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Hierarchic control for a coupled parabolic system

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(Recommended by Enrique Zuazua)

Abstract. In this paper we study a Stackelberg-Nash strategy to control systems of coupled linear parabolic partial differential equations. We assume that we can act on the system through several controls called *followers*, intended to solve a Nash multi-objective equilibrium, and a single *leader* control satisfying a null controllability objective. We obtain the existence and uniqueness of the Nash equilibrium, its characterization and the optimal leader control satisfying the null controllability problem.

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1. Introduction

The development of science and technology has motivated many branches of control theory. Initially, in the classical control theory, we encountered problems where a system must reach a predetermined target by the action of a single control, for example, find the control $v \in \mathscr{V}_{ad}$ of minimum norm such that the design specifications are met. To the extent that more realistic situations were considered, it was necessary to include several different (and even contradictory) control objectives, as well as develop theory that would handle the concepts of multicriteria optimization, where optimal decisions need to be taken in the presence of trade-offs between these different objectives. There are many points of view to deal with multi-objective problems. Notions of economics and game theory were introduced in the works of H. von Stackelberg [25], J. F. Nash [22] and V. Pareto [24], where each has a particular philosophy to solve these problems.

According to the formulation introduced by H. von Stackelberg [25], we assume the presence of various local controls, called *followers*, which have their

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own objectives, and a global control, called *leader*, with a different goal from the rest of the players (in this case, the followers). The general idea of this strategy is a game of hierarchical nature, where players compete against each other, so that the leader makes the first move and then the followers react optimally to the action of the leader. Since many followers are present and each has a specific objective, it is intended that these are in Nash equilibrium.

Up to date, in the context of partial differential equations (PDEs), there are several papers related to this topic. As a precedent, in the papers by Lions [19] and [20], the Stackelberg equilibrium was studied when control is applied on the boundary of a wave equation and control is exerted in the interior of an equation of parabolic type, respectively. Later, Díaz [9], in collaboration with Lions, studied the existence and uniqueness of Stackelberg-Nash equilibrium, as well as its characterization. In that work, the followers and the leader have both objectives of approximate controllability, the first ones locally and the second in a global way. On the other hand, Glowinski, Ramos, and Periaux [13] studied the Nash equilibrium for linear parabolic equations from a theoretical and numerical point of view. More recently, Límaco, Clark, and Medeiros [18] developed results of hierarchical control for parabolic equations with moving boundaries, while Guillén-González, Marques-Lopes, and Rojas-Medar [16] presented a Stackelberg-Nash strategy for the Stokes problem in the velocity and pressure formulation. Finally, in [4], the authors developed the first hierarchical results within the exact controllability framework for a parabolic equation.

Nevertheless, most of the previous works have one thing in common: they deal with hierarchical control of a single equation. In this paper we are interested in developing a Stackelberg-Nash strategy where the system dynamics is given by a non-scalar system of parabolic equations in which we act through a hierarchy of controls. This type of systems are particularly studied in mathematical biology. The systems analyzed in this paper represent a linear version of more complex models obtained from Chemotaxis (see for instance [7], [21]) or treatment of tumors [6]. To our knowledge, the only paper dealing with coupled systems is [5]. There, the authors study a Stackelberg-Nash strategy for two coupled equations of fluid mechanics, with the particularity that they act by means of the leaders and followers on both of the equations involved satisfying an approximate controllability constrain. In this paper, the main novelty is that we deal with a system of two coupled parabolic equations acting only in the first equation and addressing a null controllability result.

2. The problem and its formulation

Let Ω be an open and bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^2 and ω be an open and nonempty subset of Ω. Given $T > 0$, we consider the following system of coupled parabolic PDEs with leader control localized in ω and follower controls localized in $\omega_1, \omega_2 \subset \Omega$ with $\omega_i \cap \omega = \emptyset$. More precisely

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11} y_1 + a_{12} y_2 = h\chi_\omega + v^1 \chi_{\omega_1} + v^2 \chi_{\omega_2} & \text{in } Q = \Omega \times (0,T), \\ y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = 0 \quad \text{in } Q, \\ y_j = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T), \quad j = 1,2, \\ y_j(x,0) = y_j^0(x) \quad \text{in } \Omega, \quad j = 1,2, \end{cases} \tag{1}$$

where $a_{ij} = a_{ij}(x,t) \in L^\infty(Q)$ and $y_j^0 \in L^2(\Omega)$ are prescribed. Equivalently, the previous system can be written as

$$\begin{cases} y_t - \Delta y + A(x,t) y = B[h\chi_\omega + v^1 \chi_{\omega_1} + v^2 \chi_{\omega_2}] & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{2}$$

where $y^0 = (y_i^0)_{i=1,2} \in [L^2(\Omega)]^2$, $A(x,t) = \big(a_{ij}(x,t)\big)_{1 \le i,j \le 2} \in [L^\infty(Q)]^{2\times 2}$ and $B = (1 \quad 0)^*$.

In system (2), $y = (y_1, y_2)^*$ is the state, $v^i = v^i(x,t)$ and $h = h(x,t)$ are the follower and leader control functions, respectively, while χ_ω and χ_{ω_i} denote the characteristic functions of ω and ω_i. Observe that for each $h \in L^2\big(\omega \times (0,T)\big)$, $v^i \in L^2\big(\omega_i \times (0,T)\big)$, $i = 1,2$ and $y^0 \in L^2(\Omega)^2$, system (2) admits a unique weak solution, hereinafter denoted as

$$y = y(x,t;h,v^1,v^2).$$

In the case where only a leader control is exerted on ω, there exist several papers devoted to the controllability of parabolic non-scalar systems of PDEs, see for instance [1], [2], [15], [23] or [3] for a recent survey on the controllability of coupled parabolic systems. In particular, in [14] the authors proved that system (2) is indeed null controllable whenever a single control is applied in the first equation of the coupled system, as long as a_{21} has a fixed sign on an open subset of ω.

Now, we introduce the control point of view where we assume that we have a hierarchy in our wishes and we will describe the Stackelberg-Nash strategy for system (2). Let $\mathcal{O}_{1,d}, \mathcal{O}_{2,d} \subset \Omega$ be open subsets, representing the observation domains of the followers, which are localized arbitrarily in Ω. Define the functionals

$$J_i(h,v^1,v^2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |y_1 - y_{1,d}^i|^2 + |y_2 - y_{2,d}^i|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{\omega_i \times (0,T)} |v^i|^2 \, dx\, dt, \qquad i = 1,2, \tag{3}$$

and the main functional

$$J(h) = \frac{1}{2}\iint_{\omega\times(0,T)} |h|^2\,dx\,dt,$$

where $\alpha_i, \mu_i > 0$ are constants and $y_d^i = \left(y_{1,d}^i, y_{2,d}^i\right)^*$ are given functions in $L^2\left(\mathcal{O}_{i,d}\times(0,T)\right)$, $i = 1, 2$.

