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Parametric rigidness of germs of analytic unfoldings with a Hopf bifurcation

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Abstract. In this paper we prove that families of germs of one-parameter analytic differential equations with a Hopf bifurcation are *rigid* in the parameter. These systems are intrinsically real: the complex phase portrait is invariant under an antiholomorphic involution. The latter permits to identify the modulus of the analytic classification. The dynamics in the Siegel domain does not explain the existence of the real structure. Rather, these properties are meaningful in the Poincaré domain, where the fixed points are linearizable.

Mathematics Subject Classification (primary; secondary): 34M35; 34M40, 34M45

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1. Introduction

In these notes we prove that one-parameter real analytic families of ordinary differential equations of the form

$$\begin{aligned} \dot{x} &= \alpha(\eta)x - \beta(\eta)y + \sum_{j+k\geq 2} b_{jk}(\eta)x^j y^k \\ \dot{y} &= \beta(\eta)x + \alpha(\eta)y + \sum_{j+k\geq 2} c_{jk}(\eta)x^j y^k, \end{aligned} \tag{1}$$

(η is the unfolding parameter) are *rigid in the parameter*, in a precise sense defined later. The dots represent differentiation with respect to the (real) time t. The coefficients b_{jk}, c_{jk} depend analytically on the parameter. The functions $\alpha(\eta)$, $\beta(\eta)$ are real analytic and $\alpha(0) = 0$ but $\beta(0) \neq 0$. We assume that the unfolding is *generic:* $\alpha'(0) \neq 0$, see below.

It is known from the general theory that this family unfolds either a *center* (integral trajectories are closed curves with interiors containing the origin) or a

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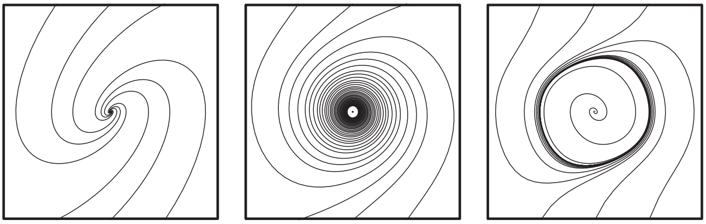

(a) Strong focus (b) Weak focus (c) Limit cycle

Figure 1. Supercritical Hopf bifurcation.

weak focus, see Figure 1. Weakness means that the convergence of integral curves to the origin is slower than that of logarithmic spirals of *strong foci*. Although our techniques apply to centers, from the viewpoint of the analytic properties of the system, the interesting case occurs when the family (1) unfolds a weak focus. (Strong foci are linearizable by Poincaré linearization theorem). In this case, the system exhibits a generic codimension-one Hopf bifurcation provided $L_1(0) \neq 0$, where

$$\begin{aligned} L_1(\eta) = 3b_{30}(\eta) + b_{12}(\eta) + c_{21}(\eta) + 3c_{03}(\eta) + \big[b_{11}(\eta)\big(b_{20}(\eta) + b_{02}(\eta)\big) \\ - c_{11}(\eta)\big(c_{20}(\eta) + c_{02}(\eta)\big) - 2b_{20}(\eta)c_{20}(\eta) + 2b_{02}(\eta)c_{02}(\eta)\big]/\beta(\eta) \end{aligned}$$

is the first Lyapunov constant of the system. This coefficient measures the sensibility of the system to small variations in the parameter. It defines, moreover, the topology of the phase portrait as follows.

Inasmuch as the system is generic, the implicit function theorem permits to take $\varepsilon := \alpha/\beta$ as a new parameter via a time scaling $t \mapsto \beta(\varepsilon)t$. The parameter ε is an invariant under analytic changes. It has been called *canonical* in other works but this denomination is both confusing and unnecessary. The eigenvalues of the linearization matrix of the vector field at zero thus become $\varepsilon + i$ and $\varepsilon - i$ and then the singular point is *elliptic* or *monodromic* [10]. The term $s = \operatorname{sign}\big(L_1(0)\big) = \pm 1$ is called the *sign* of the family. This number is an invariant which determines the nature of the singular points. If $s = +1$ the Hopf bifurcation is called *subcritical* and the cycle is present on negative values of ε. The case $s = -1$ corresponds to a supercritical Hopf bifurcation. (This is also true in higher codimension [4]). The two cases defined by s are equivalent only under the non-real rotation in 90° degrees which exchanges real and imaginary axes. (The singular points are real provided $s\varepsilon \leq 0$ and imaginary otherwise). Since complex spaces are much more rigid (in the sense of conformal geometry) than real spaces one is then naturally led to the question whether complexification is meaningful. To tackle this problem the coordinates x, y, the time t and parameter ε are extended to the complex plane. In the standard complex variables $\mathfrak{x} = x + iy$ and $\mathfrak{y} = x - iy$ the differential equation (1) takes the form

$$\begin{aligned} \dot{\mathfrak{x}} &= (\varepsilon + i)\mathfrak{x} + \cdots \\ \dot{\mathfrak{y}} &= (\varepsilon - i)\mathfrak{y} + \cdots . \end{aligned} \tag{2}$$

The complex system is invariant under the antiholomorphic involution $(\mathfrak{x}, \mathfrak{y}) \mapsto (\bar{\mathfrak{y}}, \bar{\mathfrak{x}})$ (see [2]) and hence the real phase portrait is embedded in the real plane $\{\mathfrak{x} = \bar{\mathfrak{y}}\}$. The foliation is described locally by the complexification of the Poincaré monodromy $\mathscr{P}_\varepsilon : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. The (complex) monodromy is real analytic and is the second iterate of a conformal diffeomorphism tangent to the standard *flip* $\mathfrak{x} \mapsto -\mathfrak{x}$, see [4], [7]. The monodromy is hence tangent to the identity and its form is well-suited to understand the real properties of the unfolding. The genericity assumption permits to *prepare* the family of complex differential equations via a real analytic change of coordinate [1]. For the sake of the exposition the new (prepared) complex coordinates will be denoted again by (x, y). Under the change $(\mathfrak{x}, \mathfrak{y}) \mapsto (x, y)$ the monodromy takes the prepared form

$$\mathscr{P}_\varepsilon(x) = x + x(\varepsilon + sx^2)\big(1 + b(\varepsilon) + c(\varepsilon)x^2 + x(\varepsilon + sx^2)h(x, \varepsilon)\big) \tag{3}$$

with multiplier $\mathscr{P}_\varepsilon'(0) = \exp(\varepsilon)$. The *formal invariant* $\mathbf{a}(\varepsilon)$ is defined by

$$\mathscr{P}_\varepsilon'(\pm\sqrt{-s\varepsilon}) = \exp\big(-2\varepsilon/\big(1 - s\mathbf{a}(\varepsilon)\varepsilon\big)\big). \tag{4}$$

The formal normal form of (3) is the time-one map of the vector field

$$\frac{x(\varepsilon + sx^2)}{1 + \mathbf{a}(\varepsilon)x^2}\frac{\partial}{\partial x}. \tag{5}$$

In [5] the dynamics of $\mathscr{P}_\varepsilon$ has been compared with the dynamics of its formal normal form only for those values of ε for which the singular objects (periodic orbit and equilibria) are hyperbolic and the normalization domains intersect. The corresponding invariant of analytic classification was identified only over two sectors which do not cover a full neighborhood of zero in the parameter space. The union these sectors intersects the real axis, see next section. The latter corresponds to the Poincaré domain in the parameter space. (The multiplier has absolute value different from 1 exactly when the parameter has a nonzero real part). The modulus of the analytic classification is identified when the normalizing charts on special normalization domains are compared. The denomination *Poincaré modulus* is then well-suited.

