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Typical symplectic locally constant cocycles over certain shifts of countable type have simple Lyapunov spectra

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(Recommended by Marcelo Viana)

Abstract. We show that the Lyapunov spectrum of almost every symplectic locally constant integrable cocycle over certain shifts of countable type is simple.

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1. Introduction

This note is motivated by a question of Viana [8] about the simplicity of the Lyapunov spectrum of typical cocycles over hyperbolic bases.

More concretely, our goal is to provide a partial answer to Problem 4 in [8] for symplectic/unitary locally constant integrable cocycles over certain Bernoulli shifts.

Before stating our main result (in Subsection 1.3 below), let us introduce some definitions and notations.

1.1. Full countable shifts. Given Λ a finite or countable alphabet of cardinality $\#\Lambda \geq 2$, denote by $\Sigma := \Lambda^{\mathbb{N}}$ and $\hat{\Sigma} := \Lambda^{\mathbb{Z}} := \Sigma^{-} \times \Sigma$.

We consider the natural left shift maps $f : \Sigma \to \Sigma$ and $\hat{f} : \hat{\Sigma} \to \hat{\Sigma}$ on the symbolic spaces Σ and $\hat{\Sigma}$, and we denote by $p^{+} : \hat{\Sigma} \to \Sigma$ and $p^{-} : \hat{\Sigma} \to \Sigma^{-}$ the natural projections.

The set of words of the alphabet Λ is denoted by $\Omega := \bigcup_{n \in \mathbb{N}} \Lambda^{n}$. Given $\underline{\ell} \in \Omega$, the cylinder of Σ associated to $\underline{\ell}$ is

$$\Sigma(\underline{\ell}) := \{x \in \Sigma : x \text{ starts with } \underline{\ell}\}$$

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We say that a f-invariant probability measure μ on Σ has *bounded distortion* if there exists a constant $C(\mu) > 0$ such that

$$\frac{1}{C(\mu)}\mu\big(\Sigma(\underline{\ell}_1)\big)\mu\big(\Sigma(\underline{\ell}_2)\big) \leq \mu\big(\Sigma(\underline{\ell}_1\underline{\ell}_2)\big) \leq C(\mu)\mu\big(\Sigma(\underline{\ell}_1)\big)\mu\big(\Sigma(\underline{\ell}_2)\big)$$

for all $\underline{\ell}_1, \underline{\ell}_2 \in \Omega$.

From now on, our base dynamics will be $(\hat{f}, \hat{\mu})$ where $\hat{\mu}$ is a $\hat{f}$-invariant probability measure such that $\mu := p^+(\hat{\mu})$ has bounded distortion.

Remark 1.1. It is not hard to see that if $p^+(\hat{\mu})$ has bounded distortion, then $\hat{\mu}$ is $\hat{f}$-ergodic.

1.2. Locally constant cocycles over full shifts. Let G be one of the following groups:

- symplectic group $\mathrm{Sp}(d, \mathbb{R})$, d even;
- orthogonal group $\mathrm{SO}_{\mathbb{R}}(p,q) = U_{\mathbb{R}}(p,q)$, $p+q=d$, $p \geq q$;
- complex unitary group $U_{\mathbb{C}}(p,q)$, $p+q=d$, $p \geq q$.

Given a map $A : \hat{\Sigma} \to G$, we have an associated linear cocycle $(\hat{f}, A) : \hat{\Sigma} \times \mathbb{K}^d \to \hat{\Sigma} \times \mathbb{K}^d$ defined as

$$(\hat{f}, A)(\hat{x}, v) = \big(\hat{f}(\hat{x}), A(\hat{x})v\big)$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

In this setting, a *locally constant* cocycle is a cocycle associated to a map $A : \hat{\Sigma} \to G$ such that, for each $\ell \in \Lambda$, $A(\hat{x}) = A(\hat{y})$ for any $\hat{x}, \hat{y} \in \hat{\Sigma}$ with $p^+(\hat{x})$, $p^+(\hat{y}) \in \Sigma(\ell)$.

By definition, a locally constant cocycle is determined by a map $A : \Lambda \to G$, that is, a collection of matrices $\{A_\ell\}_{\ell \in \Lambda} \in G^\Lambda$.

Given a locally constant cocycle $A : \Lambda \to G$ and a word $\underline{\ell} = (\ell_0, \ldots, \ell_{n-1}) \in \Omega$, we denote by

$$A^{\underline{\ell}} := A_{\ell_{n-1}} \ldots A_{\ell_0},$$

so that $(\hat{f}, A)^n(\hat{x}, v) = \big(\hat{f}^n(\hat{x}), A^{\underline{\ell}}v\big)$ whenever $p^+(\hat{x}) \in \Sigma(\underline{\ell})$.