The main goal is to choose h such that the following general objective (of null controllability) is achieved

$$y(\cdot, T; h, v^1, v^2) = 0 \qquad \text{in } \Omega. \tag{4}$$

The second priority is the following. Given the functions $y_{1,d}^i$ and $y_{2,d}^i$, we want to choose the controls v^i such that throughout the interval $t \in (0, T)$

$$\begin{aligned} &y(x, t; h, v^1, v^2) \text{ “do not deviate much” from } y_d^i(x, t),\\ &\text{in the observability domains } \mathcal{O}_{i,d},\ i = 1, 2. \end{aligned} \tag{5}$$

To achieve simultaneously (4) and (5) the control process can be described as follows. For a fixed leader control h, find controls $(\bar v^1, \bar v^2)$ that depend on h and the corresponding state solution $y = y(h, \bar v^1, \bar v^2)$ of equation (2) satisfying the Nash equilibrium related to (J_1, J_2) defined in (3). That is, given h, find $(\bar v^1, \bar v^2)$ such that

$$\begin{aligned} J_1(h, \bar v^1, \bar v^2) &\le J_1(h, v^1, \bar v^2), \qquad \forall v^1 \in L^2\left(\omega_1\times(0,T)\right),\\ J_2(h, \bar v^1, \bar v^2) &\le J_2(h, \bar v^1, v^2), \qquad \forall v^2 \in L^2\left(\omega_2\times(0,T)\right), \end{aligned}$$

or equivalently

$$J_1(h, \bar v^1, \bar v^2) = \min_{v^1} J_1(h, v^1, \bar v^2), \tag{6}$$

$$J_2(h, \bar v^1, \bar v^2) = \min_{v^2} J_2(h, \bar v^1, v^2). \tag{7}$$

Any pair $(\bar v^1, \bar v^2)$ satisfying (6)–(7) is called a Nash equilibrium for (J_1, J_2). Since J_1 and J_2 are strictly convex functionals, $(\bar v^1, \bar v^2)$ is a Nash equilibrium with respect to (J_1, J_2) if and only if

$$\left(\frac{\partial J_1}{\partial v^1}(h, \bar v^1, \bar v^2), v^1\right) = 0 \qquad \forall v^1 \in L^2\left(\omega_1\times(0,T)\right), \tag{8}$$

$$\left(\frac{\partial J_1}{\partial v^2}(h, \bar v^1, \bar v^2), v^2\right) = 0 \qquad \forall v^2 \in L^2\left(\omega_2\times(0,T)\right). \tag{9}$$

After identifying the Nash equilibrium and the associated state $y = y\big(h, \bar{v}^1(h), \bar{v}^2(h)\big)$ for each h, we look for an optimal control $\hat{h}$ such that

$$J(\hat{h}) = \min_{h} J\big(h, \bar{v}^1(h), \bar{v}^2(h)\big) \tag{10}$$

subject to the restriction

$$y\big(\cdot, T; h, \bar{v}^1(h), \bar{v}^2(h)\big) = 0 \quad \text{ in } \Omega. \tag{11}$$

Within this spirit, the main contribution of this paper can be stated as follows. Assume that

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d}, \tag{12}$$

denoted in the next sections as $\mathcal{O}_d$. Then we state the following:

Theorem 2.1. *Let $A = A(x,t)$ be the corresponding coupling matrix of system* (2). *Assume that $\mathcal{O}_d \cap \omega \neq \emptyset$ and that μ_i for $i = 1, 2$, are sufficiently large. If*

$$a_{12} \equiv 0 \quad \text{in } Q, \tag{13}$$

and

$$a_{21} \geq a_0 > 0 \quad \text{or} \quad -a_{21} \geq a_0 > 0 \quad \text{in } (\mathcal{O}_d \cap \omega) \times (0, T), \tag{14}$$

there exists a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that if

$$\iint_{\mathcal{O}_d \times (0,T)} \rho^2 |y^i_{j,d}|^2 \, dx \, dt < +\infty, \qquad i, j = 1, 2, \tag{15}$$

then for any $y^0 \in L^2(\Omega)^2$ there exists a control $h \in L^2\big(\omega \times (0,T)\big)$ and the corresponding Nash equilibrium $(\bar{v}^1, \bar{v}^2)$ such that the solution of (2) *satisfies* (11).

Remark 2.2. Some remarks are in order.

- The hierarchical control is largely motivated by applications where more than one objective is desirable in the behavior of the system under study. For instance, if $u = u(x,t)$ represents the concentration of a chemical product, the methodology is to reach the state 0 by means of an optimal control h acting on the domain ω, but at the same time try to keep the concentration near a reasonable quantity in $\mathcal{O}_d$ along the time interval $(0, T)$. In this paper we extend this concept to coupled parabolic systems.
- Just as in [4], condition (15) seems natural and it means that the follower objectives $y^i_{j,d}$ approach 0 as $t \to T$. This is because after computing the

follower controls $(\bar{v}^1, \bar{v}^2)$ the leader control h should not find any obstruction to control the system. Also in [4], condition (12) is required to control a single parabolic equation. It remains an open problem to verify if this condition is necessary.

- It remains an open problem, in [4] and here, to eliminate the condition $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$: intuitively it should be more difficult to drive the solution close to two different objectives in the same subset than close to two different ones in different subsets.
- Assumption (13) might seem restrictive, but this condition appears naturally in cascade systems. See Remark 6.5 below.
- Unlike other papers as [16] (in the scalar case) or [5] (in the coupled case), we are supposing that the follower controls are being applied in some sets ω_i disjoint of the leader set ω. This leads to a more realistic situation, otherwise once the followers choose a policy, the leader modifies its behavior at the same points.

The proof of Theorem 2.1 requires several steps: in Section 3 we give sufficient conditions for the existence and uniqueness of Nash equilibrium, while in Section 4 we give its characterization. In Section 5 we prove that if there exists a Nash equilibrium for the followers, then the leader solve the problem of null controllability. Lastly, we devote Section 6 to prove the observability inequality needed to establish the null controllability result.

3. Existence and uniqueness for the Nash equilibrium

In this section, we recall an existence and uniqueness result concerning the Nash equilibrium in the sense of (8)–(9) (see, for instance, [9]). We follow the same spirit as in [16] to present the result. Here, no hypotheses are required regarding the control sets ω_i and ω or the observation sets $\mathcal{O}_{i,d}$, nor the coefficient a_{12}, so we keep the notation from the problem formulation.

For this, consider the functionals given by (3) and define the functional spaces

$$\begin{aligned} \mathcal{H}_i &= L^2\big(\omega_i \times (0,T)\big), \qquad i = 1,2, \\ \mathcal{H} &= \mathcal{H}_1 \times \mathcal{H}_2. \end{aligned}$$

as well as the operator

$$\Lambda_i \in \mathcal{L}\big(\mathcal{H}_i, L^2(Q)^2\big) \quad \text{defined as} \quad \Lambda_i v^i = y^i,$$

where $y^i = (y_1^i, y_2^i)$ is solution of

$$\begin{cases} y_{1,t}^i - \Delta y_1^i + a_{11} y_1^i + a_{12} y_2^i = v^i \chi_{\omega_i} & \text{in } Q, \\ y_{2,t}^i - \Delta y_2^i + a_{21} y_1^i + a_{22} y_2^i = 0 & \text{in } Q, \\ y_j^i(0) = 0 \quad \text{in } \Omega, \quad y_j^i = 0 \quad \text{on } \Sigma, \quad j = 1, 2. \end{cases}$$

With this notation, for any $h \in L^2\big(\omega \times (0, T)\big)$ we write the solution of (2) as follows

$$y = \Lambda_1 v^1 + \Lambda_2 v^2 + q(h),$$

where $q(h) = \big(q_1(h), q_2(h)\big)$ solves the system

$$\begin{cases} q_{1,t} - \Delta q_1 + a_{11} q_1 + a_{12} q_2 = h \chi_\omega & \text{in } Q, \\ q_{2,t} - \Delta q_2 + a_{21} q_1 + a_{22} q_2 = 0 & \text{in } Q, \\ q_j(0) = y_j^0 \quad \text{in } \Omega, \quad q_j = 0 \quad \text{on } \Sigma, \quad j = 1, 2. \end{cases}$$