The Siegel domain in the parameter space corresponds to the union of two sectors of the parameter space which intersect the positive and negative directions of the imaginary axis, respectively. In the Siegel domain in the parameter space one can compute the modulus for all values of the parameter, since the union of the two sectorial neighborhoods is a disk around zero. The normalization domains (on which we compare the germ of family to the formal normal form) are crescent-like sectorial domains with vertices located at the fixed points, see next section. The analytic continuation of the normalizing chart is ramified at the fixed points. The ramification is the obstruction preventing the family of differential equations from being embeddable. (A differential equations is embeddable if its flow can be conjugated to a holomorphic flow). The corresponding invariant of the analytic classification is known as the *Siegel modulus*. The latter is a ramified function of the parameter. In particular, there are two different definitions of the Siegel modulus for the parameter values for which the Poincaré modulus can be defined. Since the modulus depends analytically on ε the Poincaré modulus, defined only on a union of two sectors in the parameter space, determines the Siegel modulus for parameter values in a full neighborhood of the origin. It is precisely this last property that allows to embed a weak conjugacy (which is topological for some isolated values of the parameter) between two families of the form (3) into a biholomorphism which is analytic in ε.

Parametric rigidity. Let f_ε denote a germ of a one-parameter family of analytic diffeomorphisms or vector fields. The expression f_ε depends *weakly analytically* on the parameter in a determined sectorial domain (of sufficiently wide opening containing $\varepsilon = 0$ at the tip) will indicate that $f(x, \varepsilon) = f_\varepsilon(x)$ is analytic in the second variable for values $\varepsilon \neq 0$ and that $f(x, \varepsilon)$ is only continuous in the second coordinate at $\varepsilon = 0$, in that sectorial domain. The family f_ε depends *strongly* analytically (or simply analytically) on ε if $f(x, \varepsilon)$ depends analytically on the parameter in that determined sectorial domain.

Two families of germs of analytic diffeomorphisms (resp. vector fields) f_ε are weakly analytically conjugate (resp. weakly analytically orbitally equivalent) if the following occurs. There exists a conjugacy (resp. orbital equivalence) between those families which depends weakly analytically on ε in a sectorial domain as above. The transformation is fibered over the parameter space and tangent to the identity. If the conjugacy (resp. orbital equivalence) depends analytically on the parameter, the families are analytically conjugate (resp. analytically orbitally equivalent).

Definition 1.1. The family of germs of real analytic diffeomorphims (resp. vector fields) f_ε is rigid in the parameter if any weak conjugacy (resp. weak orbital equivalence) between this germ and any other germ of the same form necessarily induces a real analytic biholomorphism which is holomorphic in ε.

Hence, two families of germs of real analytic diffeomorphims (resp. vector fields) which are weakly conjugate (resp. weakly orbitally equivalent) and rigid in the parameter, are indeed strongly conjugate (resp. orbitally equivalent). We prove in these notes that any germ of elliptic analytic generic unfolding with a Hopf bifurcation of codimension one is rigid in the parameter.

2. The invariant of the analytic classification

It is known that (weak or strong) orbital equivalence of two germs of generic real analytic families (2) implies (weak or strong) conjugacy of their monodromies, and *vice-versa.*

Lemma 2.1 ([1]). *Two generic germs of families of real analytic differential equations* (2) *are weakly* (*resp. strongly*) *analytically orbitally equivalent by a real equivalence if and only if the families unfolding their monodromies are weakly* (*resp. strongly*) *analytically conjugate by a real conjugacy.*

This implies that the class of the system (2) under orbital equivalence coincides with the class of the monodromy under conjugacy. As the latter is a one-dimensional germ this result is a great simplification in determining the analytic class of the system of differential equations.

2.1. The Ecalle modulus of the system (2). Two germs of real analytic families of diffeomorphisms of the form (3) with same sign s are conjugate if and only if they have the same formal invariant $\mathbf{a}(\varepsilon)$ and the same orbit space. The Ecalle modulus is a description of the latter when $\varepsilon = 0$. Indeed, the germ $\mathscr{P}_0$ is topologically the time-1 map of (5) and then a fundamental domain C_1^+ for the map $\mathscr{P}_0$ is determined by a curve ℓ_1 and its image $\mathscr{P}_0(\ell_1)$. Notice that if $x \in \ell_1$ is identified with its image $\mathscr{P}_0(x)$, then the fundamental domain is conformally equivalent to a sphere $\mathbb{S}_1^+$. The endpoints of the crescent C_1^+ limited by ℓ_1 and $\mathscr{P}_0(\ell_1)$ correspond to the points 0 and ∞ on the sphere. All orbits of $\mathscr{P}_0$ (except that of 0) are represented by at most one point of the sphere $\mathbb{S}_1^+$. However, there exist points in the neighborhood of 0 whose orbits have no representative on the sphere $\mathbb{S}_1^+$. To cover the whole orbit space we need to take three other fundamental neighborhoods C_1^-, C_2^+, C_2^- limited by curves ℓ_j and their images $\mathscr{P}_0(\ell_j)$, $j = 2, 3, 4$, respectively. As before, we identify $x \in \ell_j$ with its image $\mathscr{P}_0(x)$ and the union of these fundamental domains is also conformally equivalent to a collection of spheres $\mathbb{S}_1^-$, $\mathbb{S}_2^+$, $\mathbb{S}_2^-$. But there are points in the neighborhood of 0 (resp. ∞) which lie in different spheres but belong to the same orbit. So we need to identify a neighborhood of 0 (resp. ∞) with a neighborhood of 0 (resp. ∞) in two different spheres. This is done via a collection of analytic diffeomorphisms ψ_1^0, ψ_2^0 (resp. ψ_1^∞, ψ_2^∞) sending 0 to 0 (resp. ∞ to ∞). In this way we obtain a non-Hausdorff topological manifold endowed with a system of analytic charts given by the collection of spheres glued

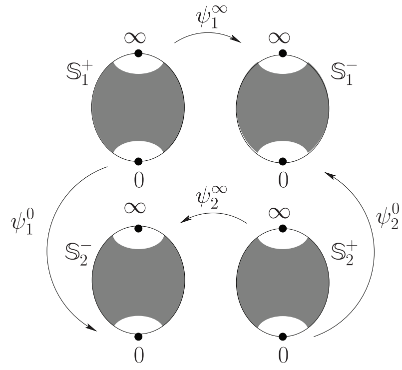

Figure 2. The flow of (5) in the cases $s = \pm 1$ and the Ecalle modulus.

