u + \lambda v), \psi\rangle \to \langle b_1(u + \lambda_0 v), \psi\rangle \qquad \text{as } \lambda \to \lambda_0.$$

On the other hand, we have

$$g_\varepsilon\big(x, u + \lambda v, \nabla(u + \lambda v)\big) \to g\big(x, u + \lambda_0 v, \nabla(u + \lambda_0 v)\big)$$

as $\lambda \to \lambda_0$ for a.e. $x \in \Omega$. This implies

$$g_\varepsilon\big(x, u + \lambda v, \nabla(u + \lambda v)\big) \to g\big(x, u + \lambda_0 v, \nabla(u + \lambda_0 v)\big) \qquad \text{in } L^1(\Omega)$$

as $\lambda \to \lambda_0$ since $\big(g_\varepsilon\big(x, u + \lambda v, \nabla(u + \lambda v)\big)\big)_\lambda$ is bounded in $L^1(\Omega)$. Therefore,

$$\langle G_\varepsilon(u + \lambda v), \psi\rangle \to \langle G_\varepsilon(u + \lambda_0 v), \psi\rangle \qquad \text{as } \lambda \to \lambda_0.$$

Moreover, thanks to the assumption (H2), we have

$$\big(B_\varepsilon(u, u) - B_\varepsilon(u, v), u - v\big) = b_1(u, u, u - v) - b_1(u, v, u - v) \geq 0.$$

Arguing as Lemma 2.2 from [14], we have the property (7).

With regards to the assertion (8), assume that

$$u_n \rightharpoonup u \qquad \text{weakly in } X \tag{9}$$

and that

$$B_\varepsilon(u_n, v) \rightharpoonup \psi \qquad \text{weakly in } X^* \text{ as } n \to +\infty. \tag{10}$$

Thanks to the compact imbedding $X \hookrightarrow L^{\bar{p}}(\Omega)$ and in view of (H1), we obtain

$$b_1(u_n, v, u_n) \to b_1(u, v, u). \tag{11}$$

By Hölder's inequality, we have

$$|b_\varepsilon(u_n, u_n - u)| \leq C_\varepsilon \|u_n - u\|_{\bar{p}} \to 0 \qquad \text{as } n \to +\infty. \tag{12}$$

Using (10) and (11), we can write

$$b_\varepsilon(u_n,u) = \big(B_\varepsilon(u_n,v),u\big) - b_1(u_n,v,u) \to (\psi,u) - b_1(u,v,u),$$

then, with (12), we have

$$b_\varepsilon(u_n,u_n) \to (\psi,u) - b_1(u,v,u) \qquad \text{as } n \to +\infty.$$

Consequently

$$\big(B_\varepsilon(u_n,v),u_n\big) = b_1(u_n,v,u_n) + b_\varepsilon(u_n,u_n) \to (\psi,u) \qquad \text{as } n \to +\infty.$$

Now, we prove that B_ε is coercive. Indeed, let i_0 be such that

$$\|D_{i_0}u\|_{p_{i_0}} = \max\{\|D_iu\|_{p_i}, i = 1,\dots,N\}.$$

Hence, thanks to (H3) and (H4) we obtain

$$\begin{aligned}
\frac{\langle (A+G_\varepsilon)u,u\rangle}{\|u\|_X} &\geq \frac{\displaystyle\int_\Omega\Big(\sum_{i=1}^N w_i|D_iu|^q + \alpha\sum_{i=1}^N |D_iu|^{p_i}\Big)\,dx}{\|u\|_X} \\
&\geq \frac{\displaystyle\int_\Omega \sum_{i=1}^N w_i|D_iu|^q\,dx + \frac{\alpha}{2}\|D_iu\|_{p_i}^{p_i} + \frac{\alpha}{2}\|D_{i_0}u\|_{p_{i_0}}^{p_{i_0}}}{\displaystyle\|u\|_{\bar p} + \|D_{i_0}u\|_{p_{i_0}} + \sum_{i=1}^N\Big(\int_\Omega w_i|D_iu|^q\,dx\Big)^{1/q}} \\
&\geq K'\,\frac{\displaystyle\int_\Omega \sum_{i=1}^N w_i|D_iu|^q\,dx + \|D_{i_0}u\|_{p_{i_0}}^{p_{i_0}} + \|u\|_{\bar p}^{\bar p}}{\displaystyle\|u\|_{\bar p} + \|D_{i_0}u\|_{p_{i_0}} + \sum_{i=1}^N\Big(\int_\Omega w_i|D_iu|^q\,dx\Big)^{1/q}},
\end{aligned}$$

where K' is a suitable positive constant. Then the coerciveness follows immediately since the following assertion holds: $\lim_{x+y+z\to\infty} \dfrac{x^q+y^t+z^s}{x+y+z} = \infty$, when $x,y,z \in \mathbb{R}_+$ and $t,s \geq 1$. $\square$

Therefore, thanks to Proposition 3.1 and Theorem 2.7 of [14], there exists $u_\varepsilon \in X$ solution of the problem

$$Au_\varepsilon + g_\varepsilon(x,u_\varepsilon,Du_\varepsilon) = f,$$

or variationally

$$\sum_{i=1}^{N}\int_{\Omega}\big(w_i|D_iu_\varepsilon|^{q-2}D_iu_\varepsilon D_iv + a_i(x,u_\varepsilon,Du_\varepsilon)D_iv\big)\,dx + \int_{\Omega} g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)v\,dx$$
$$= \langle f, v\rangle \tag{13}$$

for all $v \in X$.

`Step 2: A priori estimates.` Remark first that the assumption (3) implies that

$$m' \le \bar{p}^* = \frac{N\bar{p}}{N-\bar{p}} \qquad \text{with } \bar{p} < N.$$

Then by Lemma 3.2, we have

$$\|u\|_{m'} \le \tilde{K}\|u\|_{\bar{p}^*} \le K\|D_{i_0}u\|_{p_{i_0}},$$

where $\tilde{K}$, K are positive constants. Substituting $v = u_\varepsilon$ in (13), using (H3) and (H4), due to the result of Troisi [22] stated in Lemma 3.1 and using the fact that $m' \le \bar{p}^*$, we see that

$$\alpha\sum_{i=1}^{N}\int_{\Omega}|D_iu_\varepsilon|^{p_i}\,dx \le \|f\|_m\|u_\varepsilon\|_{m'} \le c\|f\|_m\|u_\varepsilon\|_X,$$

where c is a positive constant. Then similarly as in the proof of the coerciveness argument, we get

$$K'\frac{\displaystyle\sum_{i=1}^{N}\int_{\Omega}w_i|D_iu_\varepsilon|^q\,dx + \|D_{i_0}u_\varepsilon\|_{p_{i_0}}^{p_{i_0}} + \|u_\varepsilon\|_{\bar{p}}^{\bar{p}}}{\displaystyle\|u_\varepsilon\|_{\bar{p}} + \|D_{i_0}u_\varepsilon\|_{p_{i_0}} + \sum_{i=1}^{N}\Big(\int_{\Omega}w_i|D_iu_\varepsilon|^q\,dx\Big)^{1/q}} \le \frac{\langle (A+G_\varepsilon)u_\varepsilon, u_\varepsilon\rangle}{\|u_\varepsilon\|_X} \le c\|f\|_m.$$

If we suppose by contradiction that $\|u_\varepsilon\|_X$ is not bounded, the left hand term of the above inequality becomes unbounded. Then

$$\|u_\varepsilon\|_X \le C, \qquad \forall\varepsilon > 0 \tag{14}$$

$$\int_{\Omega} g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon\,dx \le C, \qquad \forall\varepsilon > 0 \tag{15}$$

for some constant $C > 0$ independent of ε. By the similar argument above, we can prove that A is a bounded operator, then we get

$$\|Au_\varepsilon\|_{X^*} \le C', \tag{16}$$

for some constant $C' > 0$ independent of ε.