Thus, we rewrite the functionals (3) as

$$\begin{aligned} J_i(h, v^1, v^2) = {} & \frac{\mu_i}{2} \iint_{\omega_i \times (0, T)} |v^i|^2 \, dx \, dt \\ & + \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0, T)} \|\Lambda_1 v^1 + \Lambda_2 v^2 - \tilde{y}_d^i\|^2 \, dx \, dt \qquad \text{for } i = 1, 2, \end{aligned}$$

where $\tilde{y}_d^i = y_d^i - q(h)|_{\mathcal{O}_{i,d}}$, $(i, j = 1, 2)$ and $\| \cdot \|$ stands for the usual Euclidian norm. Therefore, $(\bar{v}^1, \bar{v}^2)$ is a Nash equilibrium if and only if satisfies (8)–(9), namely

$$\mu_i \iint_{\omega_i \times (0, T)} \bar{v}^i v^i \, dx \, dt + \alpha_i \iint_{\mathcal{O}_{i,d} \times (0, T)} (\Lambda_1 \bar{v}^1 + \Lambda_2 \bar{v}^2 - \tilde{y}_d^i) \cdot \Lambda_i v^i \, dx \, dt = 0,$$

for $i = 1, 2$ and for any $(v^1, v^2) \in \mathscr{H}$. It follows that

$$\mu_i (\bar{v}^i, v^i)_{\omega_i \times (0, T)} + \alpha_i \big(\Lambda_i^* [(\Lambda_1 \bar{v}^1 + \Lambda_2 \bar{v}^2)|_{\mathcal{O}_{i,d}} - \tilde{y}_d^i], v^i\big)_{\omega_i \times (0, T)} = 0,$$

with $(\cdot, \cdot)_{\mathscr{A}}$ denoting the internal product in $L^2(\mathscr{A})$ and $\Lambda_i^* \in \mathscr{L}\big([L^2(Q)]^2, \mathscr{H}_i\big)$ is the adjoint operator of Λ_i. Hence,

$$\mu_i \bar{v}^i + \alpha_i \Lambda_i^* [(\Lambda_1 \bar{v}^1 + \Lambda_2 \bar{v}^2)|_{\mathcal{O}_{i,d}}] = \alpha_i \Lambda_i^* \tilde{y}_d^i.$$

For all $v = (v^1, v^2)$, we define the operator $R = (R_1, R_2) \in \mathscr{L}(\mathscr{H}, \mathscr{H})$ as

$$R_i v = \mu_i v^i + \alpha_i \Lambda_i^* [(\Lambda_1 v^1 + \Lambda_2 v^2)\chi_{\mathcal{O}_{i,d}}],$$

for each $i = 1, 2$. Thereby, $\bar{v} = (\bar{v}^1, \bar{v}^2)$ is a Nash equilibrium if and only if

$$R\bar{v} = \alpha_i \Lambda_i^* \tilde{y}_d^i, \qquad i = 1, 2, \tag{16}$$

where the right hand side is a given fixed element of $\mathscr{H}$. Let us calculate

$$\begin{aligned}(Rv, v)_{\mathscr{H}} = \sum_{i=1}^{2} \mu_i \|v^i\|^2_{L^2(\omega_i \times (0,T))} + \alpha_1 (\Lambda_1 v^1 + \Lambda_2 v^2, \Lambda_1 v^1)_{\mathcal{O}_{1,d} \times (0,T)} \\ + \alpha_2 (\Lambda_1 v^1 + \Lambda_2 v^2, \Lambda_2 v^2)_{\mathcal{O}_{2,d} \times (0,T)}.\end{aligned}$$

By developing the product of cross terms and applying Young's inequality to them, we finally obtain

$$\begin{aligned}(Rv, v)_{\mathscr{H}} \geq \mu_1 \|v^1\|^2_{\mathscr{H}_1} + \mu_2 \|v^2\|^2_{\mathscr{H}_2} - \frac{\alpha_1}{4} \|\Lambda_2 \chi_{\mathcal{O}_{1,d}}\|^2_{\mathscr{H}_{1,d}} \|v^2\|^2_{\mathscr{H}_2} \\ - \frac{\alpha_2}{4} \|\Lambda_1 \chi_{\mathcal{O}_{2,d}}\|^2_{\mathscr{H}_{2,d}} \|v^1\|^2_{\mathscr{H}_1},\end{aligned}$$

where $\|\cdot\|_{\mathscr{H}_{i,d}}$ denotes the norm in the space $\mathscr{L}\big(\mathscr{H}_{3-i}, L^2\big(\mathcal{O}_{i,d} \times (0,T)\big)\big)$ for $i = 1, 2$. Then, for parameters μ_1 and μ_2 sufficiently large such that

$$\begin{aligned}4\mu_1 &> \alpha_2 \|\Lambda_1 \chi_{\mathcal{O}_{2,d}}\|^2_{\mathscr{H}_{2,d}}, \\ 4\mu_2 &> \alpha_1 \|\Lambda_2 \chi_{\mathcal{O}_{1,d}}\|^2_{\mathscr{H}_{1,d}},\end{aligned}$$

we have

$$(Rv, v)_{\mathscr{H}} \geq \gamma \|v\|^2_{\mathscr{H}}, \qquad \gamma = \min_{i=1,2} \left\{ \mu_i - \frac{\alpha_{3-i}}{4} \|\Lambda_i \chi_{\omega_{3-i,d}}\|^2_{\mathscr{H}_{3-i,d}} \right\} > 0. \tag{17}$$

We define the functional $a(v, u) : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ as

$$a(v, u) = (Rv, u)_{\mathscr{H}}.$$

Then, from the definition of R and the estimation (17), a is a continuous and coercive bilinear form. Applying the Lax-Milgram theorem (see e.g. [8]), we have that for all $\phi \in \mathscr{H}$, there exists a unique element $v \in \mathscr{H}$ such that

$$a(v, u) = (\phi, u) \qquad \forall u \in \mathscr{H},$$

particularly satisfying (16). Thus we have proved the existence and uniqueness of the Nash equilibrium related to (J_1, J_2).

4. Optimality conditions for the followers

We have shown in the previous section that for μ_1 and μ_2 large enough, there exist a unique Nash equilibrium for (J_1, J_2). We want to express it in terms of a new adjoint variable. We have that $(\bar{v}^1, \bar{v}^2)$ is solution of (8)–(9) if

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (y_1 - y^i_{1,d})\hat{y}^i_1 + (y_2 - y^i_{2,d})\hat{y}^i_2 \, dx\, dt$$
$$+ \mu_i \iint_{\omega_i\times(0,T)} \bar{v}^i \hat{v}^i \, dx\, dt = 0, \qquad \forall \hat{v}^i \in L^2\big(\omega_i \times (0,T)\big),\ i = 1,2, \tag{18}$$

where $\hat{y}^i = (\hat{y}^i_1, \hat{y}^i_2)$ is the solution of system

$$\begin{cases} \hat{y}^i_{1,t} - \Delta \hat{y}^i_1 + a_{11}\hat{y}^i_1 + a_{12}\hat{y}^i_2 = \hat{v}^i \chi_{\omega_i} & \text{in } Q, \\ \hat{y}^i_{2,t} - \Delta \hat{y}^i_2 + a_{21}\hat{y}^i_1 + a_{22}\hat{y}^i_2 = 0 & \text{in } Q, \\ \hat{y}^i_j(0) = 0 \quad \text{in } \Omega, \quad \hat{y}^i_j = 0 \quad \text{on } \Sigma, \quad j = 1,2. \end{cases} \tag{19}$$

Let us introduce the adjoint state to (19), that is, $p^i = (p^i_1, p^i_2)^*$ solution of

$$\begin{cases} -p^i_{1,t} - \Delta p^i_1 + a_{11} p^i_1 + a_{21} p^i_2 = \alpha_i (y_1 - y^i_{1,d})\chi_{\mathcal{O}_{i,d}} & \text{in } Q, \\ -p^i_{2,t} - \Delta p^i_2 + a_{12} p^i_1 + a_{22} p^i_2 = \alpha_i (y_2 - y^i_{2,d})\chi_{\mathcal{O}_{i,d}} & \text{in } Q, \\ p^i_j(T) = 0 \quad \text{in } \Omega, \quad p^i_j = 0 \quad \text{on } \Sigma, \quad j = 1,2. \end{cases} \tag{20}$$

