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# Internal monoids and groups in the category of commutative cancellative medial magmas

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**Abstract.** This article considers the category of commutative medial magmas with cancellation, a structure that generalizes midpoint algebras and commutative semigroups with cancellation. In this category each object admits at most one internal monoid structure for any given unit. Conditions for the existence of internal monoids and internal groups, as well as conditions under which an internal reflexive relation is a congruence, are studied.

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## 1. Introduction

A midpoint algebra is a pair $(A, \oplus)$ where $A$ is a set and $\oplus$ a binary operation satisfying the following axioms:

$$\begin{aligned} &\text{(idempotency)} && x \oplus x = x,\\ &\text{(commutativity)} && x \oplus y = y \oplus x,\\ &\text{(cancellation)} && (\exists a \in A, x \oplus a = y \oplus a) \implies x = y,\\ &\text{(mediality)} && (x \oplus y) \oplus (z \oplus w) = (x \oplus z) \oplus (y \oplus w). \end{aligned}$$

An example of such a structure is the unit interval $A = [0, 1]$ with $x \oplus y = \frac{x+y}{2}$.

A word of motivation before entering into the details of the paper. A generalization of the above structure appeared in the study of algebraic properties of

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arbitrary spaces with a suitable notion of a geodesic path for every two points in it. We observed that, in some cases, any trajectory can be viewed as a set of points expanded from a single fixed origin with an associated monoid operation. Here, we concentrate on the study of necessary and sufficient conditions that guarantee the existence of a monoid structure with a specified identity element in the context of midpoint algebras. These algebraic structures can also be used to solve concrete problems in several fields of pure and applied mathematics, such as the so called motion planing algorithms in robotics (see for example [9]).

The category of midpoint algebras is not a Mal'tsev category. Indeed, the usual order relation on the unit interval, with the arithmetic mean as above, is clearly a subalgebra of the product $[0,1] \times [0,1]$, which is reflexive and transitive but not symmetric, contradicting the well known characterization of Mal'tsev categories [5], [6], [7]. Nevertheless, the category of midpoint algebras does have some interesting properties weaker than those of Mal'tsev categories, namely that each object admits at most one internal monoid structure for each choice of a unit. To illustrate this aspect we use another example. Let $(A, \oplus)$ be the midpoint algebra with $A = ]0,1]$, the set of positive real numbers smaller or equal than one, and the operation

$$a \oplus b = \frac{2ab}{a+b}.$$

It is clear that if $a$ and $b$ are positive real numbers then $a \oplus b$ is also positive; it is also easy to see that if $a$ and $b$ are less or equal than one then $a \oplus b$ is less or equal than one. A simple way to see it is to observe that the condition $a \oplus b \leq 1$ is equivalent to the condition

$$0 \leq a(1-b) + b(1-a).$$

Hence the operation is well defined and it is easily checked that $(A, \oplus)$ is a midpoint algebra. The diagram

$$A \times A \xrightarrow{m} A \leftarrow \{1\},$$

with

$$m(x, y) = \frac{xy}{x + y - xy},$$

is an internal monoid. This internal monoid structure is uniquely determined by the unit element $1 \in A$ and there is no internal monoid structure for any other choice of a unit element. Moreover, it is not an internal group as, for example, $\frac{1}{2}$ has no inverse since the equation

$$x = 1 + x$$

has no solution in the real numbers.

While proving some of the results presented in this paper, we have observed that, in many cases, the idempotency law could be dealt with in a separate way. This suggested to us the study of the more general structure of commutative cancellative medial magmas, which, for simplicity, we will refer to as ccm-magmas. The medial law has been widely studied over the last three quarter of a century. In the context of magmas, i.e. algebras with one binary operation (known also as groupoids), this law was also referred to as bisymmetric, entropic or transposition, among other names. It has been studied either from an algebraic point of view or from a more geometrical perspective, see for example [1], [8], [12], [14], [16], [21].

The open-closed unit interval $]0,1]$ used in the example above has several other important structures of ccm-magma: that of a midpoint algebra (with the arithmetic mean), and that of a commutative monoid with cancellation (with the usual multiplication), resulting in an interplay between different ccm-magmas on the same set. The study of such interactions is certainly worthy. Nevertheless, for the moment, we dedicate our attention to some important aspects of the internal structures in the category of ccm-magmas, such as internal monoid, internal group and internal relation.

This paper is organized as follows: Section 2 recalls some categorical notations and concepts such as the one of an internal monoid; Section 3 introduces the category of ccm-magmas, gives some examples, and some useful lemmas are proven; Section 4 observes that the category of ccm-magmas is weakly Mal'tsev and characterizes the existence of an internal monoid structure on a given object, for every choice of a unit element; Section 5 explores this property further by considering the notions of $e$-expansive, $e$-symmetric and homogeneous ccm-magmas (a short list of simple examples and counter-examples is also presented at the end of the section); and finally, Section 6 studies some properties of internal relations, namely symmetry, transitivity, reflexivity and difunctionality, which are not visible in the case of Mal'tsev categories.

## 2. Preliminaries

The basic notions and notations from Category Theory used in this paper can be found, for instance, in [17]. Let us just recall a few of them. If $\mathbb{C}$ is any category with finite limits, an internal monoid in $\mathbb{C}$ is a diagram of the shape

$$A \times A \xrightarrow{m} A \xleftarrow{e} 1$$

in which $A$ is any object in $\mathbb{C}$, $m$ and $e$ are morphisms in $\mathbb{C}$, 1 denotes the terminal object and the following diagram is commutative (where $!_A$ denotes the unique morphism from $A$ to the terminal object 1).

$$\begin{tikzcd} A^3 \arrow[r, "1_A \times m"] \arrow[d, "m \times 1_A"'] & A^2 \arrow[d, "m"'] & A \arrow[l, "\langle e!_A, 1_A \rangle"'] \arrow[dl, equal] \arrow[d, "\langle 1_A, e!_A \rangle"] \\ A^2 \arrow[r, "m"] & A & A^2 \arrow[l, "m"] \end{tikzcd}$$

An internal monoid $(A, m, e)$ is an internal group (see for example the Appendix in [3]) if and only if the diagram

$$A \overset{m}{\leftarrow} A \times A \overset{\pi_2}{\rightarrow} A$$

is a product diagram (the morphism $\pi_2$ is the canonical second projection), in other words, for every two morphisms $u, v : X \to A$ there exists a unique morphism, represented as $u - v$, from $X$ to $A$ such that

$$m\langle u - v, v\rangle = u.$$

The inverse of a generalized element $x : X \to A$ is clearly $e - x$ with $e : 1 \to A$ the unit element. This is simply another way to say that there is a morphism $t : A \to A$ such that

$$m\langle 1_A, t\rangle = e!_A = m\langle t, 1_A\rangle.$$

This paper restricts itself to quasi-varieties of universal algebra [4], that is, categories in which the objects are sets equipped with an arbitrary family of finite-arity operations, satisfying a collection of axioms which may be either expressed as identities or as implications. All the results to be proven about conditions of uniqueness are easily generalized to a category with a faithful functor into the category of sets, preserving finite limits. However, the results involving existence conditions depend on the context and do not necessarily hold in general.