`Step 3: Convergence of` u_ε. In view of Proposition 2.1, X is reflexive, then we deduce from (14) and (16) that

$$\begin{aligned} u_\varepsilon &\to u \quad \text{weakly in } X, \\ u_\varepsilon &\to u \quad \text{strongly in } L^{\bar{p}}(\Omega), \\ D_i u_\varepsilon &\to D_i u \quad \text{weakly in } L^{p_i}(\Omega), \\ D_i u_\varepsilon &\to D_i u \quad \text{weakly in } L^q_{w_i}(\Omega), \\ Au_\varepsilon &\to \chi \quad \text{weakly in } X^*. \end{aligned}$$

This implies that we can extract a subsequence still denoted by u_ε such that

$$u_\varepsilon \to u \quad \text{a.e. in } \Omega. \tag{17}$$

This is not sufficient to pass to the limit in g_ε. We need for instance

$$Du_\varepsilon \to Du \quad \text{a.e. in } \Omega. \tag{18}$$

In fact, inspired by the work [5], we prove that

$$u_\varepsilon^+ \to u^+, \quad u_\varepsilon^- \to u^- \quad \text{and} \quad Du_\varepsilon \to Du \quad \text{a.e. in } \Omega.$$

Let $k > 0$. Define $u_k^+ = u^+ \wedge k = \min\{u^+, k\}$. We shall fix k, and use the notation

$$z_\varepsilon = u_\varepsilon^+ - u_k^+.$$

Then we have $z_\varepsilon \in X$ and $z_\varepsilon^+ \in X$. Putting $v = z_\varepsilon^+$ in (13), we obtain

$$\langle Au_\varepsilon, z_\varepsilon^+ \rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) z_\varepsilon^+ \, dx = \langle f, z_\varepsilon^+ \rangle.$$

Note that if $z_\varepsilon^+ > 0$, we have $u_\varepsilon > 0$ (if not, there exists x such that $u_\varepsilon(x) \le 0$, then $u_\varepsilon^+(x) = 0$ which implies that $z_\varepsilon(x) = -u_k^+(x) = -\min\{u^+(x), k\} \le 0$, so $z_\varepsilon^+(x) = 0$). Hence from (H4), we get $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \ge 0$. Thus we have

$\langle Au_\varepsilon, z_\varepsilon^+\rangle \le \langle f, z_\varepsilon^+\rangle$. Therefore

$$\sum_{i=1}^N \int_\Omega w_i|D_i u_\varepsilon|^{q-2} D_i u_\varepsilon D_i z_\varepsilon^+ \,dx + \int_\Omega a(x,u_\varepsilon,Du_\varepsilon) Dz_\varepsilon^+ \,dx \le \langle f, z_\varepsilon^+\rangle.$$

Since $u_\varepsilon = u_\varepsilon^+$ in $\{x\in\Omega : z_\varepsilon^+ > 0\}$, we may write

$$\sum_{i=1}^N \int_\Omega w_i|D_i u_\varepsilon^+|^{q-2} D_i u_\varepsilon^+ D_i z_\varepsilon^+ \,dx + \int_\Omega a(x,u_\varepsilon,Du_\varepsilon^+) Dz_\varepsilon^+ \,dx \le \langle f, z_\varepsilon^+\rangle,$$

which implies

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega w_i|D_i u_\varepsilon^+|^{q-2} D_i u_\varepsilon^+ D_i (u_\varepsilon^+ - u_k^+)^+ \,dx \\ &\qquad + \int_\Omega [a(x,u_\varepsilon,Du_\varepsilon^+) - a(x,u_\varepsilon,Du_k^+)] D(u_\varepsilon^+ - u_k^+)^+ \,dx \\ &\quad \le -\int_\Omega a(x,u_\varepsilon,Du_k^+) D(u_\varepsilon^+ - u_k^+)^+ + \langle f, z_\varepsilon^+\rangle. \end{aligned} \tag{19}$$

As $\varepsilon\to 0$, we have

$$z_\varepsilon^+ \to (u^+ - u_k^+)^+ \quad \text{a.e. in } \Omega.$$

However z_ε^+ is bounded in X; hence

$$z_\varepsilon^+ \rightharpoonup (u^+ - u_k^+)^+ \quad \text{in } X.$$

Using the fact that

$$a(x,u_\varepsilon,Du_k^+) \to a(x,u,Du_k^+) \quad \text{in } \prod_{i=1}^N L^{p_i'}(\Omega),$$

$$|D_i u_\varepsilon|^{q-2} D_i u_\varepsilon \to |D_i u|^{q-2} D_i u \quad \text{in } L_{w_i}^{q'}(\Omega)$$

and by passing to the limit in ε in (19), we obtain

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\Big(\sum_{i=1}^N \int_\Omega w_i|D_i u_\varepsilon^+|^{q-2} D_i u_\varepsilon^+ D_i (u_\varepsilon^+ - u_k^+)^+ \,dx \\ &\qquad + \int_\Omega [a(x,u_\varepsilon,Du_\varepsilon^+) - a(x,u_\varepsilon,Du_k^+)] D(u_\varepsilon^+ - u_k^+)^+ \,dx\Big) \\ &\quad \le R_k \end{aligned} \tag{20}$$

with

$$R_k = -\int_\Omega a(x,u,Du_k^+)D(u^+ - u_k^+)^+ + \langle f, (u^+ - u_k^+)^+ \rangle.$$

Since $(u^+ - u_k^+)^+ \to 0$ in X as $k \to \infty$, we have $R_k \to 0$ as $k \to \infty$.

Now, let us prove the following assertion

$$\begin{aligned} -\liminf_{\varepsilon \to 0}\Big(&\sum_{i=1}^N \int_\Omega w_i |D_i u_\varepsilon^+|^{q-2} D_i u_\varepsilon^+ D_i (u_\varepsilon^+ - u_k^+)^- \, dx \\ &+ \int_\Omega [a(x,u_\varepsilon,Du_\varepsilon^+) - a(x,u_\varepsilon,Du_k^+)] D(u_\varepsilon^+ - u_k^+)^- \, dx\Big) \\ \le 0. \end{aligned} \tag{21}$$

Indeed, we shall use the test function $v_\varepsilon = \varphi_\lambda(z_\varepsilon^-)$ with $\varphi_\lambda(s) = se^{\lambda s^2}$ in (13) (where λ will be chosen later). We have $0 \le z_\varepsilon^- \le k$, hence $z_\varepsilon^- \in L^\infty(\Omega)$ and since $z_\varepsilon^- \in X$, clearly we have $v_\varepsilon \in X$. Then we deduce

$$\begin{aligned} \sum_{i=1}^N \int_\Omega & w_i |D_i u_\varepsilon|^{q-2} D_i u_\varepsilon D_i z_\varepsilon^- \varphi_\lambda'(z_\varepsilon^-) \, dx + \int_\Omega a(x,u_\varepsilon,Du_\varepsilon) D z_\varepsilon^- \varphi_\lambda'(z_\varepsilon^-) \, dx \\ &+ \int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\varphi_\lambda(z_\varepsilon^-) \, dx \\ &= \langle f, \varphi_\lambda(z_\varepsilon^-) \rangle. \end{aligned}$$

Define

$$E_\varepsilon = \{x \in \Omega : u_\varepsilon^+(x) \le u_k^+(x)\} \quad \text{and} \quad F_\varepsilon = \{x \in \Omega : 0 \le u_\varepsilon(x) \le u_k^+(x)\}.$$

We have

$$\int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\varphi_\lambda(z_\varepsilon^-) \, dx = \int_{E_\varepsilon} g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\varphi_\lambda(z_\varepsilon^-) \, dx.$$