If we multiply (20) by $\hat{y}^i$ in $[L^2(Q)]^2$, and integrate by parts, we obtain

$$\iint_Q \alpha_i (y_1 - y^i_{1,d})\chi_{\mathcal{O}_{i,d}} \hat{y}^i_1 - a_{21} p^i_2 \hat{y}^i_1 \, dx\, dt = \iint_Q p^i_1 (\hat{v}^i \chi_{\omega_i} - a_{12}\hat{y}^i_2)\, dx\, dt,$$
$$\iint_Q \alpha_i (y_2 - y^i_{2,d})\chi_{\mathcal{O}_{i,d}} \hat{y}^i_2 \, dx\, dt = \iint_Q (-a_{21} p^i_2 \hat{y}^i_1 + a_{12} p^i_1 \hat{y}^i_2)\, dx\, dt.$$

Adding up the above expressions and replacing on (18) we have

$$\iint_{\omega_i\times(0,T)} p^i_1 \hat{v}^i \, dx\, dt + \mu_i \iint_{\omega_i\times(0,T)} \bar{v}^i \hat{v}^i \, dx\, dt = 0,$$

which implies that

$$p_1^i \chi_{\omega_i} + \mu_i \bar{v}^i = 0.$$

Therefore, given $h \in L^2\big(\omega \times (0,T)\big)$, the pair $(\bar{v}^1, \bar{v}^2)$ is a Nash equilibrium of problem (6)–(7) if and only if

$$\bar{v}^i = -\frac{1}{\mu_i} p_1^i \chi_{\omega_i}, \qquad i = 1,2,$$

where (y, p^i) is solution of the coupled system

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11} y_1 + a_{12} y_2 = h\chi_\omega - \frac{1}{\mu_1} p_1^1 \chi_{\omega_1} - \frac{1}{\mu_2} p_1^2 \chi_{\omega_2} & \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = 0 & \text{in } Q, \\ -p_{1,t}^i - \Delta p_1^i + a_{11} p_1^i + a_{21} p_2^i = \alpha_i (y_1 - y_{1,d}^i) \chi_{\mathcal{O}_{i,d}} & \text{in } Q, \\ -p_{2,t}^i - \Delta p_2^i + a_{12} p_1^i + a_{22} p_2^i = \alpha_i (y_2 - y_{2,d}^i) \chi_{\mathcal{O}_{1,d}} & \text{in } Q, \\ y_j(0) = y_j^0, \quad p_j^i(T) = 0, \quad y_j = p_j^i = 0 \quad \text{on } \Sigma, \quad i,j = 1,2. \end{cases} \tag{21}$$

5. Null controllability

Recall that the main objective is to prove the null controllability of (y_1, y_2) at time T. Note that the computation of the follower controls satisfying conditions (8)–(9) added four additional equations coupled to the original system under study. Thus, in this section, we look for a control $h \in L^2\big(\omega \times (0,T)\big)$ such that the solution of (21) satisfies (11).

In order to accomplish this, consider the following system which is the adjoint of (21)

$$\begin{cases} -\varphi_{1,t} - \Delta \varphi_1 + a_{11} \varphi_1 + a_{21} \varphi_2 = \alpha_1 \theta_1^1 \chi_{\mathcal{O}_{1,d}} + \alpha_2 \theta_1^2 \chi_{\mathcal{O}_{2,d}} & \text{in } Q, \\ -\varphi_{2,t} - \Delta \varphi_2 + a_{12} \varphi_1 + a_{22} \varphi_2 = \alpha_1 \theta_2^1 \chi_{\mathcal{O}_{1,d}} + \alpha_2 \theta_2^2 \chi_{\mathcal{O}_{1,d}} & \text{in } Q, \\ \theta_{1,t}^i - \Delta \theta_1^i + a_{11} \theta_1^i + a_{12} \theta_2^i = -\frac{1}{\mu_i} \varphi_1 \chi_{\omega_i} & \text{in } Q, \\ \theta_{2,t}^i - \Delta \theta_2^i + a_{21} \theta_1^i + a_{22} \theta_2^i = 0 & \text{in } Q, \\ \varphi_j(T) = f_j, \quad \theta_j^i(0) = 0 \quad \text{in } \Omega, \quad \varphi_j = \theta_j^i = 0 \quad \text{on } \Sigma, \quad i,j = 1,2. \end{cases} \tag{22}$$

The main task is to prove an observability inequality for system (22). In fact, it remains an open problem if the required observability inequality holds true when $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$ and $a_{12} \neq 0$.

Taking into consideration assumptions (12) and (13) we can simplify the previous system as follows

$$\begin{cases} -\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2 = (\alpha_1\theta_1^1 + \alpha_2\theta_1^2)\chi_{\mathcal{O}_d} & \text{in } Q, \\ -\varphi_{2,t} - \Delta\varphi_2 + a_{22}\varphi_2 = (\alpha_1\theta_2^1 + \alpha_2\theta_2^2)\chi_{\mathcal{O}_d} & \text{in } Q, \\ \theta_{1,t}^i - \Delta\theta_1^i + a_{11}\theta_1^i = -\frac{1}{\mu_i}\varphi_1\chi_{\omega_i} & \text{in } Q, \\ \theta_{2,t}^i - \Delta\theta_2^i + a_{21}\theta_1^i + a_{22}\theta_2^i = 0 & \text{in } Q, \\ \varphi_j(T) = f_j, \quad \theta_j^i(0) = 0 \quad \text{in } \Omega, \quad \varphi_j = \theta_j^i = 0 \quad \text{on } \Sigma, \quad j = 1, 2. \end{cases} \tag{23}$$

The estimate is given in the following result:

Proposition 5.1. *Suppose that assumptions* (12)–(14) *hold,* $\mathcal{O}_d \cap \omega \neq \emptyset$ *and that* μ_i *are sufficiently large. There exist a positive constant* C *and a weight function* $\rho = \rho(t)$ *blowing up at* $t = T$*, such that, for every* $(f_1, f_2) \in [L^2(\Omega)]^2$*, the solution* (φ, θ^i) *to* (23) *satisfies*

$$\int_\Omega |\varphi_1(0)|^2\,dx + \int_\Omega |\varphi_2(0)|^2\,dx + \sum_{i=1}^2 \iint_Q \rho^{-2}|\theta_1^i|^2\,dx\,dt + \sum_{i=1}^2 \iint_Q \rho^{-2}|\theta_2^i|^2\,dx\,dt$$
$$\leq C \iint_{\omega\times(0,T)} |\varphi_1|^2\,dx\,dt. \tag{24}$$

The proof of Proposition 5.1 relies on various well-known arguments that will be clarified in its proof. For now, we suppose that the proposition holds and we will conclude the proof of Theorem 2.1. There are several ways to show that inequality (24) implies the existence of a null control of minimal norm. We sketch one of them. First, we can prove that

$$\|(f_1, f_2)\|_W^2 = \iint_{\omega\times(0,T)} |\varphi_1|^2\,dx\,dt,$$

where φ_1 is the corresponding (first component of the) solution to (23) defines a norm in $[L^2(\Omega)]^2$. This can be readily verified by means of Proposition 6.4 below or directly from (24), using classical results for cascade systems that provide a unique continuation property. Now, define W as the completion of $[L^2(\Omega)]^2$ with this norm and set

$$\mathcal{I}(f_1, f_2) = \frac{1}{2}\|(f_1, f_2)\|_W^2 + \int_\Omega y_1^0\varphi_1(0)\,dx + \int_\Omega y_2^0\varphi_2(0)\,dx$$
$$- \sum_{i=1}^2 \iint_{\mathcal{O}_d\times(0,T)} \alpha_i\theta_1^i y_{1,d}^i\,dx\,dt - \sum_{i=1}^2 \iint_{\mathcal{O}_d\times(0,T)} \alpha_i\theta_2^i y_{2,d}^i\,dx\,dt,$$

where (φ, θ^i) is the solution to (22). It is clear that $\mathcal{I}$ is continuous and strictly convex. Moreover, the observability inequality (24) allows to prove that