The notion of internal ccm-magma is also a natural one to be considered. It would be interesting, for example, to study topological ccm-magmas, i.e. ccm-magmas internal to the category of topological spaces [8], [15].

A congruence on an object $A$ is a subalgebra of $A \times A$, which, if considered as a relation, is reflexive, transitive and symmetric (see e.g. [3]). We will also consider difunctional relations: a relation $R \subseteq X \times Y$ is difunctional if the implication

$$xRy, zRy, zRw \implies xRw$$

holds for all $x, z \in X$ and $y, w \in Y$.

A Mal'tsev category is characterized by the property that every reflexive internal relation is a congruence relation, or equivalently, by the property that every internal relation is difunctional ([5], [6], [7] see also [3]). As we will see, this property does not hold in the categories we are considering. However, these have a weaker property: some types of reflexive internal relations are still congruences, and some types of internal relations are automatically difunctional.

## 3. Ccm-magmas

A ccm-magma (commutative cancellative medial magma) may be obtained from a midpoint algebra simply by not requiring the idempotency axiom.

**Definition 3.1.** A *ccm-magma* is an algebraic structure $(A, \oplus)$ with a binary operation $\oplus$ satisfying the following axioms:

M1 $a \oplus b = b \oplus a$

M2 $a \oplus c = b \oplus c \Rightarrow a = b$

M3 $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$

A morphism of ccm-magmas is simply a homomorphism, that is, a map preserving the binary operation.

It is known that the category of commutative cancellative magmas is weakly Mal'tsev ([19]). The third axiom has an important consequence illustrated in the following proposition.

**Proposition 3.2.** *If $f : X \to A$ and $g : Y \to A$ are homomorphisms of magmas and $(A, \oplus)$ satisfies* (M3)*, then the map $h : X \times Y \to A$ defined by*

$$h(x, y) = f(x) \oplus g(y)$$

*is a homomorphism.*

*Proof.* The result is an immediate consequence of the axiom (M3). □

Next we present a short list of examples built up from well-known algebraic structures.

**Example.** A short list of simple examples:

(1) The open unit interval $]0, 1[$ with the operation

$$x \oplus y = \frac{x + y}{2} \qquad \text{or} \qquad x \oplus y = \sqrt{xy}$$

is a ccm-magma.

(2) Every midpoint algebra is a ccm-magma.

(3) The set of natural numbers with the usual addition is a ccm-magma.

(4) Every commutative semigroup with cancellation is a ccm-magma.

(5) If $A$ is any ring, then the formula

$$x \oplus y = a(x + y) + b$$

gives a ccm-magma for every choice of $a, b \in A$, with $a$ an invertible element.

(6) If $R$ is a ring and $A$ an $R$-module, then the formula

$$x \oplus y = \alpha(x + y) + b$$

gives a ccm-magma on $A$ for every choice of an invertible scalar $\alpha \in R$ and any $b \in A$.

(7) If $A$ is the set of positive real numbers, then the formula

$$x \oplus y = \frac{axy}{x + y}$$

gives a ccm-magma for every choice of $a \in A$.

(8) If $(A, +, \times, \cdot, 0, 1)$ is a commutative and associative classical algebra over a commutative ring $R$, then the formula

$$x \oplus y = \alpha \cdot (x + y) + \beta \cdot (x \times y)$$

gives a ccm-magma on the set

$$\{x \in A \mid \alpha \cdot 1 + \beta \cdot x \text{ is invertible}\}$$

for every choice of scalars $\alpha, \beta \in R$ with $\alpha = \alpha^2$.

(9) Every loop with $a(bc) = c(ba)$ is a ccm-magma, see for instance [20].

(10) If $(A, \oplus)$ is a ccm-magma and $g : A \to A$ is a monomorphism, then the formula

$$(x, y) \mapsto g(x \oplus y) \oplus a$$

gives a ccm-magma on $A$, for every choice of $a \in A$.

The last example is obtained directly from the proposition below by letting $f = k = 1_A$ and $h = g$.

**Proposition 3.3.** *Let $(A, \oplus)$ be a ccm-magma and $f, g : A \to A$ any two maps. If $f$ and $g$ are injective and there are maps $h, k : A \to A$ such that*

$$f\big(g(x \oplus y) \oplus z\big) = \big(h(x) \oplus h(y)\big) \oplus k(z), \qquad x, y, z \in A,$$

*then the formula*

$$(x, y) \mapsto g\big(f(x) \oplus f(y)\big) \oplus a$$

*gives a ccm-magma for every choice of $a \in A$.*

*Proof.* The result is an immediate consequence of the axioms, combined with the hypotheses on the given maps. Commutativity is immediate. Cancellation follows from $f$ and $g$ being injective. To prove the medial law, on the one hand we have

$$f\big(g(fx \oplus fy) \oplus a\big) \oplus f\big(g(fz \oplus fw) \oplus a\big)$$

which by our assumptions on $f$ and $g$ simplifies to

$$\big(\big(h(fx) \oplus h(fy)\big) \oplus k(a)\big) \oplus \big(\big(h(fz) \oplus h(fw)\big) \oplus k(a)\big); \tag{3.1}$$

while on the other hand we have

$$f\big(g(fx \oplus fz) \oplus a\big) \oplus f\big(g(fy \oplus fw) \oplus a\big),$$

which by our assumptions on $f$ and $g$ simplifies to

$$\big(\big(h(fx) \oplus h(fz)\big) \oplus k(a)\big) \oplus \big(\big(h(fy) \oplus h(fw)\big) \oplus k(a)\big). \tag{3.2}$$

Finally we observe that expressions (3.1) and (3.2) are equal by a simple manipulation of axiom (M3) on the original $\oplus$, which completes the proof. $\square$

We end this section with two lemmas which will be used later on while proving that some internal relations, in the category of ccm-magmas, are automatically difunctional.

**Lemma 3.4.** *Let $f, g : A \times A \to B$ be two morphisms in the category of ccm-magmas such that $f(a,a) = g(a,a)$ for every $a \in A$. Then:*

(i) $f(a,b) = g(a,b),\ f(b,c) = g(b,c) \Rightarrow f(a,c) = g(a,c)$;

(ii) $f(a,b) = g(a,b) \Rightarrow f(b,a) = g(b,a)$.