If $u_\varepsilon \le 0$, we have $g_\varepsilon(x,u_\varepsilon,Du_\varepsilon) \le 0$ and since $\varphi_\lambda(z_\varepsilon^-) \ge 0$, using (H5), we obtain

$$\begin{aligned} \int_{E_\varepsilon} g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\varphi_\lambda(z_\varepsilon^-) \, dx &\le \int_{F_\varepsilon} g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\varphi_\lambda(z_\varepsilon^-) \, dx \\ &\le \int_{F_\varepsilon} b(|u_\varepsilon|)\Big(c(x) + \sum_{i=1}^N |D_i u_\varepsilon|^{p_i}\Big)\varphi_\lambda(z_\varepsilon^-) \, dx. \end{aligned} \tag{22}$$

Thanks to the structure assumption (H3), we can write

$$b(|u_\varepsilon|)\Big(c(x) + \sum_{i=1}^{N} |D_i u_\varepsilon|^{p_i}\Big)\varphi_\lambda(z_\varepsilon^-)$$
$$\le b(|u_\varepsilon|)\bigg(c(x) + \frac{1}{\alpha}a(x,u_\varepsilon,Du_\varepsilon)Du_\varepsilon\bigg)\varphi_\lambda(z_\varepsilon^-). \tag{23}$$

Now, using (22) and (23), we obtain

$$\int_{E_\varepsilon} g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\,dx \le \frac{1}{\alpha}b(k)\int_{F_\varepsilon} a(x,u_\varepsilon,Du_\varepsilon)Du_\varepsilon\varphi_\lambda(z_\varepsilon^-)\,dx + b(k)\int_\Omega c(x)\varphi_\lambda(z_\varepsilon^-)\,dx.$$

As in [5], by choosing $\lambda = \frac{b(k)^2}{4\alpha^2}$, we deduce that

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{N}\int_\Omega w_i|D_iu_\varepsilon^+|^{q-2}D_iu_\varepsilon^+D_i(u_\varepsilon^+-u_k^+)^-\,dx\\
&\quad-\frac{1}{2}\int_\Omega[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,Du_k^+)]D(u_\varepsilon^+-u_k^+)^-\,dx\\
&\le -\frac{1}{2}\sum_{i=1}^{N}\int_\Omega w_i|D_iu_\varepsilon^+|^{q-2}D_iu_\varepsilon^+D_i(u_\varepsilon^+-u_k^+)^-\,dx\\
&\quad+\int_\Omega[a(x,u_\varepsilon,Du_\varepsilon)-a(x,u_\varepsilon,Du_\varepsilon^+)]Du_k^+\varphi_\lambda'(u_k^+)\,dx\\
&\quad+\langle -f,\varphi_\lambda(z_\varepsilon^-)\rangle+\int_\Omega a(x,u_\varepsilon,Du_k^+)Dz_\varepsilon^-\varphi_\lambda'(z_\varepsilon^-)\,dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,u_\varepsilon,Du_\varepsilon^+)Du_k^+\varphi_\lambda(z_\varepsilon^-)\,dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,u_\varepsilon,Du_k^+)D(u_\varepsilon^+-u_k^+)\varphi_\lambda(z_\varepsilon^-)\,dx+b(k)\int_\Omega c(x)\varphi_\lambda(z_\varepsilon^-)\,dx.
\end{aligned}$$

Extracting subsequences satisfying

$$|D_iu_\varepsilon^+|^{q-2}D_iu_\varepsilon^+ \rightharpoonup v_i \qquad \text{in } L^{q'}(w_i),\ i=1,\dots,N$$

and

$$a(x,u_\varepsilon,Du_\varepsilon) \rightharpoonup \sigma_1 \qquad \text{and} \qquad a(x,u_\varepsilon,Du_\varepsilon^+) \rightharpoonup \sigma_2 \qquad \text{in } \prod_{i=1}^{N} L^{p_i'}(\Omega). \tag{24}$$

For k fixed and when ε tends to zero, using Lebesgue's dominated convergence theorem, the right side of the above inequality becomes

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{N}\int_\Omega v_i D_i(u^+-u_k^+)^-\,dx+\int_\Omega[\sigma_1(x)-\sigma_2(x)]Du_k^+\varphi_\lambda'(u_k^+)\,dx\\
&\quad+\big\langle -f,\varphi_\lambda\big((u^+-u_k^+)^-\big)\big\rangle+\int_\Omega a(x,u,Du_k^+)D(u^+-u_k^+)^-\varphi_\lambda'\big((u^+-u_k^+)^-\big)\,dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega \sigma_2(x)Du_k^+\varphi_\lambda\big((u^+-u_k^+)^-\big)\,dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,u,Du_k^+)D(u^+-u_k^+)\varphi_\lambda\big((u^+-u_k^+)^-\big)\,dx\\
&\quad+b(k)\int_\Omega c(x)\varphi_\lambda\big((u^+-u_k^+)^-\big)\,dx=\int_\Omega[\sigma_1(x)-\sigma_2(x)]Du_k^+\varphi_\lambda'(u_k^+)\,dx
\end{aligned}
$$

since $(u^+-u_k^+)^-=0$, $\varphi_\lambda(0)=0$ and $\big(a(x,u_\varepsilon,Du_\varepsilon)-a(x,u_\varepsilon,Du_\varepsilon^+)\big)(u_\varepsilon)_k^+=0$ a.e. Because, if $u_\varepsilon<0$ we have $(u_\varepsilon)_k^+=0$. This implies $\big(\sigma_1(x)-\sigma_2(x)\big)u_k^+=0$, therefore we get

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\Big(-\sum_{i=1}^{N}\int_\Omega w_i|D_iu_\varepsilon^+|^{q-2}D_iu_\varepsilon^+D_i(u_\varepsilon^+-u_k^+)^-\,dx\\
&\qquad\qquad-\int_\Omega[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,Du_k^+)]D(u_\varepsilon^+-u_k^+)^-\,dx\Big)\\
&\quad\le 0.
\end{aligned}
\tag{25}
$$

As in [5], from (20) and (25), we have

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\Big(\sum_{i=1}^{N}\int_\Omega w_i|D_iu_\varepsilon^+|^{q-2}D_iu_\varepsilon^+D_i(u_\varepsilon^+-u^+)\,dx\\
&\qquad\qquad+\int_\Omega[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,Du^+)]D(u_\varepsilon^+-u^+)\Big)\,dx\\
&\quad\le R_k+\int_\Omega[\sigma_2(x)-a(x,u,Du_k^+)]D(u_k^+-u^+).
\end{aligned}
$$

Letting $k\to\infty$ and using (Lemma 2.3, [14]), we obtain

$$
u_\varepsilon^+\to u^+\qquad\text{in } X \text{ strongly.}\tag{26}
$$

Now we want to prove the convergence of the negative part of u_ε. Indeed, as in the preceding step, we shall prove that

$$
u_\varepsilon^-\to u^-\qquad\text{in } X \text{ strongly.}\tag{27}
$$

Define $u_k^- = u^- \wedge k,\ y_\varepsilon = u_\varepsilon^- - u_k^-$, and putting $v = y_\varepsilon^+$ in (13), we get

$$\sum_{i=1}^{N} \int_\Omega w_i |D_i u_\varepsilon|^{q-2} D_i u_\varepsilon D_i y_\varepsilon^+ \, dx + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon) D y_\varepsilon^+ \, dx + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) y_\varepsilon^+ \, dx$$
$$= \langle f, y_\varepsilon^+ \rangle.$$

Since $y_\varepsilon^+ > 0$ implies $u_\varepsilon < 0$, then we have $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \le 0$. Hence $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) y_\varepsilon^+ \le 0$ a.e. in Ω. Then we get

$$\sum_{i=1}^{N} \int_\Omega w_i |D_i u_\varepsilon|^{q-2} D_i u_\varepsilon D_i y_\varepsilon^+ \, dx + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon) D y_\varepsilon^+ \, dx \ge \langle f, y_\varepsilon^+ \rangle.$$