$$
\begin{aligned}
&\mathscr{I}(f_1, f_2) \\
&\quad \geq \frac{1}{4}\|(f_1, f_2)\|_W^2 - C\Bigg(\int_\Omega |y_1^0|^2\,dx + \int_\Omega |y_2^0|^2\,dx \\
&\qquad\qquad + \sum_{i=1}^{2} \alpha_i^2 \Big(\iint_Q \rho^2 |y_{1,d}^i|^2\,dx\,dt + \iint_Q \rho^2 |y_{2,d}^i|^2\,dx\,dt\Big)\Bigg),
\end{aligned}
$$

where C and ρ are provided by Proposition 5.1. Therefore, $\mathscr{I}(f_1, f_2)$ is coercive in W. Note that here, we have used the growth assumption (15). Consequently, the existence of a minimizer $(\hat{f}_1, \hat{f}_2)$ solving

$$
\mathscr{I}(\hat{f}_1, \hat{f}_2) = \min_{(f_1, f_2) \in W} \mathscr{I}(f_1, f_2)
$$

is guaranteed. Hence, the control $h = \hat{\varphi}_1 \chi_\omega$, where $\hat{\varphi}_1$ is the solution of (23) corresponding to this minimum solves (10)–(11), see for instance [10]. This ends the proof of Theorem 2.1.

6. Proof of Proposition 5.1

We will devote this section to prove Propostion 5.1. To this end, we recall some results that will be useful to prove (24). The starting point is a well-known global Carleman inequality for solutions to scalar parabolic equations:

$$
\begin{cases}
z_t - \Delta z = F & \text{in } Q, \\
z = 0 & \text{on } \Sigma, \\
z(x, 0) = z^0(x) & \text{in } Q,
\end{cases}
\tag{25}
$$

where $F \in L^2(Q)$ and $z^0 \in L^2(\Omega)$. To formulate this inequality we first need to introduce a special weight function whose existence is guaranteed by the following result (see Lemma 1.1 in [12]).

Lemma 6.1. *Let $\mathscr{B} \subset\subset \Omega$ be a nonempty open subset. Then there exists $\eta^0 \in C^2(\overline{\Omega})$ such that*

$$
\begin{cases}
\eta^0(x) > 0 \quad \textit{all } x \in \Omega, \quad \eta^0|_{\partial\Omega} = 0, \\
|\nabla \eta^0| > 0 \quad \textit{for all } x \in \overline{\Omega \backslash \mathscr{B}}.
\end{cases}
$$

Fixing an open subset $\mathscr{B} \subset\subset \Omega$, we set

$$
\beta_0(x) = e^{2C^*\|\eta^0\|_\infty} - e^{C^*\eta^0(x)},
\tag{26}
$$

for $x \in \overline{\Omega}$, where η^0 is the function provided by Lemma 6.1 for this particular $\mathscr{B}$ and C^* is an appropriate positive constant only depending on Ω and $\mathscr{B}$.

In the following result, due to Imanuvilov and Yamamoto (see [17]), we have a global Carleman inequality for the solutions to (25):

Lemma 6.2. *Let $\mathscr{B} \subset\subset \Omega$ be a nonempty open subset. For any $m \in \mathbb{R}$, there exist constants $s_m > 0$ and $C_m > 0$ such that, for any $s \geq s_m$ and every $z^0 \in L^2(\Omega)$, the solution z to (25) satisfies*

$$\begin{aligned} I(m,z) &:= \iint_Q e^{-2s\beta}(s\gamma)^{m-2}|\nabla z|^2\,dx\,dt + \iint_Q e^{-2s\beta}(s\gamma)^m|z|^2\,dx\,dt \\ &\leq C_m\Big(\mathscr{L}_{\mathscr{B}}(m,z) + \iint_Q e^{-2s\beta}(s\gamma)^{m-3}|F|^2\,dx\,dt\Big). \end{aligned} \tag{27}$$

Furthermore, C_m only depends on Ω, $\mathscr{B}$ and m and s_m can be taken of the form $s_m = \sigma_m(T + T^2)$ where σ_m only depends on Ω, $\mathscr{B}$ and m. In inequality (27), β is defined by

$$\beta(x,t) = \frac{\beta_0(x)}{t(T-t)}, \qquad \text{for } (x,t) \in Q,$$

with β_0 the function appearing in (26) and where $\mathscr{L}_{\mathscr{B}}(m,z)$ and $\gamma = \gamma(t)$ stand for

$$\mathscr{L}_{\mathscr{B}}(m,z) := \iint_{\mathscr{B}\times(0,T)} e^{-2s\beta}(s\gamma)^m|z|^2\,dx\,dt, \qquad \gamma(t) := \frac{1}{t(T-t)}.$$

Remark 6.3. Note that by changing t for $T-t$, Lemma 6.2 remains valid for linear backwards in time systems. Therefore, we can apply it interchangeably in what follows.

Now we are in position to prove inequality (24). It is consequence of a combination of global Carleman inequalities and suitable energy estimates for the solutions to system (23). In view of assumptions (12) and (13), we may simplify (23) as

$$\begin{cases} -\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2 = \psi_1\chi_{\mathcal{O}_d} & \text{in } Q, \\ -\varphi_{2,t} - \Delta\varphi_2 + a_{22}\varphi_2 = \psi_2\chi_{\mathcal{O}_d} & \text{in } Q, \\ \psi_{1,t} - \Delta\psi_1 + a_{11}\psi_1 = -\Big(\frac{\alpha_1}{\mu_1}\chi_{\omega_1} + \frac{\alpha_2}{\mu_2}\chi_{\omega_2}\Big)\varphi_1 & \text{in } Q, \\ \psi_{2,t} - \Delta\psi_2 + a_{21}\psi_1 + a_{22}\psi_2 = 0 & \text{in } Q, \\ \varphi_j(T) = f_j, \quad \psi_j(0) = 0 \quad \text{in } \Omega, \quad \varphi_j = \psi_j = 0 \quad \text{on } \Sigma, \quad j = 1,2, \end{cases} \tag{28}$$

where $\psi_j = \alpha_1\theta_j^1 + \alpha_2\theta_j^2$ for $j = 1, 2$. Let us now present a Carleman estimate for the solutions to the non-scalar adjoint problem (28) that is the key point to accomplish the proof of Proposition 5.1:

Proposition 6.4. *Suppose that assumptions* (12)–(14) *hold and that* $\mathcal{O}_d \cap \omega \neq \emptyset$. *Then, for an adequate selection of parameters* d_i *and* $p_i \in \mathbb{R}$, *for* $i = 1, 2$, *there exist a function* $\alpha_0 \in C^2(\overline{\Omega})$ *and positive constants* C *and* σ_2 *such that, for every* $(f_1, f_2) \in [L^2(\Omega)]^2$, *the solution to system* (28) *satisfies*

$$\begin{aligned} I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(p_1,\psi_1) + I(p_2,\psi_2) \\ \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2-d_2+12}|\varphi_1|^2\,dx\,dt \end{aligned} \tag{29}$$

$$\forall s \geq s_2 = \sigma_2\big(T + T^2 + T^2 \max\{\max_{j=1,2}\|a_{jj}\|_\infty^{2/3}, \|a_{21}\|_\infty^{2/(d_2-(d_1-3))}, \|a_{21}\|_\infty^{2/(p_1-(p_2-3))}\}\big).$$

In inequality (29) $\alpha = \alpha(x,t)$ *is given by*

$$\alpha(x,t) = \frac{\alpha_0(x)}{t(T-t)}.$$

Proof. The proof is standard and relies on various well-known arguments. Define

$$\mathcal{O} := \mathcal{O}_d \cap \omega,$$

and since $\mathcal{O} \neq \emptyset$, there exists a non-empty open set $\mathcal{O}_0 \subset\subset \mathcal{O}$. Let α_0 and α be the functions associated to $\mathcal{B} = \mathcal{O}_0$ provided by Lemma 6.2. We will achieve the proof in three steps.