at the poles by the maps ψ_j^0 and ψ_j^∞, see Figure 2. The size of the neighborhoods of 0 and ∞ depends on the size of the neighborhood of the origin where $\mathscr{P}_0$ is defined, but the germs of analytic diffeomorphims:

$$\psi_1^0, \psi_2^0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$$
$$\psi_1^\infty, \psi_2^\infty : (\mathbb{C}, \infty) \to (\mathbb{C}\infty)$$

are almost intrinsic maps of the sphere: the only analytic changes of coordinates on $\mathbb{S}_j^\pm$ preserving 0 and ∞ are the linear maps. If we choose different coordinates on $\mathbb{S}_j^\pm$ we obtain different germs $(\hat{\psi}_j^0, \hat{\psi}_j^\infty)$. The equivalence relation corresponding to changes of coordinates on $\mathbb{S}_j^\pm$ and preserving the poles is

$$(\psi_1^0, \psi_2^0, \psi_1^\infty, \psi_2^\infty) \sim (\hat{\psi}_1^0, \hat{\psi}_2^0, \hat{\psi}_1^\infty, \hat{\psi}_2^\infty) \iff \exists c_1^\pm, c_2^\pm \in \mathbb{C} :$$
$$\begin{cases} c_2^- \hat{\psi}_1^0(w) = \psi_1^0(c_1^+ w), \\ c_1^- \hat{\psi}_2^0(w) = \psi_2^0(c_2^+ w), \end{cases} \quad \begin{cases} c_1^- \hat{\psi}_1^\infty(w) = \psi_1^\infty(c_1^+ w), \\ c_2^- \hat{\psi}_2^\infty(w) = \psi_2^\infty(c_2^+ w). \end{cases}$$

Inasmuch as $\mathscr{P}_0$ is the second iterate of a germ tangent to the standard flip $x \mapsto -x$, it is possible to choose representatives of the modulus such that

$$\psi_1^{0,\infty}(-w) = -\psi_2^{0,\infty}(w) \tag{6}$$

The *Ecalle-modulus* of the diffeomorphism $\mathscr{P}_0$ is given by tuples $(\psi_1^0, \psi_2^0, \psi_1^\infty, \psi_2^\infty)$, modulo the equivalence relation $\sim$. Any representative of the Ecalle modulus is hence represented by a single pair (ψ^0, ψ^∞).

On a small neighborhood $U = \mathbb{D}_r$ of zero the dynamics of $\mathscr{P}_0$ is only topologically conjugate to the dynamics of the time-one map of (5) at $\varepsilon = 0$. In the unfolding, the study of the family $\mathscr{P}_\varepsilon$ will be done over the fixed U for sufficiently small values of the parameter.

2.2. Unfolding of the Ecalle modulus in the Poincaré domain. We identify the invariant of analytic classification of (3) in the Poincaré domain of the parameter space. The latter corresponds to those values of ε for which the fixed points are hyperbolic. There are two sectorial domains for which this happens, each one intersecting a real semi-axis:

$$\begin{aligned} V_{\delta,\ell} &= \{|\varepsilon| < \rho, \arg(\varepsilon) \in (\pi/2 + \delta, 3\pi/2 - \delta)\} \\ V_{\delta,r} &= \{|\varepsilon| < \rho, \arg(\varepsilon) \in (-\pi/2 + \delta, \pi/2 - \delta)\}, \end{aligned} \tag{7}$$

where $\delta \in (0, \pi/2)$, and $\rho = \rho(\delta)$ is a small positive number. The Poincaré domain in the parameter space will be simply called the Poincaré domain or the hyperbolic direction. We assume that $V_{\delta,\ell} \cap V_{\delta,r} = \{0\}$. The number ρ is chosen so that if $\varepsilon \in (V_{\delta,\ell} \cup V_{\delta,r}) \setminus \{0\}$ then there are orbits connecting the fixed points in the neighborhood U. Thus, the fixed points are linearizable along the hyperbolic direction. The invariant is constructed by comparing the *orbit space* of (3) with the orbit space of its formal normal form. The latter is the time-one map of the formal vector field (5). Inasmuch as the fixed points are hyperbolic the family of diffeomorphisms is normalizable independently over $V_{\delta,\ell}$ and $V_{\delta,r}$. The orbit space of $\mathscr{P}_\varepsilon$ is obtained as follows. Let us denote by $\mathscr{L}_c$ the linear map $w \mapsto cw$, where $c \in \mathbb{C}$.

We take three closed curves $\{\ell_0, \ell_+, \ell_-\}$ around the fixed points along with their images $\{\mathscr{P}_\varepsilon(\ell_0), \mathscr{P}_\varepsilon(\ell_+), \mathscr{P}_\varepsilon(\ell_-)\}$. By hyperbolicity the closed regions C_0, C_+, and C_- between the curves and the images are isomorphic to three closed annuli, see Figure 3. Each curve is identified with its image $\ell \sim \mathscr{P}_\varepsilon(\ell)$ and then the quotient space is the union of three complex tori $\mathfrak{T}^0_\varepsilon$, $\mathfrak{T}^\infty_{1,\varepsilon}$, and $\mathfrak{T}^\infty_{2,\varepsilon}$. By Abel's

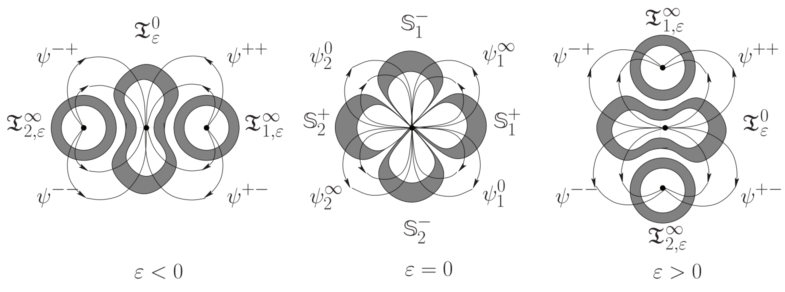

Figure 3. The orbit space of (3) in the hyperbolic direction.

Theorem [6] each torus is a quotient $\mathbb{C}^*/\mathscr{L}_c$ for some $c \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Then a natural coordinate on $\mathfrak{T}$ is the projection of a coordinate w on $\mathbb{C}^* = \mathbb{P}\backslash\{0, \infty\}$. In the quotient space each orbit of the diffeomorphism is either a fixed point or is represented by at most one point in a torus. However, some orbits may have representatives in two different tori. The identification of orbits in two different tori induce germs of families of analytic diffeomorphisms $\psi_\varepsilon : \mathbb{C}^* \mapsto \mathbb{C}^*$ such that $\psi_\varepsilon \circ \mathscr{L}_{c_1} = \mathscr{L}_{c_2} \circ \psi_\varepsilon$ if ψ_ε represents a map on an annulus in $\mathfrak{T}_1 = \mathbb{C}^*/\mathscr{L}_{c_1}$ with image on annulus in $\mathfrak{T}_2 = \mathbb{C}^*/\mathscr{L}_{c_2}$. The orbit space is hence represented by an abstract, non-Hausdorff manifold given by the union of three tori (plus the projection of the three fixed points, which represents the orbit space of the hyperbolic fixed points) glued along adjacent annuli by a collection of diffeomorphisms ψ_ε^{++}, ψ_ε^{-+}, ψ_ε^{--} and ψ_ε^{+-}.