*Proof.* (i) We will show that if $f(a,b) = g(a,b)$ and $f(b,c) = g(b,c)$ then we always have

$$f(a,c) \oplus f(b,b) = g(a,c) \oplus g(b,b) \tag{3.3}$$

and since $f(b,b) = g(b,b)$ we use (M2) and get $f(a,c) = g(a,c)$. To show (3.3) we observe:

$$\begin{aligned}
f(a,c) \oplus f(b,b) &= f(a \oplus b, c \oplus b) \\
&= f(a \oplus b, b \oplus c) \\
&= f(a,b) \oplus f(b,c) \\
&= g(a,b) \oplus g(b,c) \\
&= g(a \oplus b, b \oplus c) \\
&= g(a \oplus b, c \oplus b) \\
&= g(a,c) \oplus g(b,b).
\end{aligned}$$

(ii) Since $f(a,a) = g(a,a)$ for every $a \in A$, we always have:

$$\begin{aligned}
f(b,a) \oplus f(a,b) &= f(b \oplus a, a \oplus b) \\
&= f(a \oplus b, a \oplus b) \\
&= g(a \oplus b, a \oplus b) \\
&= g(b \oplus a, a \oplus b) \\
&= g(b,a) \oplus g(a,b)
\end{aligned}$$

now, if $f(a,b) = g(a,b)$ then, using (M2) we obtain $f(b,a) = g(b,a)$ as desired. □

**Lemma 3.5.** *Let $f, g : X \times Y \to B$ be any two morphisms in the category of ccm-magmas. Then:*

$$\begin{pmatrix} f(x,y) = g(x,y) \\ f(z,y) = g(z,y) \\ f(z,w) = g(z,w) \end{pmatrix} \implies f(x,w) = g(x,w).$$

*Proof.* Assuming $f(x, y) = g(x, y)$ and $f(z, w) = g(z, w)$ we have:

$$\begin{aligned} f(x, w) \oplus f(z, y) &= f(x \oplus z, w \oplus y) \\ &= f(x \oplus z, y \oplus w) \\ &= f(x, y) \oplus f(z, w) \\ &= g(x, y) \oplus g(z, w) \\ &= g(x \oplus z, y \oplus w) \\ &= g(x \oplus z, w \oplus y) \\ &= g(x, w) \oplus g(z, y). \end{aligned}$$

Now, if $f(z, y) = g(z, y)$ then we conclude that $f(x, w) = g(x, w)$ as desired, using axiom (M2). □

## 4. Mal'tsev and weakly Mal'tsev properties

We have already observed that the category of midpoint algebras, and hence the one of ccm-magmas, is not a Mal'tsev category. It is a weakly Mal'tsev category as a result of ccm-magmas being a subcategory of commutative cancellative magmas, known to be weakly Mal'tsev. Nonetheless, we present an alternative proof using the medial law instead of commutativity (thus the following result holds also for the category of cancellative medial magmas).

**Proposition 4.1.** *The category of ccm-magmas is a weakly Mal'tsev category.*

*Proof.* From [19] we know that if there exists a ternary term $p(x, y, z)$ such that

$$p(x, y, y) = p(y, y, x)$$

and

$$p(x, a, a) = p(y, a, a) \implies x = y$$

then the category is weakly Mal'tsev. Indeed this is the case for every ccm-magma $(A, \oplus)$ if we define

$$p(x, y, z) = (y \oplus x) \oplus (z \oplus y). \qquad \square$$

In a weakly Mal'tsev category any diagram of the shape

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[dr, "u"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "v"] & C \arrow[l, shift right, "g"'] \arrow[dl, "w"] \\ & D & \end{tikzcd} \tag{4.1}$$

with $fr = 1_B = gs$ and $ur = v = ws$, induces a bigger diagram

$$\begin{tikzcd}
 & C \arrow[dl, shift left, "\pi_2"'] \arrow[dr, shift left, "g"] \arrow[drr, bend left, "w"] & & \\
A \times_B C \arrow[ur, shift left, "e_2"] \arrow[dr, shift left, "\pi_1"] & & B \arrow[ul, shift left, "s"] \arrow[dl, shift left, "r"] \arrow[r, "v"] & D \\
 & A \arrow[ul, shift left, "e_1"] \arrow[ur, shift left, "f"] \arrow[urr, bend right, "u"'] & &
\end{tikzcd} \tag{4.2}$$

in which there is at most one morphism, say $\theta : A \times_B C \to D$, from the pullback $(A \times_B C, \pi_1, \pi_2)$ of the split epimorphism $f$ along the split epimorphism $g$ to the object $D$ such that $\theta e_1 = u$ and $\theta e_2 = w$, where $e_1$ and $e_2$ are the induced morphisms of the form

$$e_1(a) = \big(a, sf(a)\big), \qquad e_2(c) = \big(rg(c), c\big).$$

The following result states under which conditions a diagram such as (4.1) induces a morphism such as $\theta$.

**Proposition 4.2.** *Given a diagram such as* (4.2) *in the category of ccm-magmas, there is a* (*unique*) *morphism $\theta : A \times_B C \to D$, such that $\theta e_1 = u$ and $\theta e_2 = w$, if and only if, the equation*

$$x \oplus v(b) = u(a) \oplus w(c) \tag{4.3}$$

*has a solution $x \in D$ for every $a \in A$, $b \in B$ and $c \in C$ with $f(a) = b = g(c)$. When that is the case, $\theta(a, c) = x$.*

*Proof.* Suppose there is $\theta : A \times_B C \to D$ such that $\theta\big(a, s(b)\big) = u(a)$ and $\theta\big(r(b), c\big) = w(c)$, where $b = f(a) = g(c)$, then $x = \theta(a, c)$ is always a solution to the equation (4.3). Indeed, using axiom (M1), we observe that

$$\begin{aligned}
\theta(a, c) \oplus v(b) &= \theta(a, c) \oplus \theta\big(r(b), s(b)\big) \\
&= \theta\big((a, c) \oplus \big(r(b), s(b)\big)\big) \\
&= \theta\big(a \oplus r(b), c \oplus s(b)\big) \\
&= \theta\big(a \oplus r(b), s(b) \oplus c\big) \\
&= \theta\big(a, s(b)\big) \oplus \theta\big(r(b), c\big) \\
&= u(a) \oplus w(c).
\end{aligned}$$

Conversely, if the equation (4.3) has a solution (which is unique by axiom (M2)) for every $(a, c) \in A \times_B C$, then we may define a map $\theta : A \times_B C \to D$ which assigns the unique solution of (4.3) to every pair $(a, c) \in A \times_B C$. It remains to show that this map is a homomorphism. By axiom (M2), it suffices to prove that

$$\big(\theta(a, c) \oplus \theta(a', c')\big) \oplus v(b \oplus b') = \theta(a \oplus a', c \oplus c') \oplus v(b \oplus b')$$

for every $a, a' \in A$ and $c, c' \in C$ where

$$f(a) = g(c) = b \in B \quad \text{and} \quad f(a') = g(c') = b \in B.$$

Now, because $u$, $v$ and $w$ are homomorphisms and, using axiom (M3), we have

$$\begin{aligned}
\big(\theta(a, c) \oplus \theta(a', c')\big) \oplus v(b \oplus b') &= \big(\theta(a, c) \oplus \theta(a', c')\big) \oplus \big(v(b) \oplus v(b')\big) \\
&= \big(\theta(a, c) \oplus v(b)\big) \oplus \big(\theta(a', c') \oplus v(b')\big) \\
&= \big(u(a) \oplus w(c)\big) \oplus \big(u(a') \oplus w(c')\big) \\
&= \big(u(a) \oplus u(a')\big) \oplus \big(w(c) \oplus w(c')\big) \\
&= u(a \oplus a') \oplus w(c \oplus c') \\
&= \theta(a \oplus a', c \oplus c') \oplus v(b \oplus b')
\end{aligned}$$