Step 1. We begin by applying inequality (27) to each function φ_j and θ_j, solution to (28), with different real numbers d_j and p_j to be chosen later. We add such inequalities and obtain the following

$$\begin{aligned} &I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(p_1,\psi_1) + I(p_2,\psi_2) \\ &\leq C_0\Bigg(\sum_{j=1}^{2}\mathcal{L}_{\mathcal{O}_0}(d_j,\varphi_j) + \sum_{j=1}^{2}\mathcal{L}_{\mathcal{O}_0}(p_j,\psi_j) + \sum_{j=1}^{2}\iint_Q e^{-2s\alpha}(s\gamma)^{d_j-3}|\psi_j\chi_{\mathcal{O}_d}|^2\,dx\,dt \\ &\quad + \sum_{j=1}^{2}\sum_{i=1}^{j}\iint_Q e^{-2s\alpha}(s\gamma)^{d_i-3}\|a_{ji}\|_\infty^2|\varphi_j|^2\,dx\,dt \end{aligned}$$

$$+\sum_{i=1}^{2}\sum_{j=1}^{i}\iint_{Q} e^{-2s\alpha}(s\gamma)^{p_i-3}\|a_{ij}\|_{\infty}^{2}|\psi_j|^2\,dx\,dt$$

$$+\iint_{Q} e^{-2s\alpha}(s\gamma)^{p_1-3}\left|-\frac{\alpha_1}{\mu_1}\varphi_1\chi_{\omega_1}-\frac{\alpha_2}{\mu_2}\varphi_1\chi_{\omega_2}\right|^2 dx\,dt\Big), \tag{30}$$

valid for every $s \geq s_0 = \sigma_0(T + T^2)$, with C_0 and σ_0 two positive constants depending on Ω, $\mathcal{O}$, d_j and p_j. The next step is to absorb as many possible terms from the right hand side into the left hand side of (30). To accomplish this, we have to properly select each of the powers d_j and p_j involved. It is not difficult to see that if

$$d_1 - 3 < p_1 < d_1 + 3 \quad \text{and} \quad d_2 - 3 < p_2, \tag{31}$$

we get

$$I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(p_1,\psi_1) + I(p_2,\psi_2)$$
$$\leq C_0\Big(\sum_{j=1}^{2}\mathcal{L}_{\mathcal{O}_0}(d_j,\varphi_j) + \sum_{j=1}^{2}\mathcal{L}_{\mathcal{O}_0}(p_j,\psi_j)$$
$$+\sum_{j=1}^{2}\sum_{i=1}^{j}\iint_{Q} e^{-2s\alpha}(s\gamma)^{d_i-3}\|a_{ji}\|_{\infty}^{2}|\varphi_j|^2\,dx\,dt$$
$$+\sum_{i=1}^{2}\sum_{j=1}^{i}\iint_{Q} e^{-2s\alpha}(s\gamma)^{p_i-3}\|a_{ij}\|_{\infty}^{2}|\psi_j|^2\,dx\,dt\Big),$$

$\forall s \geq s_0$ and where C is a new constant only depending on Ω, $\mathcal{O}_0$, d_j, p_j and α_j. We can get rid of the two last sums in the previous inequality if we take into account that $\gamma(t)^{-1} \leq T^2/4$ in $(0, T)$, then select

$$d_1 - 3 < d_2 \quad \text{and} \quad p_2 - 3 < p_1, \tag{32}$$

and take

$$s \geq s_2 = \sigma_2\big(T + T^2 + T^2\max\{\max_{j=1,2}\|a_{jj}\|_{\infty}^{2/3}, \|a_{21}\|^{2/(d_2-(d_1-3))},$$
$$\|a_{21}\|_{\infty}^{2/(p_1-(p_2-3))}\}\big),$$

thus obtaining

$$I(d_1,\varphi_1)+I(d_2,\varphi_2)+I(p_1,\psi_1)+I(p_2,\psi_2)$$
$$\le C_2\Big(\sum_{j=1}^{2}\mathscr{L}_{\mathcal{O}_0}(d_j,\varphi_j)+\sum_{j=1}^{2}\mathscr{L}_{\mathcal{O}_0}(p_j,\psi_j)\Big),\qquad \forall s\ge s_2. \tag{33}$$

with C_2 and σ_2 two new positive constants only depending on Ω, $\mathcal{O}_0$, d_j, p_j, α_j and $\|a_{ij}\|_\infty$.

Step 2. Now, we want to eliminate the local terms corresponding to ψ_j in the right-hand side of (33). We will reason out as in [26] and [14]. Given a set $\mathcal{O}_1$ such that $\mathcal{O}_0\subset\subset\mathcal{O}_1\subset\subset\mathcal{O}$, we consider a function $\xi\in C^\infty(\mathbb{R}^N)$ verifying: $0\le\xi\le 1$ in $\mathbb{R}^N$, $\xi\equiv 1$ in $\mathcal{O}_0$, $\operatorname{supp}\xi\subset\mathcal{O}_1$ and

$$\frac{\Delta\xi}{\xi^{1/2}}\in L^\infty(\Omega)\quad\text{ and }\quad\frac{\nabla\xi}{\xi^{1/2}}\in L^\infty(\Omega)^N.$$

We set $u_j=e^{-2s\alpha}(s\gamma)^{p_j}$ for $j=1,2$. Then, we multiply the equations satisfied by φ_j in system (28) by $u_j\xi\psi_j$, respectively, and integrate in Q. We add those expressions to obtain

$$\begin{aligned}\mathscr{L}_{\mathcal{O}_0}(p_1,\psi_1)+\mathscr{L}_{\mathcal{O}_0}(p_2,\psi_2)&\le\iint_Q u_1\xi|\psi_1|^2+\iint_Q u_2\xi|\psi_2|^2\\&=\iint_Q u_1\xi\psi_1(-\varphi_{1,t}-\Delta\varphi_1+a_{11}\varphi_1+a_{21}\varphi_2)\\&\quad+\iint_Q u_2\xi\psi_2(-\varphi_{2,t}-\Delta\varphi_2+a_{22}\varphi_2).\end{aligned}\tag{34}$$

Integrating by parts several times with respect to the time and space variables in the right hand side of (34) we obtain the following expression

$$\begin{aligned}\mathscr{L}_{\mathcal{O}_0}(p_1,\psi_1)+\mathscr{L}_{\mathcal{O}_0}(p_2,\psi_2)&\le\varepsilon C_A\big(I(p_1,\psi_1)+I(p_2,\psi_2)\big)\\&\quad+C_{\varepsilon,A}\Big(\iint_{\mathcal{O}_1\times(0,T)}e^{-2s\alpha}(s\gamma)^{p_1+4}|\varphi_1|^2\\&\quad+\iint_{\mathcal{O}_1\times(0,T)}e^{-2s\alpha}(s\gamma)^{p_2+4}|\varphi_2|^2\Big),\end{aligned}\tag{35}$$

where $\varepsilon>0$, C_A, $C_{\varepsilon,A}$ are new constants only depending on Ω, $\mathcal{O}_0$, $\mathcal{O}_1$, ω_i, p_j and $\|a_{ij}\|_\infty$. Replacing (35) in (33) with $\varepsilon=\frac{1}{2C_AC_2}$, with C_2 the constant appearing in (33), we obtain

$$I(d_1,\varphi_1)+I(d_2,\varphi_2)+I(p_1,\psi_1)+I(p_2,\psi_2) \\ \leq C\Big(\iint_{\mathcal{O}_1\times(0,T)} e^{-2s\alpha}(s\gamma)^{p_1+4}|\varphi_1|^2+\iint_{\mathcal{O}_1\times(0,T)} e^{-2s\alpha}(s\gamma)^{p_2+4}|\varphi_2|^2\Big), \quad (36)$$

for a new positive constant C and valid for all $s \geq s_2$.