We will refer to the invariant of analytic classification in the hyperbolic direction simply as the Poincaré modulus. Notice that $\mathbb{P}$ deprived of the north and south poles is isomorphic to an infinite cylinder and the latter is the image of the complex plane via the exponential map $z \mapsto w = \exp(-2\pi i z)$. The four components of the Poincaré invariant yield four germs of conformal diffeomorphisms on $\mathbb{C}$ which commute with translation by one $z \mapsto z + 1$. These maps will be also denoted by $\psi_\varepsilon^{\pm\pm}$. (This is not confusing as the source space of these maps is parametrized by the complex coordinate z).

Consider the universal cover $p_\varepsilon : \mathscr{R}_\varepsilon \to U = \mathbb{D}_r$. It is known that the source space $\mathscr{R}_\varepsilon$ is a two-folded Riemann surface deprived of a countable number of holes [9] and which is parametrized by the time t of the simpler one-dimensional equation $\dot{x} = x(\varepsilon + sx^2)$. The latter is a small deformation of (5) and its time can be found by direct integration. In the x coordinate, representatives of the modulus are induced by the map $q_\varepsilon = p_\varepsilon \circ A_\varepsilon^{-1}$, where A_ε is a solution of Abel's equation

$$f\big(P_\varepsilon(\mathrm{t})\big) = f(\mathrm{t}) + 1, \tag{8}$$

and P_ε is the germ induced by (3) in the chart t, see Figure 4. The complex representatives of the modulus are respectively defined on the first, second, third and fourth quadrants, see Figure 3. In the sequel, we will drop dependence on ε when the context allows no confusion.

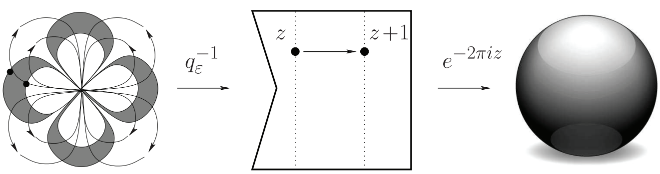

Figure 4. On the left, the 4 domains of sectorial trivialization.

2.2.1. Schwarz reflection and analytic continuation on $\mathbb{P}$. It is always possible to choose a *real* representative of the Poincaré modulus, in the sense that it sends the real equator of $\mathbb{P}$ into itself. This particular choice in turn reflects the real character of (3) and is expressed in terms of certain symmetries. The latter are indeed consequence of the invariance of the foliation under the anitholomorphic involution $(x, y) \mapsto (\bar{y}, \bar{x})$, see [2]. This is in notorious contrast with the case of other non-real conformal germs (e.g. parabolic or saddle diffeomorphisms). We discuss further details.

Let $\psi_\varepsilon^{\pm\pm}(w) = \sum_{n\in\mathbb{Z}} a_n^{\pm\pm}(\varepsilon)w^n$ be the expansions in Laurent series of the four components of any representative of the Poincaré modulus in the annulus. The complex conjugation $w \mapsto \overline{w}$ generates a multiplicative property.

Proposition 2.2. *There exist representatives $\psi_\varepsilon^{\pm\pm}$ of the modulus in the hyperbolic direction such that*

$$\sum_{\mathbb{Z}} a_n^{+-}(\varepsilon)w^n \cdot \sum_{\mathbb{Z}} \overline{a_{-n}^{++}(\bar{\varepsilon})}w^n = 1, \qquad \sum_{\mathbb{Z}} a_n^{--}(\varepsilon)w^n \cdot \sum_{\mathbb{Z}} \overline{a_{-n}^{-+}(\bar{\varepsilon})}w^n = 1.$$

Proof. Any given representative of the Poincaré invariant $\psi_\varepsilon^{\pm\pm} = \psi_\varepsilon^{\pm\pm}(z)$ in the z coordinate is a composite function of the form $\psi_\varepsilon^{\pm\pm} = B_\varepsilon^{\pm} \circ (A_\varepsilon^{\pm})^{-1}$ where $A_\varepsilon^{\pm}$ and $B_\varepsilon^{\pm}$ are two one-parameter solutions of Abel's equation (8). The left index of $\psi_\varepsilon^{\pm\pm}$ corresponds to the index of $A_\varepsilon^{\pm}$ and the right one, to the index of the map $B_\varepsilon^{\pm}$. Theorem 5.1 in the reference [5] proves that the solutions $A_\varepsilon^{\pm}$, $B_\varepsilon^{\pm}$ can be chosen such that $A_\varepsilon^{\pm}(t) = \overline{A_{\bar{\varepsilon}}^{\pm}(\bar{t})}$ and $B_\varepsilon^{\pm}(t) = \overline{B_{\bar{\varepsilon}}^{\mp}(\bar{t})}$. The complex conjugation $z \mapsto \bar{z}$ yields

$$\psi_\varepsilon^{+-}(\bar{z}) = \overline{\psi_{\bar{\varepsilon}}^{++}(z)}, \qquad \psi_\varepsilon^{--}(\bar{z}) = \overline{\psi_{\bar{\varepsilon}}^{-+}(z)}. \tag{9}$$

Inasmuch as $\exp(-2\pi i\bar{z}) = \overline{1/\exp(-2\pi iz)}$, composition of (9) with the exponential map yields

$$\psi_\varepsilon^{+-}(w) = 1/\overline{\psi_{\bar{\varepsilon}}^{++}(1/\overline{w})}, \qquad \psi_\varepsilon^{--}(w) = 1/\overline{\psi_{\bar{\varepsilon}}^{-+}(1/\overline{w})}$$

in the w variable. The conclusion follows. □

Corollary 2.3. *For real values of the parameter the equator of the Riemann sphere is invariant under the Poincaré modulus.*

Proof. Define the map $\psi_\varepsilon^{+} = \psi_\varepsilon^{++}$ on $\mathrm{dom}(\psi_\varepsilon^{++})$. By Proposition 2.2 the map $m(w) = 1/\overline{\psi_{\bar{\varepsilon}}^{++}(1/\overline{w})}$ coincides with ψ_ε^{+-} on $\mathrm{dom}(\psi_\varepsilon^{+-})$. By the Schwarz reflection principle, m extends ψ_ε^+ to all of $\operatorname{dom}(\psi_\varepsilon^{++}) \cup \operatorname{dom}(\psi_\varepsilon^{+-})$. That is, ψ_ε^{++} and ψ_ε^{+-} are analytic continuations of each other. Inasmuch as

$$\psi_\varepsilon^+(w) = 1/\overline{\psi_{\bar{\varepsilon}}^+(1/\overline{w})}$$

the representative ψ_ε^+ fixes the *real* equator $\mathbb{R}P = \{|w| = 1\}$ of the Riemann sphere provided ε be real. (That is, if $w\overline{w} = 1$ then $|\psi_\varepsilon^+| = 1$ as well).

An analogous reasoning proves that ψ_ε^{--} and ψ_ε^{-+} are analytic continuations of each other. Proposition 2.2 proves again that the map

$$\psi_\varepsilon^- = \begin{cases} \psi_\varepsilon^{--} & \text{on } \operatorname{dom}(\psi_\varepsilon^{--}), \\ \psi_\varepsilon^{-+} & \text{on } \operatorname{dom}(\psi_\varepsilon^{-+}) \end{cases}$$

fixes the real equator of the Riemann sphere on real values of the parameter. □

The pair $(\psi_\varepsilon^+, \psi_\varepsilon^-)$ of *real* germs of diffeomorphisms is a representative of the invariant in the hyperbolic direction. It is also known [5] that the four components of the Poincaré invariant are related by the linear map $w \mapsto -w$. The latter generates an additive property.