as desired, which completes the proof. □

In particular, any internal reflexive graph admits, at most, one structure of internal category. This is easily seen from the above result by choosing $D = A = C$, $r = s = v$, and $u$ and $w$ to be the identity morphisms. Even more particularly, by choosing $B$ to be a singleton, and if $(A, \oplus)$ is a ccm-magma then, for every idempotent $e \in A$, there is, at most, one internal monoid structure on $A$ which is compatible with the binary operation, that is, there exists at most one monoid $(A, *_e, e)$ such that

$$(x *_e y) \oplus (z *_e w) = (x \oplus z) *_e (y \oplus w). \tag{4.4}$$

Note that $e \in A$ must be an idempotent, so that the inclusion $\{e\} \to A$ may be a homomorphism.

**Corollary 4.3.** *Let $(A, \oplus)$ be a ccm-magma and $e \in A$ an idempotent element in $A$. There is a (unique) internal monoid structure $(A, *_e, e)$ in $A$, if and only if the equation*

$$\theta \oplus e = x \oplus y$$

*has a solution $\theta = \theta(x, y) \in A$ for every $x, y \in A$. In that case $x *_e y$ is given by $\theta(x, y)$.*

*Proof.* It follows from the previous proposition with $A = D = C$, $B = 1$ the terminal algebra, $f$ and $g$ uniquely determined while $r = s = v$ send the unique element in 1 to the chosen element $e$ in $A$ (which is a homomorphism as soon as $e \oplus e = e$), and $u = w$ is the identity morphism. The fact that the operation $*_e$ is associative and has a unit $e$ follows from general arguments used in [18] but may also be demonstrated directly:

(i) As $\big(x *_e (y *_e z)\big) \oplus e = x \oplus (y *_e z)$ and $\big((x *_e y) *_e z\big) \oplus e = (x *_e y) \oplus z$, we have, by (M1) and (M3),
$\big(\big(x *_e (y *_e z)\big) \oplus e\big) \oplus (e \oplus e) = (x \oplus e) \oplus (y \oplus z)$ and
$\big(\big((x *_e y) *_e z\big) \oplus e\big) \oplus (e \oplus e) = (x \oplus y) \oplus (z \oplus e) = (x \oplus e) \oplus (y \oplus z)$ which, by (M2), imply that
$x *_e (y *_e z) = (x *_e y) *_e z$.

(ii) $(x *_e e) \oplus e = x \oplus e$ which, by (M2), implies that $x *_e e = x$. □

## 5. Internal monoids and internal groups

It is well known [5], [6], [7] that, in Mal'tsev categories, every internal monoid is necessarily an internal group. This property does not apply to ccm-magmas. In this section, we will study some sufficient conditions for the existence of an internal monoid or group structure within a ccm-magma with a chosen unit element, which is necessarily an idempotent. For that we introduce the following notions, which have already been considered in the literature for different purposes, see for example [2], [12]:

**Definition 5.1.** Let $(A, \oplus)$ be a ccm-magma and consider any element $e \in A$. We will say that:

(i) $A$ is $e$-expansive if for every $a \in A$ there exists $2_e(a) \in A$ such that $2_e(a) \oplus e = a$;

(ii) $A$ is $e$-symmetric if for every $a \in A$ there exists $-_e(a) \in A$ such that $-_e(a) \oplus a = e$;

(iii) $A$ is homogeneous if it is $e$-expansive (or $e$-symmetric) for every $e \in A$.

In fact, a homogeneous ccm-magma is the same as a commutative medial quasigroup.

A sufficient condition for a ccm-magma to admit an internal monoid structure with an idempotent element $e$ as its unit is to be $e$-expansive. When that is the case, then the internal monoid is a group if and only if the algebra is $e$-symmetric. Moreover, if every element is an idempotent, that is, if we have a midpoint algebra, then it is $e$-expansive if and only if there exists a monoid structure over $e$. Also every internal monoid (or group) is commutative and admits cancellation.

**Proposition 5.2.** *Let $(A, \oplus)$ be a ccm-magma and consider any idempotent element $e \in A$. If $A$ is e-expansive then $(A, *_e, e)$ is a monoid with*

$$x *_e y = 2_e(x \oplus y).$$

*Moreover, it is a group if and only if $A$ is e-symmetric.*

*Proof.* If $A$ is $e$-expansive then in particular $2_e(x \oplus y)$ is a solution to the equation

$$\theta \oplus e = x \oplus y,$$

for every $x, y \in A$. From Corollary 4.3 we may conclude that $(A, *_e, e)$ is an internal monoid. Now, if moreover $A$ is $e$-symmetric then $-_e(a)$ is the inverse of $a$, for every $a \in A$. Indeed we have that

$$-_e(a) *_e a = e$$

is equivalent (by (M2)) to

$$\big(-_e(a) *_e a\big) \oplus e = e \oplus e$$

which holds because

$$\big(-_e(a) *_e a\big) \oplus e = 2_e\big(-_e(a) \oplus a\big) \oplus e = -_e(a) \oplus a = e = e \oplus e.$$

Conversely, if $(A, *_e, e)$ is a group, then, for every $a \in A$, its inverse element, say $a' \in A$, is a solution to the equation $x \oplus a = e$. Indeed, since $a'$ is such that $a' *_e a = e$, or equivalently $2_e(a' \oplus a) = e$, and $e$ is an idempotent, we conclude that

$$e = e \oplus e = 2_e(a' \oplus a) \oplus e = a' \oplus a. \qquad \square$$

In the case when the operation $\oplus$ has the geometrical meaning of midpoint, the formula $a *_e b = 2_e(a \oplus b)$ is intuitively illustrated via the following diagram.

$$\begin{tikzcd}
a \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & a *_e b \arrow[dd, dash] \arrow[dl, dash] \\
& a \oplus b & \\
e \arrow[rr, dash] \arrow[ur, dash] & & b \arrow[ul, dash]
\end{tikzcd}$$

As a more concrete example, let $A$ be any real vector space. Then, by defining

$$a \oplus b = \frac{1}{2}(a+b)$$

we obtain a ccm-magma which is 0-symmetric and 0-expansive with the usual interpretation of $-a$ and $2a$, as illustrated for the particular case of the real line.

$$\begin{tikzcd}
{} \arrow[r, dash] & (-a) \arrow[r, dash] \arrow[d, equal] & (-a)\oplus a \arrow[r, dash] \arrow[d, equal] & a \arrow[r, dash] \arrow[d, equal] & 2a \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r, dash] & (-a) \arrow[r, dash] & 0 \arrow[r, dash] & 0\oplus(2a) \arrow[r, dash] & 2a \arrow[r] & {}
\end{tikzcd}$$

More generally, for every $e \in A$, this structure of ccm-magma is $e$-symmetric and $e$-expansive, with $2_e(a) = 2a - e$ and $-_e(a) = 2e - a$. This fact is related to the affine transformation $x \mapsto x + e$.