Step 3. Thanks to assumption (14) and by selecting appropriately the parameter s we will eliminate the local term corresponding to φ_2 in the previous inequality. To this end, consider a new set $\mathcal{O}_2$ such that $\mathcal{O}_1 \subset\subset \mathcal{O}_2 \subset \mathcal{O}$ and a function $\tilde{\xi} \in C^\infty(\mathbb{R}^N)$ with properties analogous to that of Step 2.

We set $u = e^{-2s\alpha}(s\gamma)^{p_2+4}$. Recall that the coefficient a_{21} satisfies (14) and, for simplicity, assume that $a_{21} \geq a_0$ in $\mathcal{O}\times(0,T)$. We multiply the equation satisfied by φ_1 in system (28) by $u\tilde{\xi}\varphi_2$ and integrate in Q. We obtain

$$a_0\mathscr{L}_{\mathcal{O}_1}(p_2+4,\varphi_2) \leq \iint_Q u\tilde{\xi}a_{21}|\varphi_2|^2 = \iint_Q (\varphi_{1,t}+\Delta\varphi_1 - a_{11}\varphi_1)u\tilde{\xi}\varphi_2 \\ + \iint_Q \psi_1\chi_{\mathcal{O}_d}u\eta\varphi_2. \quad (37)$$

Again, reasoning out as in [14], we integrate by parts several times with respect to the time and space variables in the right hand side of (37) to obtain

$$a_0\mathscr{L}_{\mathcal{O}_1}(p_2+4,\varphi_2) \leq 6\varepsilon I(d_2,\varphi_2) + C\left(1+\frac{1}{\varepsilon}\right)\iint_{\mathcal{O}_2\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2-d_2+12}|\varphi_1|^2 \\ + C\iint_Q e^{-2s\alpha}(s\gamma)^{p_2+4}|\psi_1|^2, \quad (38)$$

for a new positive constant C and $\varepsilon > 0$. Replacing (38) in (36) with $\varepsilon = \frac{a_0}{12C}$ where C is the constant appearing in (36) we get

$$I(d_1,\varphi_1)+I(d_2,\varphi_2)+I(p_1,\psi_1)+I(p_2,\psi_2) \\ \leq C\Big(\iint_{\mathcal{O}_2\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2-d_2+12}|\varphi_1|^2+\iint_Q e^{-2s\alpha}(s\gamma)^{p_2+4}|\psi_1|^2\Big), \quad (39)$$

valid for all $s \geq s_2$. Selecting $p_2+4 < p_1$ and increasing the parameter s_2, if necessary, we obtain

$$I(d_1,\varphi_1)+I(d_2,\varphi_2)+I(p_1,\psi_1)+I(p_2,\psi_2) \\ \leq C\iint_{\mathcal{O}_2\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2-d_2+12}|\varphi_1|^2, \qquad \forall s \geq s_2.$$

Therefore the proof is complete. □

Remark 6.5. So far, we have not been able to remove the assumption (13) from Theorem 2.1. Indeed, consider that $a_{12} \neq 0$ in Q in system (28). Following the ideas of the proof of Proposition 6.4, it is not difficult to see that two new terms appear in the right-hand side of (30), more precisely

$$\iint_Q e^{-2s\alpha}(s\gamma)^{d_2-3}\|a_{12}\|_\infty^2|\varphi_1|^2 \quad \text{and} \quad \iint_Q e^{-2s\alpha}(s\gamma)^{p_1-3}\|a_{12}\|_\infty^2|\psi_2|^2.$$

Note that (31) and (32) remain to be a valid choice, but we have to add the additional constraints

$$d_2 - 3 < d_1 \quad \text{and} \quad p_1 - 3 < p_2, \tag{40}$$

in order to absorb this terms. We proceed as before and eliminate the local terms associated to ψ_j. Then, we estimate the local term corresponding to φ_2 and by replacing as in (39) we note that we cannot longer choose $p_2 + 4 < p_1$ since it interferes with the previous selection (40).

As mentioned earlier, the observability inequality (24) is a combination of a global Carleman estimate and suitable energy estimates. To this aim, we set $\bar{s} = s_2$ where s_2 is the constant furnished by Proposition 6.4. Let us consider the function

$$l(t) = \begin{cases} T^2/4 & \text{for } 0 \le t \le T/2, \\ t(T-t) & \text{for } T/2 \le t \le T, \end{cases}$$

and the following associated weight functions

$$\bar{\alpha}(x,t) = \frac{\alpha_0(x)}{l(t)}, \qquad \bar{\gamma}(t) = \frac{1}{l(t)}.$$

Let $\eta \in C^1([0,T])$ be a function satisfying

$$\eta = 1 \quad \text{in } [0, T/2], \qquad \eta = 0 \quad \text{in } [3T/4, T], \qquad |\eta'(t)| \le C/T. \tag{41}$$

Hereinafter C will denote a generic positive constant that may vary from line to line. Arguing as in [11], we proceed to obtain energy estimates for φ_j with this cut-off function, this is,

$$\int_\Omega (-\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2)\eta\varphi_1\,dx = \int_\Omega \psi_1\chi_{\mathcal{O}_d}\eta\varphi_1\,dx,$$
$$\int_\Omega (-\varphi_{2,t} - \Delta\varphi_2 + a_{22}\varphi_2)\eta\varphi_2\,dx = \int_\Omega \psi_2\chi_{\mathcal{O}_d}\eta\varphi_2\,dx,$$

Then, by adding up and integrating by parts we obtain that

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \eta(|\varphi_1|^2+|\varphi_2|^2)\,dx\Big)+\int_\Omega \eta(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2)\,dx\\
&\quad=-\int_\omega \eta a_{11}|\varphi_1|^2\,dx-\int_\Omega \eta a_{22}|\varphi_2|^2\,dx-\int_\Omega \eta a_{21}\varphi_2\varphi_1\,dx\\
&\qquad-\frac{1}{2}\int_\Omega \eta'(|\varphi_1|^2+|\varphi_2|^2)\,dx+\int_\Omega \psi_1\chi_{\mathcal{O}_d}\eta\varphi_1\,dx+\int_\Omega \psi_2\chi_{\mathcal{O}_d}\eta\varphi_2\,dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
&-\frac{d}{dt}\Big(\int_\Omega \eta(|\varphi_1|^2+|\varphi_2|^2)\,dx\Big)+2\int_\Omega \eta(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2)\,dx\\
&\quad\le \frac{C}{T}\int_\Omega(|\varphi_1|^2+|\varphi_2|^2)\,dx+2(1+\|A\|)\int_\Omega \eta(|\varphi_1|^2+|\varphi_2|^2)\,dx\\
&\qquad+\int_\Omega \eta(|\psi_1|^2+|\psi_2|^2)\,dx
\end{aligned}
$$

where $\|A\| := \sum_{i=1}^2\sum_{j=1}^i \|a_{ij}\|_\infty$. We multiply by $e^{2(1+\|A\|)t}$ the above expression and integrate in $[0,T]$. Then, in view of (41), we obtain for a new constant that

$$
\begin{aligned}
&\int_\Omega\big(|\varphi_1(0)|^2+|\varphi_2(0)|^2\big)\,dx+\int_0^{T/2}\int_\Omega(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2)\,dx\,dt\\
&\quad\le C\Big(\int_{T/2}^{3T/4}\int_\Omega(|\varphi_1|^2+|\varphi_2|^2)\,dx\,dt+\int_0^{3T/4}\int_\Omega(|\psi_1|^2+|\psi_2|^2)\,dx\,dt\Big).
\end{aligned}
$$

Using Poincaré's inequality it is not difficult to see that

$$
\begin{aligned}
&\int_\Omega\big(|\varphi_1(0)|^2+|\varphi_2(0)|^2\big)\,dx+\int_0^{T/2}\int_\Omega(|\varphi_1|^2+|\varphi_2|^2+|\nabla\varphi_1|^2+|\nabla\varphi_2|^2)\,dx\,dt\\
&\quad\le C\Big(\int_{T/2}^{3T/4}\int_\Omega(|\varphi_1|^2+|\varphi_2|^2)\,dx\,dt+\int_0^{3T/4}\int_\Omega(|\psi_1|^2+|\psi_2|^2)\,dx\,dt\Big).
\end{aligned}
$$