Proposition 2.4. *There exist representatives of the Poincaré invariant such that*

$$\sum_{\mathbb{Z}} a_n^{++}(\varepsilon)w^n + \sum_{\mathbb{Z}} (-1)^n a_n^{--}(\varepsilon)w^n = 0,$$
$$\sum_{\mathbb{Z}} a_n^{+-}(\varepsilon)w^n + \sum_{\mathbb{Z}} (-1)^n a_n^{-+}(\varepsilon)w^n = 0.$$

As a corollary, any representative of the invariant of the analytic classification in the hyperbolic direction is completely determined by only one of its four components. Such a representative will be denoted by $\psi_{\varepsilon,P}$.

2.3. Unfolding of the Ecalle modulus in the Siegel domain. In the Siegel domain of the parameter space (also called the nonhyperbolic direction or simply the Siegel domain) the values of the parameter are taken in the union of two sectorial domains of the parameter space

$$\begin{aligned} V_{\delta,+} &= \{|\varepsilon| < \rho, \arg(\varepsilon) \in (-\delta, \pi + \delta)\} \\ V_{\delta,-} &= \{|\varepsilon| < \rho, \arg(\varepsilon) \in (\pi - \delta, 2\pi + \delta)\} \end{aligned} \tag{10}$$

where $\delta \in (0, \pi/2)$, and $\rho = \rho(\delta)$ is a small positive number. When the parameter is restricted independently to each of these sectors, it is known that a suitable

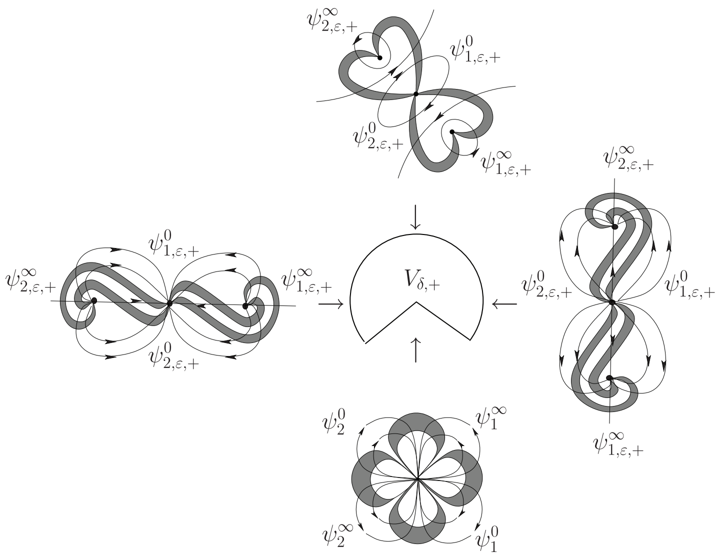

Figure 5. The crescents (gray strips) and some flow lines in the case $s = +1$ for values $\varepsilon \in V_{\delta,+}$.

choice of ρ guarantees the existence of four crescents $C^{\pm}_{1,\varepsilon}$, $C^{\pm}_{2,\varepsilon}$ in the source space of (3), with endpoints at the fixed points. (The crescents corresponding to the other sector are obtained by revolving the former crescents along the imaginary axis, compare Figures 5 and 6). On each sector in (10) the crescents are bounded by curves ℓ_j, $j = 1, 2$, and their images $\mathscr{P}_\varepsilon(\ell_j)$, see Figure 5. If ℓ_j and its image are identified, then the quotient space is conformally equivalent to the union of four spheres $S^{\pm}_{j,\varepsilon}$ with the fixed points located at the south and north poles. It is known [9] that there exists an analytic chart

$$\phi^{\pm}_{j,\varepsilon} : S^{\pm}_{j,\varepsilon} \to \mathbb{P}, \tag{11}$$

sending poles into poles and a chosen point x_0 (called the *base point* and which depends analytically on ε) into 1. These coordinates are constructed in the universal cover $\mathscr{R}_\varepsilon$ and are known as *Fatou coordinates* [9]. The construction is standard and can be reviewed in several articles and monographs. The pullback of (11) via the quotient projection $C^{\pm}_{j,\varepsilon} \to S^{\pm}_{j,\varepsilon}$ generates analytic charts

$$\varrho^{\pm}_{j,\varepsilon} : C^{\pm}_{j,\varepsilon} \to \mathbb{C}, \qquad j = 1, 2 \tag{12}$$

on the crescents. Points in neighborhoods of the poles of the spheres are identified via analytic diffeomorphisms $\psi^{0}_{j,\varepsilon}, \psi^{\infty}_{j,\varepsilon} : \mathbb{P} \to \mathbb{P}$ defined by

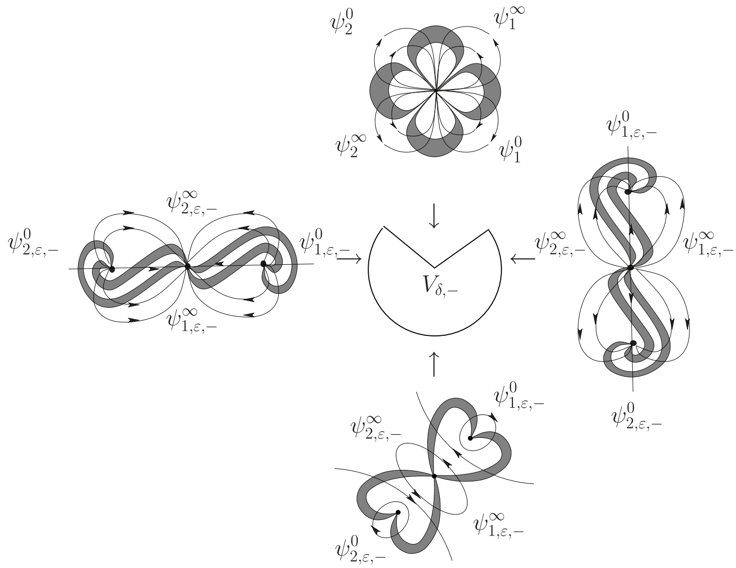

Figure 6. The crescents in the case $s = +1$ for values $\varepsilon \in V_{\delta,-}$.