We also notice that if a ccm-magma is $e$-expansive and $e$-symmetric, for some element $e$, then it is so for all elements, in other words it is homogeneous. This result will be used in the proof of Corollary 6.6.

**Proposition 5.3.** *Let $(A, \oplus)$ be a ccm-magma and $e \in A$ an element in it. If it is $e$-expansive and $e$-symmetric then it is homogeneous.*

*Proof.* We have to prove that for every $u, v \in A$, there exists a solution $x$ to the equation $x \oplus u = v$. Indeed,

$$\begin{aligned}
x \oplus u = v &\iff (x \oplus u) \oplus \big(e \oplus -_e(u)\big) = v \oplus \big(e \oplus -_e(u)\big) \\
&\iff (x \oplus e) \oplus \big(u \oplus -_e(u)\big) = v \oplus \big(e \oplus -_e(u)\big) \\
&\iff (x \oplus e) \oplus e = v \oplus \big(e \oplus -_e(u)\big) \\
&\iff x \oplus e = 2_e\big(v \oplus \big(e \oplus -_e(u)\big)\big) \\
&\iff x = 2_e\big(2_e\big(v \oplus \big(e \oplus -_e(u)\big)\big)\big),
\end{aligned}$$

which gives the desired solution to the equation. □

As already referred, every homogeneous ccm-magma is a commutative medial quasigroup. This means that for homogeneous ccm-magmas, Proposition 5.2 is a special case (commutative) of the well known Toyoda Theorem [21]. This theorem has been generalized for medial magmas with cancellation (see for instance [12]), but the result is no longer comparable with the one of an internal monoid structure.

Restricting the study to homogeneous ccm-magmas has some advantages but it forces the category to be Mal'tsev (and hence there is no longer the distinction between internal monoid and internal group). Indeed, adapting the well-known formulas describing the category of quasigroups as a Mal'tsev category, say from [10], we can conclude that if a ccm-magma is expansive for every element then

$$p(x, y, z) = 2_{2_y(y)}(x) \oplus 2_y(z)$$

is a Mal'tsev term.

We continue with another aspect of ccm-magmas which will be used in Proposition 5.5. If a given ccm-magma is expansive with respect to some idempotent element $e$ then the respective map $2_e$ is a homomorphism. In general, we have:

**Proposition 5.4.** *Let $(A, \oplus)$ be a ccm-magma. If it is u-expansive and v-expansive for some $u, v \in A$, then it is also $(u \oplus v)$-expansive, and moreover,*

$$2_u(a) \oplus 2_v(b) = 2_{u\oplus v}(a \oplus b).$$

*Proof.* Suppose there exists $2_{u\oplus v}(a \oplus b)$, then we necessarily have

$$2_u(a) \oplus 2_v(b) = 2_{u\oplus v}(a \oplus b)$$

because, by *adding* $u \oplus v$ in each term, we obtain $a \oplus b$. It remains to prove that $2_{u\oplus v}$ exists. In other words we have to prove that for every $a \in A$, there is $x \in A$ such that $x \oplus (u \oplus v) = a$. We claim that

$$x = 2_u\big(2_u(a)\big) \oplus 2_v(u)$$

is the needed solution. Indeed,

$$\begin{aligned}\big(2_u\big(2_u(a)\big) \oplus 2_v(u)\big) \oplus (u \oplus v) &= \big(2_u\big(2_u(a)\big) \oplus u\big) \oplus \big(2_v(u) \oplus v\big)\\ &= 2_u(a) \oplus u\\ &= a.\end{aligned}$$

So, we get $2_{u\oplus v}(a) = 2_u\big(2_u(a)\big) \oplus 2_v(u) = 2_v\big(2_v(a)\big) \oplus 2_u(v)$. □

The next result explains the connection between two induced monoid structures for two different idempotents $u$ and $v$.

**Proposition 5.5.** *Let $(A, \oplus)$ be a ccm-magma which is u-expansive and v-expansive for some idempotent elements $u, v \in A$. The two monoid structures on $A$, induced by u and v, are isomorphic. Moreover the two structures are related as follows:*

$$a *_u b = (a *_v b) *_u v.$$

*Proof.* Let $a *_u b = 2_u(a \oplus b)$ and $a *_v b = 2_v(a \oplus b)$ be the two monoid operations induced, respectively by $u$ and $v$, assuming $u$ and $v$ to be idempotent and $A$ to be $u$-expansive and $v$-expansive. The map $f : (A, *_u, u) \to (A, *_v, v)$, such that $f(a) = 2_u(a \oplus v)$, $a \in A$, is a monoid homomorphism. Clearly, the units are preserved, since

$$f(u) = 2_u(u \oplus v) = 2_u(v \oplus u) = 2_u(v) \oplus 2_u(u) = 2_u(v) \oplus u = v.$$

From

$$\begin{aligned} f(a *_u b) \oplus v &= f\big(2_u(a \oplus b)\big) \oplus v \\ &= 2_u\big(2_u(a \oplus b) \oplus v\big) \oplus v \\ &= 2_u\big(2_u(a \oplus b) \oplus v\big) \oplus \big(u \oplus 2_u(v)\big) \\ &= \big(2_u\big(2_u(a \oplus b)\big) \oplus 2_u(v)\big) \oplus \big(u \oplus 2_u(v)\big) \\ &= \big(2_u\big(2_u(a \oplus b)\big) \oplus u\big) \oplus \big(2_u(v) \oplus 2_u(v)\big) \\ &= 2_u(a \oplus b) \oplus \big(2_u(v \oplus v)\big) \\ &= 2_u(a \oplus b) \oplus \big(2_u(v)\big) \\ &= 2_u\big((a \oplus b) \oplus v\big) \end{aligned}$$

and

$$\begin{aligned} \big(f(a) *_v f(b)\big) \oplus v &= 2_v\big(f(a) \oplus f(b)\big) \oplus v \\ &= f(a) \oplus f(b) \\ &= 2_u(a \oplus v) \oplus 2_u(b \oplus v) \\ &= 2_u\big((a \oplus v) \oplus (b \oplus v)\big) \\ &= 2_u\big((a \oplus b) \oplus (v \oplus v)\big) \\ &= 2_u\big((a \oplus b) \oplus v\big) \end{aligned}$$

we may conclude that $f(a *_u b) = f(a) *_v f(b)$. The inverse homomorphism of $f$ is $g : (A, *_v, v) \to (A, *_u, u)$ with $g(a) = 2_v(a \oplus u)$. Indeed,

$$gf(a) = 2_v\big(2_u(a \oplus v) \oplus u\big) = 2_v(a \oplus v) = 2_v(a) \oplus 2_v(v) = 2_v(a) \oplus v = a$$

and similarly we prove $fg(a) = a$. Finally, we prove $a *_u b = (a *_v b) *_u v$ by observing that

$$(a *_u b) \oplus u = a \oplus b = \big((a *_v b) *_u v\big) \oplus u. \qquad \square$$

In some cases, a ccm-magma $(A, \oplus)$ does not only admit an internal monoid structure over some idempotent element $e$, but also the structure itself is a monoid with unit element $e$, that is $a \oplus e = a$. This property is summarized in the following proposition.