Since $\bar\alpha$ and $\bar\gamma$ have lower and upper bounds for $(x,t)\in\overline{\Omega}\times[0,T/2]$, we can introduce the corresponding weight functions in the above expression, namely

$$
\begin{aligned}
&\int_\Omega\big(|\varphi_1(0)|^2+|\varphi_2(0)|^2\big)\,dx+\bar I_{[0,T/2]}(d_1,\varphi_1)+\bar I_{[0,T/2]}(d_2,\varphi_2)\\
&\quad\le C\Big(\int_{T/2}^{3T/4}\int_\Omega(|\varphi_1|^2+|\varphi_2|^2)\,dx\,dt+\int_0^{3T/4}\int_\Omega(|\psi_1|^2+|\psi_2|^2)\,dx\,dt\Big), \qquad (42)
\end{aligned}
$$

where

$$\bar{I}_{[a,b]}(d,z) = \int_a^b \int_\Omega e^{-2\bar{s}\bar{\alpha}} \bar{\gamma}^d |z|^2 \, dx \, dt + \int_a^b \int_\Omega e^{-2\bar{s}\bar{\alpha}} \bar{\gamma}^{d-2} |\nabla z|^2 \, dx \, dt.$$

We add to both sides of the inequality (42) the terms $\bar{I}_{[0,T/2]}(p_1,\psi_1)$ and $\bar{I}_{[0,T/2]}(p_2,\psi_2)$, therefore

$$\begin{aligned} &\int_\Omega |\varphi_1(0)|^2 \, dx + \int_\Omega |\varphi_2(0)|^2 \, dx + \bar{I}_{[0,T/2]}(d_1,\varphi_1) \\ &\quad + \bar{I}_{[0,T/2]}(d_2,\varphi_2) + \bar{I}_{[0,T/2]}(p_1,\psi_1) + \bar{I}_{[0,T/2]}(p_2,\psi_2) \\ &\le C\Big(\bar{I}_{[0,T/2]}(p_1,\psi_1) + \bar{I}_{[0,T/2]}(p_2,\psi_2) \\ &\quad + \int_{T/2}^{3T/4} \int_\Omega (|\varphi_1|^2 + |\varphi_2|^2 + |\psi_1|^2 + |\psi_2|^2) \Big). \end{aligned} \tag{43}$$

To get rid of the terms corresponding to $\bar{I}_{[0,T/2]}(p_1,\psi_1)$ and $\bar{I}_{[0,T/2]}(p_2,\psi_2)$ in the right-hand side, first we see that from classical energy estimates for the third and fourth equation in (28), we have

$$\int_0^{T/2} \int_\Omega (|\psi_1|^2 + |\psi_2|^2 + |\nabla\psi_1|^2 + |\nabla\psi_2|^2) \, dx \, dt \le C\left(\frac{\alpha_1^2}{\mu_1^2} + \frac{\alpha_2^2}{\mu_2^2} \right) \int_0^{T/2} \int_\Omega |\varphi_1|^2 \, dx \, dt$$

where C is independent of μ_1 and μ_2. We introduce the weight functions in this new expression and obtain

$$\begin{aligned} &\bar{I}_{[0,T/2]}(p_1,\psi_1) + \bar{I}_{[0,T/2]}(p_2,\psi_2) \\ &\quad \le C\left(\frac{\alpha_1^2}{\mu_1^2} + \frac{\alpha_2^2}{\mu_2^2} \right) \left(\frac{T^2}{4} \right)^{d_1} \int_0^{T/2} \int_\Omega |\varphi_1|^2 e^{-2\bar{s}\bar{\alpha}} \bar{\gamma}^{d_1} e^{2\bar{s}\bar{\alpha}} \, dx \, dt. \end{aligned} \tag{44}$$

where we used the fact that $\bar{\gamma}^{-1} \le T^2/4$. Replacing (44) in the right-hand side of (43) and noting that for μ_i, $i = 1,2$, sufficiently large, we can absorb the new term into the left hand side to obtain

$$\begin{aligned} &\int_\Omega |\varphi_1(0)|^2 \, dx + \int_\Omega |\varphi_2(0)|^2 \, dx + \bar{I}_{[0,T/2]}(d_1,\varphi_1) \\ &\quad + \bar{I}_{[0,T/2]}(d_2,\varphi_2) + \bar{I}_{[0,T/2]}(p_1,\psi_1) + \bar{I}_{[0,T/2]}(p_2,\psi_2) \\ &\le C\Big(\int_{T/2}^{3T/4} \int_\Omega (|\varphi_1|^2 + |\varphi_2|^2 + |\psi_1|^2 + |\psi_2|^2) \, dx \, dt \Big). \end{aligned}$$

Then, using (29) and the upper bounds on the weight functions, we obtain

$$\begin{aligned}\int_\Omega |\varphi_1(0)|^2\,dx &+ \int_\Omega |\varphi_2(0)|^2\,dx + \bar{I}_{[0,T/2]}(d_1,\varphi_1)\\ &+ \bar{I}_{[0,T/2]}(d_2,\varphi_2) + \bar{I}_{[0,T/2]}(p_1,\psi_1) + \bar{I}_{[0,T/2]}(p_2,\psi_2)\\ &\le C\big(I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(p_1,\psi_1) + I(p_2,\psi_2)\big)\\ &\le C\iint_{\omega\times(0,T)} |\varphi_1|^2\,dx\,dt. \end{aligned} \tag{45}$$

On the other hand, since $\bar{\alpha} = \alpha$ and $\bar{\gamma} = \gamma$ in $\Omega \times (T/2, T)$ we use again inequality (29) to obtain

$$\begin{aligned}\bar{I}_{[T/2,T]}(d_1,\varphi_1) &+ \bar{I}_{[T/2,T]}(d_2,\varphi_2) + \bar{I}_{[T/2,T]}(p_1,\psi_1) + \bar{I}_{[T/2,T]}(p_2,\psi_2)\\ &\le I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(p_1,\psi_1) + I(p_2,\psi_2)\\ &\le C\iint_{\omega\times(0,T)} |\varphi_1|^2\,dx\,dt. \end{aligned} \tag{46}$$

We put together (45) and (46)

$$\begin{aligned}\int_\Omega |\varphi_1(0)|^2\,dx &+ \int |\varphi_2(0)|^2\,dx + \bar{I}_{[0,T]}(d_1,\varphi_1)\\ &+ \bar{I}_{[0,T]}(d_2,\varphi_2) + \bar{I}_{[0,T]}(p_1,\psi_1) + \bar{I}_{[0,T]}(p_2,\psi_2)\\ &\le C\iint_{\omega\times(0,T)} |\varphi_1|^2\,dx\,dt. \end{aligned} \tag{47}$$

Now we conclude the proof of Proposition 5.1. Set $\bar{\alpha}^*(t) = \max_{\bar{\Omega}} \bar{\alpha}(x,t)$ and define $\rho(t) := e^{\bar{s}\bar{\alpha}^*}$. Thus, $\rho(t)$ is a non-decreasing strictly positive function blowing up at $t = T$. We obtain energy estimates with this new weight for θ_j^i solution to the third equation and fourth equation in (23) (recall that at this point we are assuming that (12) and (13) hold). More precisely,

$$\iint_Q \rho^{-2}(|\theta_1^i|^2 + |\theta_2^i|^2)\,dx\,dt \le C\iint_{\omega_i\times(0,T)} \rho^{-2}|\varphi_1|^2\,dx\,dt, \qquad i = 1,2.$$

We note that the right hand side of the previous inequality is indeed comparable to $\bar{I}_{[0,T]}(d_1,\varphi_1)$ up to a multiplicative constant and by replacing accordingly in (47) we finally obtain inequality (24).

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