Since $u_\varepsilon = -u_\varepsilon^-$ on the set $\{x \in \Omega : y_\varepsilon^+ > 0\}$, we can write

$$-\sum_{i=1}^{N} \int_\Omega w_i |D_i u_\varepsilon|^{q-2} D_i u_\varepsilon D_i y_\varepsilon^+ \, dx + \int_\Omega a(x, u_\varepsilon, -Du_\varepsilon^-) D y_\varepsilon^+ \, dx \ge \langle f, y_\varepsilon^+ \rangle,$$

which implies

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega w_i |D_i u_\varepsilon|^{q-2} D_i u_\varepsilon D_i (u_\varepsilon^- - u_k^-)^+ \, dx \\ &\qquad - \int_\Omega [a(x, u_\varepsilon, -Du_\varepsilon^-) - a(x, u_\varepsilon, -Du_k^-)] D(u_\varepsilon^- - u_k^-)^+ \, dx \\ &\quad \le \int_\Omega a(x, u_\varepsilon, -Du_k^-) D(u_\varepsilon^- - u_k^-)^+ \, dx - \langle f, y_\varepsilon^+ \rangle. \end{aligned}$$

As $\varepsilon \to 0$ we have $y_\varepsilon^+ \to (u^- - u_k^-)^+$ a.e. in Ω. Since y_ε^+ is bounded in X, $y_\varepsilon^+ \rightharpoonup (u^- - u_k^-)^+$ in X (for k fixed).

Passing to the limit in ε we obtain

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \Big(\sum_{i=1}^{N} \int_\Omega w_i |D_i u_\varepsilon|^{q-2} D_i u_\varepsilon D_i (u_\varepsilon^- - u_k^-)^+ \, dx \\ &\qquad - \int_\Omega [a(x, u_\varepsilon, -Du_\varepsilon^-) - a(x, u_\varepsilon, -Du_k^-)] D(u_\varepsilon^- - u_k^-)^+ \, dx \Big) \\ &\quad \le \tilde{R}_k, \end{aligned} \tag{28}$$

with

$$\tilde{R}_k = \int_\Omega a(x, u, -Du_k^-) D(u^- - u_k^-)^+ - \langle f, (u^- - u_k^-)^+ \rangle,$$

and $\tilde{R}_k \to 0$ as $k \to +\infty$ since $(u^- - u_k^-)^+ \to 0$ in X as $k \to \infty$.

By considering again as test function $v_\varepsilon = \varphi_\lambda(y_\varepsilon^-)$, we show as above that

$$\limsup_{\varepsilon\to 0}\Big(\sum_{i=1}^N \int_\Omega w_i |D_i u_\varepsilon|^{q-2} D_i u_\varepsilon D_i (u_\varepsilon^- - u_k^-)^- \, dx + \int_\Omega [a(x,u_\varepsilon,-Du_\varepsilon^-) - a(x,u_\varepsilon,-Du_k^-)] D(u_\varepsilon^- - u_k^-)^- \, dx\Big) \le 0. \tag{29}$$

Finally combining (28) and (29), we see that

$$u_\varepsilon^- \to u^- \qquad \text{strongly in } X. \tag{30}$$

Therefore, since g is continuous, we get the conclusion

$$g_\varepsilon(x,u_\varepsilon,Du_\varepsilon) \to g(x,u,Du) \qquad \text{a.e. in } \Omega$$

and

$$g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon \to g(x,u,Du)u \qquad \text{a.e. in } \Omega.$$

From (15), the assumption (H4), and in view of Fatou's lemma, we obtain

$$\int_\Omega g(x,u,Du)u\,dx \le \lim_{\varepsilon\to 0}\int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon\,dx \le C, \tag{31}$$

which implies that

$$g(x,u,Du)u \in L^1(\Omega).$$

Now let $\delta > 0$. In view of Hölder inequality we can write

$$\begin{aligned}\int_E |g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)|\,dx &\le \int_{E\cap\{|u_\varepsilon|\le\delta\}} |g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)|\,dx \\ &\quad + \delta^{-1}\int_{E\cap\{|u_\varepsilon|>\delta\}} g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon\,dx \\ &\le h(\delta)\int_E \Big(c(x) + \sum_{i=1}^N |D_i u_\varepsilon|^{p_i}\Big)\,dx + \delta^{-1}C,\end{aligned} \tag{32}$$

where E is any measurable subset of Ω and C is the constant of (15) which is independent of k. Thanks to (14), the above inequality implies the equi-integrability of $g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)$.

Thanks to (24), (26), (32) and Vitali's theorem we get

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \to g(x, u, Du) \qquad \text{strongly in } L^1(\Omega).$$

Hence it follows that $g(x, u, Du) \in L^1(\Omega)$.

Passing to the limit in (13), we obtain

$$\langle \chi, v \rangle + \int_\Omega g(x, u, Du) v \, dx = \langle f, v \rangle \qquad \text{for all } v \in X.$$

It remains to show that $Au = \chi$. For this purpose, note that since A is bounded, hemicontinuous and monotone, then A is pseudo-monotone.

Now, by substituting $v = u_\varepsilon$ in (13), and in view of (31) we get

$$\limsup_{\varepsilon \to 0} \langle Au_\varepsilon, u_\varepsilon \rangle \le \langle f, u \rangle - \int_\Omega g(x, u, Du) u \, dx.$$

This implies

$$\limsup_{\varepsilon \to 0} \langle Au_\varepsilon, u_\varepsilon \rangle \le \langle \chi, v \rangle.$$

Since A is a pseudo-monotone operator, then $\chi = Au$. Finally, we conclude that

$$\begin{cases} g(x, u, Du) \in L^1(\Omega), \quad g(x, u, Du) u \in L^1(\Omega) \\ \langle Au, v \rangle + \int_\Omega g(x, u, Du) v \, dx = \langle f, v \rangle \quad \text{for all } v \in X. \end{cases}$$

This completes the proof of Theorem 3.1. □

Before stating our second main result of this paper, let us first define the Banach space Y obtained as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_Y = \|u\|_{\bar{\kappa}} + \max_{i=1}^{N} \left[\|D_i u\|_{\kappa_i} \vee \left(\int_\Omega w_i |D_i u|^r \, dx \right)^{1/r} \right],$$

where κ_i and r are as in the following theorem.

Theorem 3.6. *Let $p_i \ge 2$, $i = 1, \dots, N$. Assume* (H1)–(H5) *hold and $f \in L^1(\Omega)$. Furthermore, assume that*

$$1 \le \kappa_i < p_i \frac{N(\bar{p} - 1)}{\bar{p}(N - 1)}, \qquad i = 1, \dots, N. \tag{33}$$

Then for all $1 < r < \dfrac{q\bar{\kappa}^*}{1 + \bar{\kappa}^*}$, there exists $u \in Y$ solution of problem (P).

Proof. For fixed $k > 0$, we define the truncation T_k at levels $\mp k, k > 0$ by

$$T_k(s) = \begin{cases} s, & |s| \le k; \\ k \ \mathrm{sign}(s), & |s| > k. \end{cases}$$

Let $f_n = T_n(f)$ be a sequence of bounded functions, such that

$$f_n \to f \quad \text{strongly in } L^1(\Omega) \quad \text{and} \quad |f_n| \le |f|.$$

Let us define the sequence of approximate problems (P_n) by

$$(P_n) \qquad \begin{cases} Au_n + g(x, u_n, Du_n) = f_n, & \text{in } \Omega \\ u_n = 0, & \text{on } \partial\Omega. \end{cases}$$

Thanks to Theorem 3.1, there exists a solution $u_n \in X$ of the problem (P_n).

Let us choose as a test function in (P_n)

$$\phi(u_n) = \phi_\delta(u_n) = (\delta - 1) \int_0^{u_n} \frac{dt}{(1 + |t|)^\delta}, \qquad \delta > 1.$$

Then, using the assumption (H3) and the fact that $|\phi(u_n)| \le 1$, we deduce

$$\int_\Omega \frac{w_i |D_i u_n|^q}{(1 + |u_n|)^\delta} dx \le C \quad \text{and} \quad \int_\Omega \frac{|D_i u_n|^{p_i}}{(1 + |u_n|)^\delta} dx \le C. \tag{34}$$

To carry on the proof, we need the following Lemmas 3.7–3.10.