**Proposition 5.6.** *Let $(A, \oplus)$ be a ccm-magma and $e \in A$ an idempotent. The following conditions are equivalent:*

(i) *the operation $\oplus$ is associative;*

(ii) *the element $e \in A$ is a unit element for $\oplus$;*

(iii) *the ccm-magma is $e$-expansive with $2_e(a) = a$;*

(iv) *the structure $(A, \oplus, e)$ is an internal monoid.*

*Proof.* If the operation $\oplus$ is associative and $e \in A$ is idempotent then

$$(a \oplus e) \oplus e = a \oplus (e \oplus e) = a \oplus e$$

and hence $a \oplus e = a$. If $e \in A$, an idempotent, is also a unit element for $\oplus$ then $2_e(a) = a$ by definition of $2_e$. We already know that if the ccm-magma is $e$-expansive then $(A, *_e, e)$ is an internal monoid structure with $x *_e y = 2_e(x \oplus y)$. When $2_e$ is the identity map, the operation $*_e$ is simply the original $\oplus$. This proves (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), whereas (iv) $\Rightarrow$ (i) is obvious. $\square$

Note also that if a ccm-magma $(A, \oplus)$ is associative, then it has, at most, one idempotent. Indeed, if $e_1$ and $e_2$ are two idempotents, and $\oplus$ is associative, then

$$e_1 \oplus (e_1 \oplus e_2) = e_1 \oplus e_2 = e_2 \oplus (e_2 \oplus e_1)$$

from which it follows that $e_1 = e_2$.

In the case of midpoint algebras, that is, when every element is an idempotent, there is no distinction between having a monoid structure for some element $e \in A$ and being $e$-expansive.

**Proposition 5.7.** *Let $(A, \oplus)$ be a midpoint algebra with $e \in A$. The following conditions are equivalent:*

(i) *there is an internal monoid structure over $e$;*

(ii) *for every $a, b \in A$, there is $x \in A$ such that $x \oplus e = a \oplus b$;*

(iii) *the midpoint algebra is $e$-expansive.*

*Proof.* The two conditions (i) and (ii) are already equivalent in the context of ccm-magmas (Corollary 4.3). Also, in the more general case of ccm-magmas,

(iii) implies (i), as it is proven in Proposition 5.2. We are left to prove that (i) implies (iii). With an internal monoid structure $(A, *_e, e)$ we can define $2_e(a) = a *_e a$. Indeed, using the fact that every element $a \in A$ is idempotent ($A$ is a midpoint algebra), we have

$$(a *_e a) \oplus e = a \oplus a = a. \qquad \square$$

If, in a midpoint algebra $(A, \oplus)$, there is an internal monoid structure $(A, *_e, e)$, then $*_e$ is distributive over $\oplus$, as it immediately follows from (4.4). This is not true in general for ccm-magmas.

**Proposition 5.8.** *Let $(A, \oplus)$ be a ccm-magma with an internal monoid structure $(A, *_e, e)$. The following two conditions are equivalent:*

(i) *the ccm-magma is a midpoint algebra (every element $a \in A$ is an idempotent);*

(ii) *for every $x, y, z \in A$,*

$$x *_e (y \oplus z) = (x *_e y) \oplus (x *_e z). \tag{5.1}$$

*Proof.* If $(A, \oplus)$ is a midpoint algebra then (5.1) follows from (4.4). Conversely, if we have an internal monoid structure $(A, *_e, e)$ together with (5.1), then every element $a \in A$ is an idempotent

$$a \oplus a = (a *_e e) \oplus (a *_e e) = a *_e (e \oplus e) = a *_e e = a.$$

Note that $e$ is an idempotent because it is the unit element of an internal monoid. $\square$

**Some examples illustrating the properties discussed in the previous results.** This section concludes with a short list of examples and counter-examples showing several particular cases and properties, which a given ccm-magma may or may not have for some specific choice of an idempotent element $e$ in it. First we observe that if the four basic properties in the left column on the table below are considered, then there are only six possible combinations between them, namely the ones expressed in the other columns and denoted by I to VI:

| | I | II | III | IV | V | VI |
|---|---|---|---|---|---|---|
| $e$-expansive | yes | yes | no | no | no | no |
| $e$-symmetric | yes | no | yes | no | no | yes |
| internal monoid over $e$ | yes | yes | no | no | yes | yes |
| internal group over $e$ | yes | no | no | no | no | yes |

The properties I to VI defined in the table above have an obvious interpretation. For example, a ccm-magma has property III if and only if it is $e$-symmetric but not $e$-expansive for a specific choice of $e$, it does not have an internal group structure over $e$ or even an internal monoid; while the ccm-magmas with property V have an internal monoid structure over some idempotent element $e$, they are not $e$-expansive nor $e$-symmetric, and consequently do not possess an internal group structure. Now, combining the previous properties with the existence of one or more, none, or even all idempotents, and also with associativity, we observe the following list of ccm-magmas $(A, \oplus)$.

(1) Ccm-magmas with no idempotent elements (in this case there is no interaction with properties I to VI from above):
  (a) Non-associative:
    (i) $a \oplus b = \frac{a+b}{2} + 1$, $A = \mathbb{R}$
    (ii) $a \oplus b = \frac{3ab}{a+b}$, $A = \mathbb{R}^+$
    (iii) $a \oplus b = 2(a+b)$, $A = \mathbb{R}^+$
  (b) Associative:
    (i) $a \oplus b = a + b + 1$, $A = [0, +\infty]$
    (ii) $a \oplus b = \frac{ab}{a+b}$, $A = \mathbb{R}^+$ or $A = ]0, 1]$
    (iii) $a \oplus b = a + b + ab$, $A = \mathbb{R}^+$
    (iv) $a \oplus b = a + b$, $A = \mathbb{R}^+$
    (v) $a \oplus b = \frac{a+b}{1+ab}$, $A = ]0, 1[$
    (vi) $a \oplus b = \log(\exp(a) + \exp(b))$, $A = \mathbb{R}$