$$\begin{aligned}\psi^\infty_{j,\varepsilon} &= \phi^-_{j,\varepsilon} \circ (\phi^+_{j,\varepsilon})^{-1}\\ \psi^0_{j,\varepsilon} &= \phi^-_{j,\varepsilon} \circ (\phi^+_{j+1,\varepsilon})^{-1}\end{aligned} \tag{13}$$

for $j = 1, 2$ and where we identify $\phi^+_{3,\varepsilon} = \phi^+_{1,\varepsilon}$. Evidently the maps (13) fix the poles and are uniquely determined up to linear maps. The degree of freedom in the coordinate determines an equivalence relation,

$$(\psi^0_{j,\varepsilon}, \psi^\infty_{j,\varepsilon}) \sim (\hat{\psi}^0_{j,\varepsilon}, \hat{\psi}^\infty_{j,\varepsilon}) \iff \exists c_\varepsilon, c'_\varepsilon \in \mathbb{C}^* : \tag{14}$$
$$\begin{cases} c'_\varepsilon \hat{\psi}^0_{j,\varepsilon}(w) = \psi^0_{j,\varepsilon}(c_\varepsilon w),\\ c'_\varepsilon \hat{\psi}^\infty_{j,\varepsilon}(w) = \psi^\infty_{j,\varepsilon}(c_\varepsilon w),\end{cases}$$

where c_ε and c'_ε depend weakly analytically on $\varepsilon \in V_{\delta,-} \cup V_{\delta,+}$ and $c_0, c'_0 \neq 0$. Moreover, the $\mathbb{Z}_2$ action of the map $x \mapsto -x$ on (3) implies that different tuples $(\psi^0_{j,\varepsilon}, \psi^\infty_{j,\varepsilon})$ inherit (6); that is $\psi^{0,\infty}_{1,\varepsilon}(-w) = -\psi^{0,\infty}_{2,\varepsilon}(w)$. Therefore, any representative of an equivalence class is indeed completely determined by any pair $(\psi^0_{j,\varepsilon}, \psi^\infty_{j,\varepsilon})$. In principle, the invariant of the analytic classification of (3) in the Siegel direction (or simply the Siegel modulus) consists of the sign s, the formal invariant $\mathbf{a}(\varepsilon)$—defined implicitly in (4)—and a representative $(\psi^0_\varepsilon, \psi^\infty_\varepsilon)$. However, in the next section we prove that the Siegel modulus can be retrieved from the Poincaré modulus on the union of two subsectors in the intersection $V_{\delta,-} \cap V_{\delta,+}$ (hyperbolic direction). Since the formal invariant depends analytically on ε it can be further continued along the sectors (10). Therefore, the Siegel modulus consists of the sign s and the functional class defined by (14). It is worthwhile noticing that if we restrict to values $c_\varepsilon = c'_\varepsilon$ then a suitable choice of the global coordinate on $\mathbb{P}$ permits to take $(\psi_\varepsilon^0)'(0) = (\psi_\varepsilon^\infty)'(\infty)$.

2.4. Renormalized return maps. The Siegel modulus allows to study the dynamics of the germ near each of the fixed points by means of *renormalized return maps* when the multiplier is on the unit circle [11]. It is known that the renormalized return maps still exist for nonzero values of ε in the Siegel direction. They are given by the composition of the Siegel modulus with a global *transition* map $l_\varepsilon : \mathbb{P} \to \mathbb{P}$. The latter is an analytic automorphism of $\mathbb{P}$ fixing 0 and ∞. Hence it is linear and then the nonlinear part of a renormalized return map comes from the Siegel modulus. More precisely, for nonzero values of $\varepsilon \in V_{\delta,-} \cup V_{\delta,+}$ the renormalized return maps are the composite maps

$$\begin{cases} \kappa_{\varepsilon,\pm}^0 = l_{\varepsilon,\pm} \circ \psi_{\varepsilon,\pm}^0, \\ \kappa_{\varepsilon,\pm}^\infty = l_{\varepsilon,\pm} \circ \psi_{\varepsilon,\pm}^\infty \end{cases} \tag{15}$$

where the subscript $+, -$ makes reference to the respective sector $V_{\delta,+}$ or $V_{\delta,-}$ on which the definition takes place. The dynamics of these maps can be interpreted as follows. In the neighborhood of each singular point the germ (3) is iterated until it reaches back the crescent $C_{j,\varepsilon}^\pm$. For example, given $x \in C_{1,\varepsilon}^+$ in the neighborhood of a nonzero fixed point, and given $\varepsilon \in V_{\delta,+}$, we let w be the coordinate of x on $\mathbb{P}_1^+$ and we denote by $n = \min\{m \in \mathbb{N} : \mathscr{P}_\varepsilon^{\circ m}(x) \in C_{1,\varepsilon}^+\}$. Then $\kappa_{\varepsilon,+}^\infty(w)$ is the coordinate of the iterate $\mathscr{P}_\varepsilon^{\circ n}(x)$ on the sphere $\mathbb{P}$. Of course, the multipliers of the renormalized return maps are analytic invariants:

$$\begin{aligned} (\kappa_{\varepsilon,+}^0)'(0) = (\kappa_{\varepsilon,-}^\infty)'(\infty) = \exp(\mathfrak{a}_0), \\ (\kappa_{\varepsilon,+}^\infty)'(\infty) = (\kappa_{\varepsilon,-}^0)'(0) = \exp(\mathfrak{a}_\infty), \end{aligned} \tag{16}$$

where $\mathfrak{a}_0 = 4\pi^2/\varepsilon$ and $\mathfrak{a}_\infty = -2\pi^2\big(1 - s\mathbf{a}(\varepsilon)\varepsilon\big)/\varepsilon$.

3. The Siegel invariant in the Poincaré domain

The sectors (10) intersect in two smaller sectors $\tilde{V}_\ell \subset V_{\delta,\ell}$ and $\tilde{V}_r \subset V_{\delta,r}$ over which the renormalized maps (15) are linearizable [8], [11]. More precisely, for values of $\varepsilon \in \tilde{V}_\ell \cup \tilde{V}_r \subset V_{\delta,+}$, there exist charts $\varphi_{\varepsilon,+}^0$, $\varphi_{\varepsilon,+}^\infty$ such that

$$\begin{aligned} \varphi_{\varepsilon,+}^0 \circ \kappa_{\varepsilon,+}^0 = \mathscr{L}_{\exp(\mathfrak{a}_0)} \circ \varphi_{\varepsilon,+}^0, \\ \varphi_{\varepsilon,+}^\infty \circ \kappa_{\varepsilon,+}^\infty = \mathscr{L}_{\exp(\mathfrak{a}_\infty)} \circ \varphi_{\varepsilon,+}^\infty. \end{aligned} \tag{17}$$

(Note that the choice of the constants in the linear coordinates is not arbitrary: it is fixed by (16)). For values of the parameter $\varepsilon \in \tilde{V}_\ell \cup \tilde{V}_r \subset V_{\delta,-}$, there exist charts $\varphi^0_{\varepsilon,-}$, $\varphi^\infty_{\varepsilon,-}$ on $\mathbb{P}$ such that

$$\begin{aligned} \varphi^0_{\varepsilon,-} \circ \kappa^0_{\varepsilon,-} &= \mathscr{L}_{\exp(\mathfrak{a}_\infty)} \circ \varphi^0_{\varepsilon,-}, \\ \varphi^\infty_{\varepsilon,-} \circ \kappa^\infty_{\varepsilon,-} &= \mathscr{L}_{\exp(\mathfrak{a}_0)} \circ \varphi^\infty_{\varepsilon,-}. \end{aligned} \tag{18}$$