Lemma 3.7. *Let p_i and κ_i be such that*

$$p_i \ge 2 \quad \text{and} \quad \kappa_i \in \left[1, p_i \frac{N(\bar{p} - 1)}{\bar{p}(N - 1)}\right), \qquad i = 1, \dots, N$$

where $\bar{p} < N$ and $\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}$ and let $1 < r < \frac{q\bar{\kappa}^*}{1 + \bar{\kappa}^*}$. Then

- *$(u_n)_n$ remains in a bounded set of $L^{\bar{\kappa}}(\Omega)$.*
- *$(D_i u_n)_n$ remains in a bounded set of $L^{\kappa_i}(\Omega)$.*
- *$(D_i u_n)_n$ remains in a bounded set of $L^r_{w_i}(\Omega)$.*

Proof. We can assume that $\kappa_i / p_i = \bar{\kappa} / \bar{p}$. If not, we set $\theta = \max\{\kappa_i / p_i, i = 1, \dots, N\}$ and replace κ_i by θp_i. Then, since $\theta p_i \ge \kappa_i$, the fact that $(D_i u_n)_n$ remains in a bounded set of $L^{\theta p_i}(\Omega)$ implies the result.

From now on, we set $\kappa_i = \theta p_i$, with $\theta \in \left(0, \frac{N(\bar{p}-1)}{\bar{p}(N-1)}\right) \subset (0,1)$. Since $\kappa_i < p_i \frac{N(\bar{p}-1)}{\bar{p}(N-1)}$ then we have $\bar{\kappa} < \frac{N(\bar{p}-1)}{N-1}$. This implies that $\frac{\theta}{1-\theta} < \bar{\kappa}^*$, $\theta = \bar{\kappa}/\bar{p}$. Hence there exists $\delta > 1$ such that

$$\frac{\delta\theta}{1-\theta} \leq \bar{\kappa}^*, \qquad \bar{\kappa}^* = \frac{N\bar{\kappa}}{N-\bar{\kappa}}. \tag{35}$$

By using Hölder inequality, we obtain

$$\begin{aligned}\int_\Omega |D_i u_n|^{\kappa_i}\,dx &= \int_\Omega \frac{|D_i u_n|^{\kappa_i}}{(1+|u_n|)^{\delta\theta}}(1+|u_n|)^{\delta\theta}\,dx \\ &\leq \left(\int_\Omega \frac{|D_i u_n|^{p_i}}{(1+|u_n|)^{\delta}}\,dx\right)^{\theta}\left(\int_\Omega (1+|u_n|)^{\delta(\theta/(1-\theta))}\,dx\right)^{1-\theta}. \end{aligned} \tag{36}$$

In view of (34) and (36), we get

$$\begin{aligned}\int_\Omega |D_i u_n|^{\kappa_i}\,dx &\leq C\left(\int_\Omega (1+|u_n|)^{\delta(\theta/(1-\theta))}\,dx\right)^{1-\theta} \\ &\leq C\left(\int_\Omega (1+|u_n|)^{\bar{\kappa}^*}\,dx\right)^{1-\theta}. \end{aligned} \tag{37}$$

Hence

$$\left(\int_\Omega |D_i u_n|^{\kappa_i} dx\right)^{1/N\kappa_i} \leq C\left(\int_\Omega (1+|u_n|)^{\bar{\kappa}^*}\,dx\right)^{1/N\kappa_i - 1/Np_i}.$$

Now we apply Lemma 3.2 and using (35), we can write

$$\begin{aligned}\|u_n\|_{\bar{\kappa}^*} &\leq C\left(\int_\Omega (1+|u_n|)^{\bar{\kappa}^*}\,dx\right)^{1/\bar{\kappa}-1/\bar{p}} \\ &= C + C\|u_n\|_{\bar{\kappa}^*}^{\bar{\kappa}^*(1/\bar{\kappa}-1/\bar{p})}. \end{aligned} \tag{38}$$

Therefore, since $\bar{p} < N$, we have $\bar{\kappa}^*\left(\frac{1}{\bar{\kappa}} - \frac{1}{\bar{p}}\right) < 1$ and by the last inequality, we deduce that the sequence $(u_n)_n$ is bounded in $L^{\bar{\kappa}^*}(\Omega)$, so that $(u_n)_n$ is bounded in $L^{\bar{\kappa}}(\Omega)$. The combination of this and (37), implies that $(D_i u_n)_n$ is bounded in $L^{\kappa_i}(\Omega)$ for all $i = 1, \ldots, N$. Concerning the boundedness of $(D_i u_n)_n$ in $L^r_{w_i}(\Omega)$, using again Hölder inequality, we have for $0 < \theta < 1$ such that $r = \theta q$ $(r < q)$

$$\begin{aligned}\int_\Omega w_i |D_i u_n|^r\,dx &= \int_\Omega \left(\frac{w_i^\theta |D_i u_n|^{\theta q}}{(1+|u_n|)^{\delta\theta}}\right)\left(w_i^{1-\theta}(1+|u_n|)^{\delta\theta}\right)dx \\ &\leq \left(\int_\Omega \frac{w_i |D_i u_n|^q}{(1+|u_n|)^{\delta}}\,dx\right)^{\theta}\left(\int_\Omega w_i (1+|u_n|)^{\delta(\theta/(1-\theta))}\,dx\right)^{1-\theta}. \end{aligned}$$

Since $r < \frac{q\bar{\kappa}^*}{1+\bar{\kappa}^*}$, then we have $\frac{\theta}{1-\theta} < \bar{\kappa}^*$. Hence, there exists $\delta > 1$ such that $\frac{\delta\theta}{1-\theta} \leq \bar{\kappa}^*$. As above, we deduce, thanks to (38), that $\int_\Omega w_i |D_i u_n|^r \, dx \leq C$ since $w_i \in L^\infty(\Omega)$. Therefore

$$D_i u_n \to D_i u \quad \text{weakly in } L^r_{w_i}(\Omega),$$
$$u_n \to u \quad \text{strongly in } L^{\bar{\kappa}}(\Omega).$$

Lemma 3.8. *There exists a constant* $C > 0$ *such that for all* $n \geq 1$

$$\|T_k(u_n)\|_X \leq Ck \tag{39}$$

and

$$\|g(., u_n, Du_n)\|_1 \leq C, \tag{40}$$

where T_k *is the truncation defined by*

$$T_k(s) = \begin{cases} s, & |s| \leq k; \\ k \ \mathrm{sign}(s), & |s| > k, \end{cases}$$

for fixed $k > 0$.