We say that a locally constant cocycle $A : \Lambda \to G$ over $(\hat{f}, \hat{\mu})$ is *integrable* if

$$\sum_{\ell \in \Lambda} \mu\big(\Sigma(\ell)\big) \log\|A_\ell^{\pm 1}\| < \infty$$

Note that the space of locally constant integrable cocycles has a natural (product) topology and a natural (product left Haar) measure coming from the product space G^{Λ}. In what follows, the space of locally constant integrable cocycles will always be equipped with this natural topology and natural measure.

Since $\hat{\mu}$ is $\hat{f}$-ergodic (cf. Remark 1.1), it follows from Oseledets theorem that any locally constant integrable cocycle $A : \Lambda \to G$ over $(\hat{f}, \hat{\mu})$ has a well-defined Lyapunov spectrum

$$\lambda_1(A) \geq \cdots \geq \lambda_d(A)$$

describing the fiber dynamics of the iterates of $(\hat{x}, v) \in \hat{\Sigma} \times \mathbb{K}^d$ under $(\hat{f}, A)$ for $\hat{\mu}$-almost every $\hat{x} \in \hat{\Sigma}$ and for all $v \in \mathbb{K}^d$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

The Lyapunov spectrum of $A : \Lambda \to G$ is affected by the structure of G:

- $\lambda_{d-i+1}(A) = -\lambda_i(A)$ for all $1 \leq i \leq d$ when $G = \mathrm{Sp}(d, \mathbb{R})$;
- $\lambda_{d-i+1}(A) = -\lambda_i(A)$ for all $1 \leq i \leq q$ and $\lambda_j(A) = 0$ for all $q < j \leq p$ when $G = U_{\mathbb{K}}(p, q)$, $q \leq p$, $p + q = d$.

Also, the unstable and stable Oseledets subspaces associated to A are always *isotropic*.

We say that A has *simple* Lyapunov spectrum whenever the multiplicities of its Lyapunov exponents are the lowest possible given the constraints imposed by G, i.e.,

- $\lambda_i(A) > \lambda_{i+1}(A)$ for all $1 \leq i \leq d - 1$ when $G = \mathrm{Sp}(d, \mathbb{R})$;
- $\lambda_i(A) > \lambda_{i+1}(A)$ for all $1 \leq i \leq q$ when $G = U_{\mathbb{K}}(p, q)$, $q \leq p$, $p + q = d$.

For later use, we say that an integer k is *admissible* (for G) if

- $1 \leq k < d$ when $G = \mathrm{Sp}(d, \mathbb{R})$;
- $1 \leq k \leq q$ or $p \leq k < d$ when $G = U_{\mathbb{K}}(p, q)$, $q \leq p$, $d = p + q$.

In this setup, given an admissible integer k (for $G = \mathrm{Sp}(d, \mathbb{R})$ or $U_{\mathbb{K}}(p, q)$), we denote by $\mathrm{Gr}(k)$ the Grassmanian of k-planes (over $\mathbb{K}$) which are

- isotropic (if $1 \leq k \leq d/2$) or coisotropic (if $d/2 \leq k < d$) when $G = \mathrm{Sp}(d, \mathbb{R})$ (and thus $\mathbb{K} = \mathbb{R}$);
- isotropic (if $1 \leq k \leq q$) or coisotropic (if $p \leq k < d$) when $G = U_{\mathbb{K}}(p, q)$, $q \leq p$, $d = p + q$.

1.3. Statement of the main result. The purpose of this note is to prove the following theorem:

Theorem 1.2. *Let* $G = \mathrm{Sp}(d, \mathbb{R})$ *or* $U_{\mathbb{K}}(p, q)$, $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *The Lyapunov spectrum of almost every locally constant integrable cocycle* $A : \Lambda \to G$ *over* $(\hat{f}, \hat{\mu})$ *is simple.*

The (short) proof of this theorem is presented in Section 2 below. For now, let us close this introductory section with the following comment on Theorem 1.2.

Remark 1.3. The argument used in the proof of our main result is specific to the case of locally constant cocycles. In particular, it is not clear to us whether our strategy can be adapted to handle other natural classes of cocycles over countable shifts (such as dominated Hölder cocycles, cf. [8]).