It is clear from (15) that tuples $(\kappa^0_{\varepsilon,\pm}, \kappa^\infty_{\varepsilon,\pm})$ are representatives of the Siegel modulus. On the other hand, the linearizing charts provide representatives of the invariant along the hyperbolic direction. Indeed, representatives of the Poincaré modulus in the intersection of the Siegel domains (10) are given by the composite maps (again, the subscript $\pm$ makes reference to the corresponding Siegel sector $V_{\delta,+}$ or $V_{\delta,-}$),

$$\psi_{\varepsilon,P,\pm} = \begin{cases} \varphi^0_{\varepsilon,\pm} \circ (\varphi^\infty_{\varepsilon,\pm})^{-1}, & \varepsilon \in \tilde{V}_\ell \\ \varphi^\infty_{\varepsilon,\pm} \circ (\varphi^0_{\varepsilon,\pm})^{-1}, & \varepsilon \in \tilde{V}_r. \end{cases} \tag{19}$$

The linearizing coordinates are unique up to composition on the left with linear maps. In this section we solve the converse: given any representative $\psi_{\varepsilon,P,\pm}$ of the Poincaré invariant on $\mathbb{P}$ we retrieve charts $(V_0, \varphi^0_{\varepsilon,\pm})$ and $(V_\infty, \varphi^\infty_{\varepsilon,\pm})$ on $\mathbb{P}$ which depend weakly analytically on $\varepsilon \in \tilde{V}_\ell \cup \tilde{V}_r$. We drop dependence on ε and on the Siegel subscript $\pm$ whenever possible.

Theorem 3.1. *The Siegel invariant can be determined from the Poincaré modulus.*

Proof. Consider two positive numbers $r_1 < 1 < r_0$ and the standard covering of the Riemann sphere $\mathbb{P}$ by an atlas of two charts $(\mathbb{D}_0, id_0)$ and $(\mathbb{D}_\infty, t)$. The set $\mathbb{D}_0 = \{|w| < r_0\} \subset \mathbb{R}^2$ is an open disk (id_0 is the identity on $\mathbb{D}_0$) and $\mathbb{D}_\infty = \{|w| > r_1\} \cup \{\infty\}$ with the chart $t = 1/w$, in which it also becomes an open disk. The intersection $\mathbb{D}_0 \cap \mathbb{D}_\infty$ in $\mathbb{P}$ is the circular annulus $A_{0,\infty} = \{r_1 < |w| < r_0\}$. The latter contains the equator $\{|w| = 1\}$ and induces two annuli $A_0 \subset \mathbb{D}_0$ and $A_\infty \subset \mathbb{D}_\infty$. Inasmuch as the choice of the two numbers r_1, r_0 is irrelevant we can assume that A_∞ is the domain of the Poincaré invariant $\psi_{\varepsilon,P}$ and A_0 is the target space for the values $\varepsilon \in \tilde{V}_\ell$ and that the modulus has the opposite source and target spaces for $\varepsilon \in \tilde{V}_r$. In either case, the source will be denoted by $A_{\mathfrak{S}}$ and the target, $A_{\mathfrak{T}}$. From this standard structure (with transition $w \mapsto 1/w$) we construct a nonstandard atlas on $\mathbb{P}$.

Consider the disks $\mathbb{D}_0$, $\mathbb{D}_\infty$ as disjoint sets in $\mathbb{R}^2$. The invariant $\psi_{\varepsilon,P}$ induces a map $\psi_{\varepsilon,P}$ on $A_{\mathfrak{S}} \subset \mathbb{R}^2$. Let us define $M = \mathbb{D}_0 + \mathbb{D}_\infty$ (ie., the set $\mathbb{D}_0 \cup \mathbb{D}_\infty$ with topology: $U \subset \mathbb{D}_0 + \mathbb{D}_\infty$ is open if and only if $U \cap \mathbb{D}_0$ is open in $\mathbb{D}_0$ and $U \cap \mathbb{D}_\infty$ is open in $\mathbb{D}_\infty$). Then M has the structure of a smooth manifold and an equivalence relation R is defined by identifying $w \in A_{\mathfrak{S}}$ with $\psi_{\varepsilon,P}(w) \in A_{\mathfrak{T}}$.

By Corollary 2.3 the disks are glued along the unit circle (equator). Hence the quotient M/R is well defined and corresponds to a simply connected Riemann surface equipped with an analytic structure. By the Uniformization Theorem the quotient is conformally equivalent to the complex projective space. The canonical inclusions $i_0 : \mathbb{D}_0 \hookrightarrow M/R$ and $i_\infty : \mathbb{D}_\infty \hookrightarrow M/R$ are injective because the invariant is a diffeomorphism. (These maps depend analytically on the parameter). It is evident that

$$\psi_{\varepsilon,P} = \begin{cases} i_0^{-1} \circ i_\infty, & \varepsilon \in \tilde{V}_\ell \\ i_\infty^{-1} \circ i_0, & \varepsilon \in \tilde{V}_r \end{cases} \tag{20}$$

up to linear maps $c \mapsto cw$ in the *middle* space, where the constants c do not vanish at $\varepsilon = 0$ and depend weakly analytically on the parameter. If $V_0 = i_0(\mathbb{D}_0)$ and $V_\infty = i_\infty(\mathbb{D}_\infty)$ then we have the charts (V_0, i_0^{-1}) and $(V_\infty, i_\infty^{-1})$ on the quotient manifold. It is clear that the family of all such charts is an atlas for $\mathbb{P}$. It is also evident that the transition function between the charts coincides with the representative $\psi_{\varepsilon,P}$ of the Poincaré invariant.

We set $\varphi_\varepsilon^0 = id_0 \circ i_0^{-1}$ and $\varphi_\varepsilon^\infty = t^{-1} \circ i_\infty^{-1}$. These maps are diffeomorphisms respectively defined in neighborhoods of the poles of the projective space. Hence they are unique up to left composition with linear maps. We will use this degree of freedom: we will consider only those constants which depend weakly analytically on ε on the union of the Siegel domains (10). A chart $(\varphi_\varepsilon^0, \varphi_\varepsilon^\infty)$ with this restriction is called *normalized*. It is clear that such a chart fixes the corresponding pole and depends weakly analytically on the parameter in the Poincaré domain. By (20) the composite maps

$$\begin{cases} \varphi_\varepsilon^0 \circ (\varphi_\varepsilon^\infty)^{-1}, & \varepsilon \in \tilde{V}_\ell \\ \varphi_\varepsilon^\infty \circ (\varphi_\varepsilon^0)^{-1}, & \varepsilon \in \tilde{V}_r \end{cases}$$

yield representatives $\psi_{\varepsilon,P} : A_{0,\infty} \to A_{0,\infty}$ for values $\varepsilon \in \tilde{V}_\ell \cup \tilde{V}_r$. In turn, families of conformal maps $\kappa_{\varepsilon,\pm}^0$, $\kappa_{\varepsilon,\pm}^\infty$ can be respectively defined for those values of ε via (17) and (18). The families thus defined trivially satisfy (16) and are conjugate to linear maps along the hyperbolic direction. By the *almost* unicity of the renormalized return maps they can be analytically continued (in the parameter) along the non-hyperbolic direction. Indeed, the extension to a punctured neighborhood of zero in the parameter space is holomorphic because the chart $(\varphi_\varepsilon^0, \varphi_\varepsilon^\infty)$ is normalized. Since the maps $(\varphi_\varepsilon^0, \varphi_\varepsilon^\infty)$ are defined up to additive constants on the left the maps $(\kappa_{\varepsilon,\pm}^0, \kappa_{\varepsilon,\pm}^\infty)$ are renormalized return maps which depend weakly analytically on ε. A representative of the Siegel modulus on the Riemann sphere can be thus recovered from equation (15) for values of $\varepsilon \in V_{\delta,-} \cup V_{\delta,+}$. The proof is complete. □

4. Parametric rigidity of the Hopf bifurcation

A *symmetry* of (3) is the germ of a one-parameter analytic family of diffeomorphism which commutes with $\mathscr{P}_\varepsilon$ on a neighborhood of the origin containing the fixed points. A symmetry is real if the coefficients of its asymptotic expansion are real for real values of ε. This notion is helpful in the proof of the following result.