Proof of Lemma 3.8. Taking $T_k(u_n)$ as a test function in (P_n), we find

$$\begin{aligned} \sum_{i=1}^N \int_\Omega \big(w_i |D_i u_n|^q T_k'(u_n) + a_i(x, u_n, Du_n) D_i u_n T_k'(u_n)\big) \, dx \\ + \int_\Omega g(x, u_n, Du_n) T_k(u_n) \, dx \\ \leq k\|f\|_1. \end{aligned} \tag{41}$$

On one hand, using (H3), (H4) and the fact that $(T_k')^{p_i} = T_k'$ and $(T_k')^q = T_k'$ we obtain (39). The estimate (41) also leads to

$$\int_{|u_n| \leq k} g(x, u_n, Du_n) u_n \, dx + k \int_{|u_n| > k} |g(x, u_n, Du_n)| \, dx \leq k\|f\|_1.$$

On the other hand, thanks to (H4), we obtain

$$\int_{|u_n| > k} |g(x, u_n, Du_n)| \, dx \leq \|f\|_1.$$

This estimate implies that

$$\int_\Omega |g(x,u_n,Du_n)|\,dx = \int_{|u_n|>k} |g(x,u_n,Du_n)|\,dx + \int_{|u_n|\le k} |g(x,u_n,Du_n)|\,dx$$
$$\le C + \int_\Omega b(k)\Big(c(x) + \sum_{i=1}^N |D_iT_k(u_n)|^{p_i}\Big)\,dx.$$

This inequality and (39) imply (40). □

Now let us prove that for all $0<\varepsilon<1$ small enough, we have also

$$D_iu_n \to D_iu \quad \text{in } L^{r-\varepsilon}_{w_i}(\Omega). \tag{42}$$

Indeed, by (P_n), we have for any $k>0$,

$$\sum_{i=1}^N \int_\Omega w_i[|D_iu_n|^{q-2}D_iu_n - |D_iu_m|^{q-2}D_iu_m]D_iT_k(u_n-u_m)\,dx$$
$$+ \int_\Omega [a(x,u_n,Du_n) - a(x,u_m,Du_m)]D_iT_k(u_n-u_m)\,dx$$
$$+ \int_\Omega [g(x,u_n,Du_n) - g(x,u_m,Du_m)]T_k(u_n-u_m)\,dx$$
$$= \int_\Omega (f_n - f_m)T_k(u_n-u_m)\,dx.$$

This yields using (H2) and (40)

$$\sum_{i=1}^N \int_{|u_n-u_m|\le k} w_i[|D_iu_n|^{q-2}D_iu_n - |D_iu_m|^{q-2}D_iu_m]D_i(u_n-u_m)\,dx$$
$$\le Ck + Ck\|f\|_1 = C(1+\|f\|_1)k.$$

Hence in view of the monotonicity properties of q-Laplacian operator, with $q>2$, we get for $i=1,\dots,N$

$$\int_{|u_n-u_m|\le k} w_i|D_iu_n - D_iu_m|^q\,dx \le Ck.$$

Then using Hölder inequality, the fact that $w_i \in L^\infty(\Omega)$ and that $(D_iu_n)_n$ is bounded in $L^r_{w_i}(\Omega)$, we obtain

$$
\begin{aligned}
&\int_\Omega w_i|D_i(u_n-u_m)|^{r-\varepsilon}\,dx\\
&\quad=\int_{|u_n-u_m|\le k} w_i^{(r-\varepsilon)/q}|D_i(u_n-u_m)|^{r-\varepsilon}w_i^{(q-r+\varepsilon)/q}\,dx\\
&\quad\quad+\int_{|u_n-u_m|>k} w_i^{(r-\varepsilon)/r}|D_i(u_n-u_m)|^{r-\varepsilon}w_i^{\varepsilon/r}\,dx\\
&\quad\le C\Big(\int_{|u_n-u_m|\le k} w_i|D_i(u_n-u_m)|^{q}\,dx\Big)^{(r-\varepsilon)/q}\\
&\quad\quad+\Big(\int_{|u_n-u_m|>k} w_i|D_i(u_n-u_m)|^{r}\,dx\Big)^{(r-\varepsilon)/r}\Big(\int_{|u_n-u_m|>k} w_i\,dx\Big)^{\varepsilon/r}\\
&\quad\le Ck^{(r-\varepsilon)/q}+Cmes\{|u_n(x)-u_m(x)|>k\}^{\varepsilon/r}.
\end{aligned}
$$

Since k is an arbitrary positive number and $(u_n)_n$ is a Cauchy sequence in measure, we deduce that $(D_iu_n)_n$ is a Cauchy sequence in $L^{r-\varepsilon}_{w_i}(\Omega)$. Then by (42) we have

$$
w_i|D_iu_n|^{q-2}D_iu_n\to w_i|D_iu|^{q-2}D_iu\qquad\text{in } L^1(\Omega).
$$

Corollary. *There exist $u\in Y$ and a subsequence, still denoted by u_n, such that u_n (resp. $T_k(u_n)$) weakly converges to u (resp. $T_k(u)$) in Y and a.e.*

Lemma 3.9. *For all $k>0$, there exists a function θ such that for all $\varepsilon>0$, we have*

$$
\begin{aligned}
\int_{\{|u_n-T_k(u)|\le\varepsilon\}}\Big(&\sum_{i=1}^N w_i|D_iu_n|^{q-2}D_iu_nD_i\big(u_n-T_k(u)\big)\\
&+a(x,u_n,Du_n)D\big(u_n-T_k(u)\big)\Big)\,dx\le\theta(\varepsilon),
\end{aligned}
$$

with $\lim_{\varepsilon\to0}\theta(\varepsilon)=0$.

Proof. We choose $T_\varepsilon\big(u_n-T_k(u)\big)$ as a test function in (P_n), we have

$$
\begin{aligned}
\int_{\{|u_n-T_k(u)|\le\varepsilon\}}\Big(&\sum_{i=1}^N w_i|D_iu_n|^{q-2}D_iu_nD_i\big(u_n-T_k(u)\big)\\
&+a(x,u_n,Du_n)D\big(u_n-T_k(u)\big)\Big)\,dx\\
\le\varepsilon\big(&\|g(.,u_n,Du_n)\|_1+\|f\|_1\big).
\end{aligned}
$$

This inequality and (40) give the result. □

Lemma 3.10. *Let*

$$r_i \in \left[\frac{1}{p_i - 1}, \frac{p_i N(\bar{p} - 1)}{(p_i - 1)\bar{p}(N - 1)}\right), \qquad i = 1, \dots, N,$$

then $\left(a_i(x, u_n, Du_n)\right)_n$ *is bounded in* $L^{r_i}(\Omega)$ *for all* $i = 1, \dots, N$.

Proof. Note that we can choose $r_i > 1$, thanks to

$$\frac{\bar{p}(N - 1)}{N(\bar{p} - 1)} < p_i < \frac{\bar{p}(N - 1)}{N - \bar{p}} \qquad \text{and} \qquad \bar{p} < N.$$

As in Lemma 2.11 of [11], let σ be such that

$$\frac{r_i(p_i - 1)}{p_i} < \sigma < \frac{N(\bar{p} - 1)}{\bar{p}(N - 1)} < 1.$$

This is possible since we have

$$1 < r_i < \frac{p_i N(\bar{p} - 1)}{(p_i - 1)\bar{p}(N - 1)} \qquad \text{and} \qquad \bar{p} < N.$$

Hence

$$\sigma p_i \in \left[1, \frac{p_i N(\bar{p} - 1)}{\bar{p}(N - 1)}\right) \qquad \text{and} \qquad \frac{(p_i - 1)r_i}{\sigma p_i} < 1. \tag{43}$$

Using (H1), we can write

$$|a_i(x, u_n, Du_n)|^{r_i} \leq C\Big(k(x)^\sigma + |u|^{\sigma\bar{p}} + \sum_{j=1}^{N} |D_j u_n|^{p_j \sigma}\Big)^{(p_i - 1)r_i/\sigma p_i}.$$

In view of this inequality, Lemma 3.7 and (43), we get Lemma 3.10. □

Now, by using Lemmas 3.7–3.9, and the compactness result as in [20], there exists a subsequence (still denoted u_n) such that

$$Du_n \to Du \qquad \text{a.e in } \Omega,$$

and for all $i = 1, \dots, N$, we have

$$D_i u_n \to D_i u \qquad \text{strongly in } L^s(\Omega), \qquad \forall s \in [1, q_i),\ \forall q_i \in \left[p_i, \frac{N(\bar{p} - 1)}{\bar{p}(N - 1)}\right).$$

By Lemma 3.10, we deduce that

$$a(x, u_n, Du_n) \to a(x, u, Du) \qquad \text{strongly in } \left(L^1(\Omega)\right)^N.$$

Consequently, using the fact that

$$w_i|D_i u_n|^{q-2}u_n \to w_i|D_i u|^{q-2}u \qquad \text{strongly in } L^1(\Omega),$$

a is a Carathéodory function, so that, $a_i(x, u_n, Du_n) \to a_i(x, u, Du)$ a.e in Ω, $g(x, u_n, Du_n) \to g(x, u, Du)$ a.e in Ω, since g is a Carathéodory function and the fact that $f_n \to f$ strongly in $L^1(\Omega)$, then by letting $n \to \infty$ in (P_n), we deduce that problem (P) has a weak solution in Y.