2. Proof of Theorem 1.2

The starting point of the proof of Theorem 1.2 is the following variant in [7] of the simplicity criterion due to Avila and Viana [2], [3]:

Theorem 2.1 ([2], [3], [7]). *Denote by* $G = \mathrm{Sp}(d, \mathbb{R})$ *or* $U_{\mathbb{K}}(p, q)$, $q \leq p$, $p + q = d$, $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *Let* $A : \Lambda \to G$ *be a locally constant integrable cocycle over* $(\hat{f}, \hat{\mu})$. *Assume that:*

- *A is pinching: there exists a word* $\underline{\ell}^* \in \Omega$ *such that* $A^{\underline{\ell}^*}$ *is a pinching matrix, i.e., the logarithms of the moduli of the eigenvalues of the matrix* $A^{\underline{\ell}^*}$ *satisfy the inequalities in the definition of simple Lyapunov spectrum above;*
- *A is twisting: for some pinching matrix* $A^{\underline{\ell}^*}$, $\underline{\ell}^* \in \Omega$, *and for each admissible integer* k *(for* G*), there exists a word* $\underline{\ell}(k) \in \Omega$ *such that* $A^{\underline{\ell}(k)}$ *is* k*-twisting relative to* $A^{\underline{\ell}^*}$, *i.e.,*

$$A^{\underline{\ell}(k)}(F) \cap F' = \{0\}$$

  for any $A^{\underline{\ell}^*}$*-invariants subspaces* $F \in \mathrm{Gr}(k)$, $F' \in \mathrm{Gr}(d - k)$.

Then, the Lyapunov spectrum of A is simple.

Remark 2.2. The condition that a matrix D is k-twisting with respect to a pinching matrix C is equivalent to the fact that all $k \times k$ algebraic minors associated to isotropic C-invariant k-planes do not vanish.

This simplicity result suggests to prove Theorem 1.2 by showing that a typical $A : \Lambda \to G$ is pinching and twisting.

For this sake, we need the following two particular cases of results due to Breuillard (see Lemma 6.8 in [5] and also Lemma 7.7 in [1]) and Benoist [4]:

Theorem 2.3 (Breuillard). *Let* $G = \mathrm{Sp}(d, \mathbb{R})$ *or* $U_{\mathbb{K}}(p, q)$, $q \le p$, $p + q = d$, $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *There exists a proper algebraic subvariety* $\mathscr{W}$ *of* $G \times G$ *such that any pair of elements* $x, y \in G$ *generated a Zariski dense monoid of* G *whenever* $(x, y) \notin \mathscr{W}$.

Theorem 2.4 (Benoist). *Let* $\Gamma \subset G$ *be a Zariski-dense monoid* (*where* $G = \mathrm{Sp}(d, \mathbb{R})$ *or* $U_{\mathbb{K}}(p, q)$, $q \le p$, $p + q = d$, $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$). *Then,* Γ *contains a pinching matrix.*

At this point, we are ready to finish the proof of Theorem 1.2. In fact, let us fix two distinct letters $\ell_0, \ell_1 \in \Lambda$. By Theorem 2.3, for almost every $A : \Lambda \to G$, the monoid Γ generated by A_{ℓ_0} and A_{ℓ_1} is Zariski dense in G.

We claim that any such $A : \Lambda \to G$ is pinching and twisting. Indeed, by Theorem 2.4, there exists a word $\underline{\ell}^* \in \Omega$ (on the letters ℓ_0 and ℓ_1) such that $C = A^{\underline{\ell}^*} \in \Gamma$ is a pinching matrix. Furthermore, for each admissible integer k, we can find a word $\underline{\ell}(k) \in \Omega$ (on ℓ_0 and ℓ_1) such that $D_k := A^{\underline{\ell}(k)} \in \Gamma$ is k-twisting with respect to C: this is a consequence of the Zariski density of Γ and the fact that the condition of k-twisting with respect to C is Zariski open in G (cf. Remark 2.2). This proves the claim.

By the simplicity criterion in Theorem 2.1, the proof of Theorem 1.2 is now complete.

Remark 2.5. This argument provides a full measure set of locally constant integrable cocycles $A : \Lambda \to G$ with simple Lyapunov spectrum which is *independent* of the choice of the probability measure $\hat{\mu}$ (with $p^+(\hat{\mu})$ having bounded distortion).

Remark 2.6. The statement of Theorem 1.2 can be extended to the case of subshifts of countable type. Indeed, the inducing technique explained in Appendix A of [3] relates a subshift of countable type of $\Lambda^{\mathbb{Z}}$ to a full shift by studying the first return map to an appropriate cylinder $\Sigma(\underline{\ell})$ in $\Lambda^{\mathbb{Z}}$. Moreover, the Lyapunov spectrum of a locally constant cocycle A over the subshift coincides with the Lyapunov spectrum of the induced locally constant cocycle A^w over the full shift on $\mathscr{I}^{\mathbb{Z}}$ where $\mathscr{I}$ consists of certain "minimal" words w of Λ with fixed prefix $\underline{\ell}$. On the other hand, given two distinct words $w_1, w_2 \in \mathscr{I}$, the monoid generated by A^{w_1} and A^{w_2} is Zariski-dense in G for almost every choice of A_ℓ, $\ell \in \Lambda$: this is a direct consequence of Theorem 4.1 in [6]. Therefore, the proof of Theorem 1.2 applies to A^w, and, *a fortiori*, we can extend the conclusions of Theorem 1.2 to a locally constant A over subshifts of countable type.

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