Theorem 4.1. *The class of germs of families of real analytic differential equations of the form* (2) *is rigid in the parameter.*

Proof. Lemma 2.1 implies, in particular, that germs of analytic families of elliptic vector fields of the form (2) are rigid in the parameter if and only if their monodromies are rigid in the parameter. The latter has been proved in Theorem 1.2 of the reference [3] but we include the basic steps for the sake of completeness. Through this result we prove that any germ of generic family (2) with a Hopf bifurcation is rigid in the parameter.

Let us assume that, for values of the parameters in the union of the sectors (7) (hyperbolic direction) the two prepared families of diffeomorphisms (monodromies) are weakly conjugate on the neighborhood $U = \mathbb{D}_r$. In the prepared form the parameters are analytic invariants. Thus we can consider the conjugacy over the identity and it suffices to compare the two families for a given $\varepsilon \in \tilde{V}_\ell \cup \tilde{V}_r = V_{\delta,-} \cap V_{\delta,+}$ and extend by analyticity. The weak conjugacy will be denoted by $h_{\varepsilon,\ell}$ or by $h_{\varepsilon,r}$ provided $\varepsilon \in V_{\delta,\ell}$ or $\varepsilon \in V_{\delta,r}$, respectively.

It is known that the existence of $h_{\varepsilon,\ell}$ and $h_{\varepsilon,r}$ is a sufficient condition for the Poincaré moduli of the diffeomorphisms to coincide [5]. By Theorem 3.1, representatives of the Siegel invariant for the two families are determined. There is no reason for these moduli to coincide, though. However, as Fatou coordinates (11) are unique up to composition on the left with linear maps, we can assume that the associated crescents (the domains of the pullbacks (12)) are the same for the two diffeomorphisms. Moreover, if we restrict the values of the parameter to the hyperbolic direction, the Fatou coordinates (11) associated with the two monodromies coincide up to right composition with the conjugacy $h_{\varepsilon,\ell}$, $h_{\varepsilon,r}$ and up to left composition with linear maps. (Both solve the same Abel's equation in the Poincaré domain). It is easy to prove then that the Siegel invariants of the two families coincide over the subsectors $\tilde{V}_\ell \cup \tilde{V}_r$. Inasmuch as the Siegel modulus is analytic in the parameter on the larger subsectors (10), the Siegel moduli of the two diffeomorphisms coincide. It is known that this is a necessary and sufficient condition for the two diffeomorphisms to be (strongly) analytically conjugate. Back in the x coordinate this conjugacy induces two conjugacies $h_{\varepsilon,+}$ and $h_{\varepsilon,-}$ on the union of the crescents. (The subscript $\pm$ makes reference to the Siegel sector

$V_{\delta,+}$ or $V_{\varepsilon,-}$). However, the crescents are fundamental domains and then $h_{\varepsilon,+}$ and $h_{\varepsilon,-}$ extend to $\mathbb{D}_r$ via iterations. The fact that the families have the same Siegel modulus ensures that the extension is non-ramified. It remains to prove that such a conjugacy induces a *real* biholomorphism analytic in ε.

A sufficient condition for the Poincaré and Siegel moduli to unfold the Ecalle invariant is that the Fatou coordinates be *normalized:* they have the same limit as $\varepsilon \to 0$. Since Fatou coordinates are defined up to additive constant such a normalization is always possible. This condition ensures that $h_{0,r} \equiv h_{0,-} \equiv h_{0,+}$. For nonzero values of the parameter the composite maps $H_\varepsilon^+ = (h_{\varepsilon,+})^{-1} \circ h_{\varepsilon,r}$ and $H_\varepsilon^- = (h_{\varepsilon,-})^{-1} \circ h_{\varepsilon,r}$ are two symmetries of the monodromy of the second system on the sector $V_{\delta,r}$, which *a priori* depend weakly analytically on ε. It is known [7] that along the hyperbolic direction, the symmetries of the monodromy are conjugate to the time-$t(\varepsilon)$ flow of (5) for a continuous time $t(\varepsilon)$ which is unique provided it unfolds the identity. Up to composition of $h_{\varepsilon,r}$, $h_{\varepsilon,+}$ and $h_{\varepsilon,\ell}$ on the left with real symmetries of (3) (which are chosen to be conjugate to time-$t_{r,\pm}(\varepsilon)$ flows of (5) for times $t_{r,\pm}(\varepsilon) = \ln h'_{\varepsilon,r,\pm}(0)/\varepsilon$) it is always possible to assume that the multipliers

$$h'_{\varepsilon,r}(0) = h'_{\varepsilon,+}(0) = h'_{\varepsilon,-}(0) = 1$$

in the Poincaré domain. Hence $(H_\varepsilon^\pm)'(0) = \exp\big(-\varepsilon t(\varepsilon)\big) \equiv 1$ and thus $t(\varepsilon) = 2\pi i N/\varepsilon$ for an integer N. Since $t(\varepsilon)$ is continuous at zero, $N = 0$ and $H_\varepsilon^\pm$ is the identity. That is, $h_{\varepsilon,r} \equiv h_{\varepsilon,+} \equiv h_{\varepsilon,-}$ for values $\varepsilon \in V_{\delta,r}$. In particular, the maps $h_{\varepsilon,+}$, $h_{\varepsilon,-}$ are real for real values of the parameter. This also implies the remarkable fact that $h_{\varepsilon,r}$ depends analytically on the parameter.

For values of $\varepsilon \in V_{\delta,\ell}$ we prove in analogous way that, up to left composition with real symmetries of (3), $h_{\varepsilon,\ell} \equiv h_{\varepsilon,+} \equiv h_{\varepsilon,-}$ on $\varepsilon \in V_{\delta,\ell}$. The maps $h_{\varepsilon,+}$, $h_{\varepsilon,-}$, are hence real and $h_{\varepsilon,\ell}$ depends analytically on ε.

Inasmuch as the two maps $h_{\varepsilon,+}$, $h_{\varepsilon,-}$ conjugate the monodromies, they coincide on $V_{\delta,-} \cap V_{\delta,+}$. □

Corollary 4.2. *Two real germs of families of conformal diffeomorphisms of the form* (3) *with the same sign s are strongly conjugate along the hyperbolic direction if and only if they have the same Poincaré invariant.*

Lemma 2.1 and Corollary 4.2 thus imply the following (stronger) result.

Corollary 4.3. *Two germs of generic families of real analytic vector fields of the form* (2) *are strongly analytically orbitally equivalent by a real equivalence if and only if their Poincaré monodromies have the same sign s and the Poincaré moduli of the associated prepared families* (3) *coincide.*

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