Let us prove now that $g(x, u_n, Du_n)$ is uniformly equi-integrable.

Let $\gamma > 0$. We define $\varphi_\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\gamma(\sigma) = \begin{cases} \phi(\sigma - \gamma), & \sigma > \gamma \\ 0, & |\sigma| \leq \gamma \\ \phi(\sigma + \gamma), & \sigma < -\gamma. \end{cases}$$

By choosing $\varphi_\gamma(u_n)$ as a test function in (P_n), in view of (H2) and using the fact that $|\phi(s)| \leq 1$, we obtain

$$\int_{\{|u_n| \geq \gamma\}} g(x, u_n, Du_n)\varphi_\gamma(u_n)\, dx \leq \int_{\{|u_n| \geq \gamma\}} |f|\, dx \to 0 \qquad \text{as } \gamma \to +\infty \tag{44}$$

uniformly with respect to n. Now by using the properties of the function ϕ, and the fact that $g(x, u_n, Du_n)u_n \geq 0$, we get

$$\begin{aligned} &\int_{\{|u_n| \geq \gamma\}} |g(x, u_n, Du_n)|\, dx \\ &\qquad \leq \frac{1}{\varphi_\gamma(2\gamma)} \int_{\{|u_n| \geq \gamma\}} g(x, u_n, Du_n)\varphi_\gamma(u_n)\, dx \to 0 \qquad \text{as } \gamma \to +\infty. \end{aligned}$$

Thanks to (H5), we can also write for $E \subset \Omega$

$$\begin{aligned} &\int_{E \cap \{|u_n| \leq \gamma\}} |g(x, u_n, Du_n)|\, dx \\ &\qquad \leq h(\gamma) \int_E c(x)\, dx + \sum_{i=1}^{N} |E|^{1-\alpha_i/p_i} \left(\int_\Omega |D_i T_\gamma(u_n)|^{p_i}\right)^{\alpha_i/p_i} dx. \end{aligned}$$

This inequality and (39) give the equi-integrability of g on Ω. So it is simple to pass to the limit in (P_n). Therefore, Theorem 3.1 is proved. □

Remark 3.11. In order to obtain the equality in (33), one has to impose a stronger assumption on the datum f; more precisely we will require that $f \in L^1 \log L^1(\Omega)$, i.e.,

$$\int_\Omega |f| \log(1 + |f|)\, dx < +\infty.$$

The following result hold.

Theorem 3.12. *Let* $p_i \geq 2$, $i = 1, \ldots, N$. *Assume* (H1)–(H5) *hold, and* $f \in L^1 \log L^1(\Omega)$. *Then the problem* (P) *has at least one solution* $u \in Y$, *for all*

$$\kappa_i = p_i \frac{N(\bar{p} - 1)}{\bar{p}(N - 1)}, \qquad i = 1, \ldots, N.$$

Proof. We modify the previous proof with the help of techniques used in [6]. Using $\log(1 + |u_n|)\,\mathrm{sgn}(u_n)$ as test function in (P_n), the inequality becomes

$$\begin{aligned} \int_\Omega \frac{|D_i u_n|^{p_i}}{1 + |u_n|}\, dx &\leq \int_\Omega f \log(1 + |u_n|)\, dx \\ &\leq \int_\Omega |f| \log(1 + |f|)\, dx + \int_\Omega (1 + |u_n|)\, dx \leq C. \end{aligned} \tag{45}$$

Similarly, we have

$$\int_\Omega \frac{w_i |D_i u_n|^q}{1 + |u_n|}\, dx \leq C.$$

This gives a change in inequality (35),

$$\delta = \frac{(1 - \theta)\bar{\kappa}^*}{\theta} = 1, \qquad \bar{\kappa}^* = \frac{N\bar{\kappa}}{N - \bar{\kappa}}.$$

Then $(Du_n)_n$ is bounded in $\prod_{i=1}^N L^{\kappa_i}(\Omega)$ and in $\prod_{i=1}^N L^r_{w_i}(\Omega)$ with $\kappa_i = p_i \frac{N(\bar{p}-1)}{\bar{p}(N-1)}$ and $r = \frac{q\bar{\kappa}^*}{1+\bar{\kappa}^*}$. As in Lemma 3.10, we have $\big(a_i(x, u_n, Du_n)\big)_n$ is bounded in $L^{r_i}(\Omega)$ for

$$r_i = \frac{p_i N(\bar{p} - 1)}{(p_i - 1)\bar{p}(N - 1)}, \qquad i = 1, \ldots, N.$$

So that

$$a_i(x, u_n, Du_n) \rightharpoonup a_i(x, u, Du) \qquad \text{weakly in } L^{r_i}(\Omega). \qquad \square$$

4. Uniqueness result

Let us first note that the result of this section is motivated by the work of Antontsev and Chipot [1]. Herein, we suppose that the function $g = g(x, u)$ depends only on u.

Let us consider the problem

$$(P_u) \qquad \begin{cases} -\sum_{i=1}^{N} D_i\big(w_i|D_iu|^{q-2}D_iu + a_i(x,u,Du)\big) + g(x,u) = f, & \text{in } \Omega \\ u = 0, & \text{in } \partial\Omega. \end{cases}$$

First, we assume instead of the condition (H2) that for some constant $\eta > 0$, a.e. $x \in \Omega$, $\forall u, v \in \mathbb{R}$, $\forall \xi, \xi' \in \mathbb{R}^N$ we have

$$\begin{aligned} &\big(a_i(x,u,\xi) - a_i(x,v,\xi'), \xi_i - \xi_i'\big) \\ &\qquad \geq \eta|\xi_i - \xi_i'|^{p_i} - \mu(|u-v|)(|\xi_i| + |\xi_i'|)^{p_i-1}|\xi_i - \xi_i'|, \end{aligned} \tag{46}$$

where the function μ satisfies

$$\int_{0^+} \frac{1}{\mu(x)^\theta}\,dx = +\infty, \qquad 1 < \theta \leq \min_{i=1}^{N} \frac{p_i}{p_i - 1}. \tag{47}$$

We also assume that

$$u \mapsto g(x,u) \qquad \text{is increasing.} \tag{48}$$

Here the exponents $p_i > 2$, $i = 1, \ldots, N$.

Theorem 4.1. *Assume* (H1)–(H5), (46) *and* (48) *hold, and that the exponents p_i and m are restricted as in* (3) *and let $f \in L^m(\Omega)$. Then the weak solution $u \in X$ to (P_u) is unique.*

Proof. As in [1], thanks to (47), there exists $\varrho > 0$ such that

$$\forall \sigma \in (0, \varrho), \qquad \mu(\sigma) \leq 1, \tag{49}$$

and for any $0 < \varepsilon < \varrho$, we get

$$\int_t^\varepsilon \frac{1}{\mu(x)^\theta}\,dx \to +\infty \qquad \text{as } t \to 0^+.$$

Hence, for all $\varepsilon < \varrho$, there exists $\delta_\varepsilon > 0$ such that

$$\int_{\delta_\varepsilon}^{\varepsilon} \frac{1}{\mu(x)^\theta}\,dx = 1.$$

Now let us define the function S_ε for all $s \in \mathbb{R}$ by

$$S_\varepsilon(s) = \begin{cases} 0, & \text{if } s < \delta_\varepsilon \\ \int_{\delta_\varepsilon}^{s} \frac{1}{\mu(x)^\theta}\,dx, & \text{if } s \in [\delta_\varepsilon, \varepsilon] \\ 1, & \text{if } s > \varepsilon. \end{cases} \tag{50}$$