(2) Ccm-magmas with every element an idempotent, that is, midpoint algebras (due to Proposition 5.7 the properties V and VI do not apply):

| Property | $a \oplus b$ | $A$ | $e \in A$ |
|---|---|---|---|
| I | $\frac{a+b}{2}$ | $\mathbb{R}$ | every |
| I | $\sqrt[3]{\frac{a^3+b^3}{2}}$ | $\mathbb{R}$ | every |
| II | $\frac{a+b}{2}$ | $[0, +\infty[$ | $e = 0$ |
| II | $\frac{2ab}{a+b}$ | $]0, 1]$ | $e = 1$ |
| III | $\frac{a+b}{2}$ | $[0, 1]$ | $e = \frac{1}{2}$ |
| III | $\frac{2ab}{a+b}$ | $]1, +\infty[$ | $e = 2$ |
| IV | $\frac{a+b}{2}$ | $\mathbb{R}^+$ | N/A |
| IV | $\frac{2ab}{a+b}$ | $\mathbb{R}^+$ | N/A |

(3) Ccm-magmas with at least one idempotent element which are not midpoint-algebras (in this case again we distinguish between associative and non-

associative and give examples of each one of the properties I to VI from above):

(a) non-associative:

| Property | $a \oplus b$ | $A$ | $e \in A$ |
|---|---|---|---|
| I | $2(a+b)$ | $\mathbb{R}$ | $e=0$ |
| I | $2\sqrt[3]{a^3+b^3}$ | $\mathbb{R}$ | $e=0$ |
| II | $2(a+b)$ | $[0,+\infty[$ | $e=0$ |
| III | $\frac{a+b}{3}$ | $[-1,1]$ | $e=0$ |
| IV | $\frac{a+b}{3}$ | $[0,1]$ | $e=0$ |
| V | $2(a+b)$ | $\mathbb{N}_0$ | $e=0$ |
| VI | $2(a+b)$ | $\mathbb{Z}$ | $e=0$ |

(b) associative (in this case we only distinguish properties I and II from above, as it follows from Proposition 5.6).

| Property | $a \oplus b$ | $A$ | $e \in A$ |
|---|---|---|---|
| I | $a+b$ | $\mathbb{R}$ | $e=0$ |
| II | $a+b$ | $[0,+\infty[$ | $e=0$ |
| II | $a+b-ab$ | $[0,1[$ | $e=0$ |

(4) Finite ccm-magmas. Every finite ccm-magma is homogeneous. Some examples are as follows. The matrix

$$A_2 \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

shows an example of the *multiplication table* for a non-associative ccm-magma with three idempotents. The matrix

$$A_3 \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

shows an example of a non-associative ccm-magma with no idempotents. However there are no ccm-magmas with a finite and even number of idempotents. This is due to the fact that if $(A, \oplus)$ is a ccm-magma then the subset $\{a \in A \mid a \oplus a = a\}$ is a subalgebra of ccm-magmas and there are no commutative idempotent quasigroups (homogeneous ccm-magmas) of even order [20].

## 6. Internal relations

As we have observed in the introduction, the category of ccm-magmas is not Mal'tsev, and so there is no hope of having every internal reflexive relation automatically as a congruence, nor having any internal relation as a difunctional one. Nevertheless, as it is shown in [11] a category is weakly Mal'tsev if and only if every strong relation is difunctional. In particular, if $f, g : X \times Y \to B$ are any two morphisms of ccm-magmas, then the relation $R \subseteq X \times Y$, defined by

$$xRy \iff f(x, y) = g(x, y) \tag{6.1}$$

is a strong relation.

Hence the following results:

**Proposition 6.1.** *Let $f, g : A \times A \to B$ be two morphisms in the category of ccm-magmas such that $f(a, a) = g(a, a)$ for every $a \in A$. The internal relation defined by*

$$xRy \iff f(x, y) = g(x, y)$$

*is a congruence.*

*Proof.* It is an immediate consequence of Lemma 3.4. □

**Proposition 6.2.** *Let $f, g : X \times Y \to B$ be two morphisms in the category of ccm-magmas. The internal relation defined by*

$$xRy \iff f(x, y) = g(x, y)$$

*is difunctional.*

*Proof.* It is an immediate consequence of Lemma 3.5. □

We now present some results involving another special type of internal relations, namely the ones constructed from a subalgebra of a given ccm-magma.

**Proposition 6.3.** *Let $(A, \oplus)$ be a ccm-magma with a subalgebra $X \subseteq A$ and an idempotent element $e \in X$. The relation*

$$aRb \iff \exists x \in X, \qquad a \oplus e = x \oplus b \tag{6.2}$$

(i) *is an internal relation;*

(ii) *is reflexive;*

(iii) *is transitive whenever $X$ has a (unique) internal monoid structure with $e \in X$ as its unit;*

(iv) *is symmetric if and only if $X$ is e-symmetric.*

*Proof.* (i) The relation is an internal relation if and only if $R \subseteq A \times A$ is a subalgebra of the product, or equivalently, if and only if

$$aRb, a'Rb' \implies (a \oplus a')R(b \oplus b').$$

It is the case because if there exist $x, x' \in X$ such that $a \oplus e = x \oplus b$ and $a' \oplus e = x' \oplus b'$ then we have

$$\begin{aligned}(a \oplus a') \oplus e &= (a \oplus a') \oplus (e \oplus e)\\ &= (a \oplus e) \oplus (a' \oplus e)\\ &= (x \oplus b) \oplus (x' \oplus b')\\ &= (x \oplus x') \oplus (b \oplus b')\end{aligned}$$

showing that there is $(x \oplus x') \in X$ such that

$$(a \oplus a') \oplus e = (x \oplus x') \oplus (b \oplus b'),$$

and so $(a \oplus a')R(b \oplus b')$.

(ii) The reflexivity of $R$ follows from the observation that $a \oplus e = e \oplus a$ and $e \in X$.

(iii) Suppose there is an internal monoid structure in $X$ with $e$ as its unit element. Corollary 4.3 tells us that $X$ has a monoid structure with $e$ as unit if and only if for every $x, y \in X$ there is $(x *_e y) \in X$ such that $(x *_e y) \oplus e = x \oplus y$. In this case, we prove transitivity by showing that if $aRb$ and $bRc$, that is

$$a \oplus e = x \oplus b, \qquad b \oplus e = y \oplus c,$$

then $aRc$, because

$$a \oplus e = (x *_e y) \oplus c.$$

It is straightforward to prove the above equality, by composing with $(e \oplus e)$ and then using cancellation:

$$\begin{aligned}
(a \oplus e) \oplus (e \oplus e) &= (x \oplus b) \oplus (e \oplus e) \\
&= (x \oplus e) \oplus (b \oplus e) \\
&= (x \oplus e) \oplus (y \oplus c) \\
&= (x \oplus y) \oplus (e \oplus c) \\
&= \big((x *_e y) \oplus e\big) \oplus (c \oplus e) \\
&= \big((x *_e y) \oplus c\big) \oplus (e \oplus e).
\end{aligned}$$

(iv) Let us show that $R$ is symmetric if and only if $X$ is $e$-symmetric. By definition of $R$, for every $x \in X$, we have $xRe$. Hence, if the relation is symmetric we also have $eRx$ from which we conclude the existence of $y \in X$ such that $e = e \oplus e = y \oplus x$. This shows that $X$ is $e$-symmetric. Conversely, if $X$ is $e$-symmetric then for every $x \in X$ there is $y \in X$ such that $x \oplus y = e$ and consequently the relation $R$ is symmetric. Indeed, given $aRb$, that is $a \oplus e = x \oplus b$ for some $x \in X$, we have, with $y = -_e(x)$:

$$(y \oplus a) \oplus e = (y \oplus e) \oplus (a \oplus e) = (y \oplus x) \oplus (e \oplus b) = e \oplus (e \oplus b)$$

from which we conclude

$$y \oplus a = b \oplus e$$

and so $bRa$. □

The condition on the existence of a monoid structure in item (iii) above is sufficient for the relation to be transitive but it is not necessary. Indeed we only have the operation $(x *_e y)$ well defined for certain pairs $(x, y) \in X \times X$, namely the ones for which given any $c \in A$ there are $a, b \in A$ such that $a = (y *_e c)$ and $b = x *_e (y *_e c)$.