This is a Lipschitz function satisfying $S_\varepsilon(0) = 0$. Let u, v be two solutions to (P_u). Taking $S_\varepsilon(u - v)$ as a test function in (P_u), we have

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega \big(\big(|D_iu|^{q-2}D_iu - |D_iv|^{q-2}D_iv\big) \\ + \big(a_i(x,u,Du) - a_i(x,v,Dv)\big)\big)(D_iu - D_iv)S_\varepsilon'(u-v)\,dx \\ + \int_\Omega \big(g(x,u) - g(x,v)\big)S_\varepsilon(u-v)\,dx = 0.\end{aligned}$$

Due to the algebraic expression $(|x|^{q-2}x - |y|^{q-2}y, x-y) \geq 0$, $\forall x, y \in \mathbb{R}^N$, using the structure condition (46) and the definition of S_ε, we get

$$\begin{aligned}\eta \sum_{i=1}^{N} \int_{\mu_\varepsilon} \frac{|D_iu - D_iv|^{p_i}}{\omega(|u-v|)^\theta}\,dx + \int_\Omega \big(g(x,u) - g(x,v)\big)S_\varepsilon(u-v)\,dx \\ \leq \sum_{i=1}^{N} \int_{\Omega_\varepsilon} \frac{|D_iu - D_iv|}{\mu(|u-v|)^{\theta-1}}(|D_iu| + |D_iv|)^{p_i-1}\,dx,\end{aligned} \tag{51}$$

where

$$\Omega_\varepsilon = \{x \,|\, \delta_\varepsilon < (u-v)(x) < \varepsilon\}.$$

An application of Young's inequality gives

$$\begin{aligned}&\frac{|D_iu - D_iv|}{\mu(|u-v|)^{\theta-1}}(|D_iu| + |D_iv|)^{p_i-1} \\ &\quad = (p_i\eta/2)^{1/p_i}\frac{|D_iu - D_iv|}{\mu(|u-v|)^{\theta-1}}(p_i\eta/2)^{-1/p_i}(|D_iu| + |D_iv|)^{p_i-1} \\ &\quad \leq \frac{\eta}{2}\frac{|D_iu - D_iv|^{p_i}}{\mu(|u-v|)^{p_i(\theta-1)}} + C_i(|D_iu| + |D_iv|)^{p_i}.\end{aligned} \tag{52}$$

So, by relation (49) and the fact that

$$1 < \theta \leq \min_{i=1}^{N} \frac{p_i}{p_i - 1} \implies p_i(\theta - 1) \leq \theta \quad \text{for all } i = 1, \ldots, N,$$

we deduce that

$$\forall s \in [\delta_\varepsilon, \varepsilon], \qquad \mu(s)^{p_i(\theta-1)} \geq \mu(s)^{\theta}. \tag{53}$$

Now, in view of (51), (52) and (53), we obtain

$$\begin{aligned} &\frac{\eta}{2}\sum_{i=1}^{N} \int_{\Omega_\varepsilon} \frac{|D_i u - D_i v|^{p_i}}{\mu(|u-v|)^{\theta}}\, dx + \int_{\Omega} \big(g(x,u) - g(x,v)\big) S_\varepsilon(u-v)\, dx \\ &\quad \leq C \sum_{i=1}^{N} \int_{\Omega_\varepsilon} (|D_i u| + |D_i v|)^{p_i}\, dx. \end{aligned}$$

We deduce in particular that

$$\int_{\Omega} \big(g(x,u) - g(x,v)\big) S_\varepsilon(u-v)\, dx \leq C \sum_{i=1}^{N} \int_{\Omega} \chi_{\Omega_\varepsilon} (|D_i u| + |D_i v|)^{p_i}\, dx, \tag{54}$$

where $\chi_{\Omega_\varepsilon}$ denotes the characteristic function of a set Ω_ε. By using the properties of the function S_ε, the fact that

$$\chi_{\Omega_\varepsilon} \to 0 \text{ a.e.}, \qquad \text{and} \qquad S_\varepsilon(u-v) \to 1 \qquad \text{on } u - v > 0$$

and by applying Lebesgue's dominated convergence theorem, we get after passing to the limit $\varepsilon \to 0$ in (54)

$$\int_{u-v>0} \big(g(x,u) - g(x,v)\big)\, dx \leq 0.$$

Finally and according to (48), we have $u = v$.

This achieves the Proof of Theorem 4.1. □

4.1. Application. Let us consider the problem

$$(P_0) \quad \begin{cases} -\displaystyle\sum_{i=1}^{N} D_i\big(w_i |D_i u|^{q-2} D_i u + m_i(x,u) |D_i u|^{p_i - 2} D_i u\big) + g(x,u) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where $g(x,s) = \sum_{i=1}^{N} |s|^{p_i-2}s$ and the exponents $p_i > 2$ for $i = 1, \dots, N$. m_i is a Carathéodory valued functions satisfying

$$|m_i(x,u) - m_i(x,v)| \le C|u - v|, \qquad \forall u, v \in \mathbb{R}, \text{ a.e. } x \in \Omega \tag{55}$$

and

$$0 < \rho \le m_i(x,u) \le \sigma < +\infty, \qquad \forall u \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{56}$$

for all $i = 1, \dots, N$.

Theorem 4.2. *Assume* (55)–(56) *hold, and that the exponents p_i and m are restricted as in* (3) *and let $f \in L^m(\Omega)$. Then the weak solution $u \in X$ to* (P_0) *is unique.*

Proof. The proof needs the following lemma.

Lemma 4.3 ([4]). *There exists a constant $\lambda > 0$ for which*

$$(|\xi_i|^{p_i-2}\xi_i - |\xi_i'|^{p_i-2}\xi_i').(\xi_i - \xi_i') \ge \lambda|\xi_i - \xi_i'|^{p_i}$$

holds true for all $\xi = (\xi_1, \dots, \xi_N)$, $\xi' = (\xi_1', \dots, \xi_N') \in \mathbb{R}^N$.

We aim to prove that $a_i(x,u,Du) = m_i(x,u)|D_iu|^{p_i-2}D_iu$ satisfies the property (46). For all $u, v \in X$, we have

$$\begin{aligned}
&\big(a_i(x,u,Du) - a_i(x,v,Dv)\big) \cdot (D_iu - D_iv) \\
&\quad = m_i(x,u)(|D_iu|^{p_i-2}D_iu - |D_iv|^{p_i-2}D_iv)(D_iu - D_iv) \\
&\qquad + \big(m_i(x,u) - m_i(x,v)\big)|D_iv|^{p_i-2}D_iv(D_iu - D_iv).
\end{aligned}$$

Using Lemma 4.3 and the assumption (56), we get

$$\begin{aligned}
&\big(a_i(x,u,Du) - a_i(x,v,Dv)\big) \cdot (D_iu - D_iv) \\
&\quad \ge \rho\lambda|D_iu - D_iv|^{p_i} - |m_i(x,u) - m_i(x,v)|\,|D_iv|^{p_i-1}|D_iu - D_iv|.
\end{aligned}$$

Now, by the asumption (55), we obtain

$$\begin{aligned}
&\big(a_i(x,u,Du) - a_i(x,v,Dv)\big) \cdot (D_iu - D_iv) \\
&\quad \ge \rho\lambda|D_iu - D_iv|^{p_i} - C|u - v|\,|D_iv|^{p_i-1}|D_iu - D_iv|.
\end{aligned}$$

Hence

$$\begin{aligned}&\big(a_i(x,u,Du)-a_i(x,v,Dv)\big)\cdot(D_iu-D_iv)\\&\qquad\geq\rho\lambda|D_iu-D_iv|^{p_i}-\mu(|u-v|)(|D_iu|+|D_iv|)^{p_i-1}|D_iu-D_iv|,\end{aligned}$$

with $\mu(x)=Cx$. Since $u\mapsto|u|^{p_i-2}u$ is increasing, Theorem 4.2 follows from Theorem 4.1. □

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