This suggests the following necessary and sufficient condition for the transitivity of this type of internal relations.

**Proposition 6.4.** *Let $(A, \oplus)$ be a ccm-magma with a subalgebra $X \subseteq A$ and an idempotent element $e \in X$. The relation*

$$aRb \iff \exists x \in X, \qquad a \oplus e = x \oplus b$$

*is transitive if and only if:*

*for all $x, y \in X$ and $c \in A$, if there exist $a, b \in A$ such that*

$$a \oplus e = x \oplus b \quad \text{and} \quad b \oplus e = y \oplus c \tag{6.3}$$

*then there is $z \in X$ such that $z \oplus e = x \oplus y$.*

*Proof.* Assume $R$ is transitive. If we have solutions $a$ and $b$ for the equations (6.3) then we also have $aRb$ and $bRc$ which, by transitivity, gives us the desired $z \in X$ such that $a \oplus e = z \oplus c$. It is now a simple calculation to check that $z \oplus e = x \oplus y$. Indeed we have

$$\begin{aligned}(z \oplus e) \oplus (c \oplus e) &= (z \oplus c) \oplus (e \oplus e)\\ &= (a \oplus e) \oplus (e \oplus e)\\ &= (x \oplus b) \oplus (e \oplus e)\\ &= (x \oplus e) \oplus (b \oplus e)\\ &= (x \oplus e) \oplus (y \oplus c)\\ &= (x \oplus y) \oplus (e \oplus c)\end{aligned}$$

and the result follows from cancellation. Conversely, if $aRb$ and $bRc$ then we also have $x, y \in X$ as in (6.3) and hence there is an element $z \in X$ such that $z \oplus e = x \oplus y$ from which we conclude that $a \oplus e = z \oplus c$. Indeed

$$\begin{aligned}(a \oplus e) \oplus (e \oplus e) &= (x \oplus b) \oplus (e \oplus e)\\ &= (x \oplus e) \oplus (b \oplus e)\\ &= (x \oplus e) \oplus (y \oplus c)\\ &= (x \oplus y) \oplus (e \oplus c)\\ &= (z \oplus e) \oplus (c \oplus e)\\ &= (z \oplus c) \oplus (e \oplus e),\end{aligned}$$

this shows that $aRc$ and concludes the proof. □

When $A$ is $e$-expansive, item (iii) in Proposition 6.3 can now be reformulated so to relate the transitivity of $R$ with the property of $X$ being $e$-expansive.

**Corollary 6.5.** *Let $(A, \oplus)$ be a ccm-magma which is e-expansive for some idempotent element $e \in A$ and let $X$ be a subalgebra with $e \in X$. The relation*

$$aRb \iff \exists x \in X, \quad a \oplus e = x \oplus b$$

*is transitive if and only if $X$ is e-expansive.*

*Proof.* If $X$ is $e$-expansive then in particular it has an internal monoid structure with $x * y = 2_e(x \oplus y)_X$. Hence the relation is transitive (Proposition 6.3(iii)). Conversely, if the relation is transitive, then for all $x, y \in X$ the element $2_e(x \oplus y) \in A$ is in fact an element in $X$. Indeed, $2_e(x \oplus y)Ry$ and $yRe$ implies $2_e(x \oplus y)Re$ which is equivalent to $2_e(x \oplus y) \in X$. This shows that $X$ is $e$-expansive. □

Combining the two previous results on symmetry and transitivity with $X$ being $e$-expansive and $e$-symmetric we also observe:

**Corollary 6.6.** *Let $(A, \oplus)$ be a homogeneous ccm-magma with a subalgebra $X \subseteq A$ and an idempotent element $e \in X$. The relation*

$$aRb \iff \exists x \in X, \qquad a \oplus e = x \oplus b$$

*is a congruence, if and only if $X$ is homogeneous.*

*Proof.* If $X$ is homogeneous then it is $e$-expansive with $2_e(x) \in X$ and it is $e$-symmetric with $-_e(x) = 2_x(e) \in X$, for every $x \in X$. As a consequence the relation $R$ is transitive and symmetric, and hence it is a congruence, since it is always an internal reflexive relation. Conversely, let us suppose $R$ is a congruence. By Corollary 6.5 and Proposition 6.3(iv) we already know that $X$ is $e$-expansive and $e$-symmetric, hence the result in Proposition 5.3 concludes the proof. □

## 7. Conclusion

This work shows that the category of ccm-magmas admits several classifications for its objects. One possibility is to differentiate between those ccm-magmas admitting an internal monoid structure and those who don't. A ccm-magma $(A, \oplus)$ with a given idempotent, $e$, admits an internal monoid structure with $e$ as its unit if and only if the equation

$$x \oplus e = a \oplus b$$

has a solution $x$ for every $a$, $b$ in $A$. This condition is weaker than $A$ being $e$-expansive. However, the two conditions are equivalent when every element $a \in A$ can be decomposed as $a = x_1 \oplus x_2$. This property is considered, for instance, in [13].

It is shown that every relation $R$ of the form (6.1), constructed with the two homomorphisms $f$ and $g$, is necessarily a difunctional relation. This result is also a consequence of the fact that the relation $R$ is a strong relation and the category of ccm-magmas is weakly Mal'tsev. More generally, when $f$ and/or $g$

are not homomorphisms, it might happen that $R$ is still an internal relation but not a difunctional one. In a similar way, if $f$ and $g$ are as in Proposition 6.1, with $f(a,a) = g(a,a)$, but not homomorphisms, then we may have a reflexive internal relation which is not a congruence. For example, the relation $R$ in Proposition 6.3, is equivalently defined as $aRb$ if and only if $f(a,b) = g(a,b)$, where, for all $a, b \in A$, $f(a,b) = a \oplus e$ while $g(a,b) = a \oplus e$ if there exists $x \in X$ such that $a \oplus e = x \oplus b$, otherwise $g(a,b) = e \oplus b$.

Every finite ccm-magma is necessarily homogeneous (Definition 5.1), and since axiom (M3) is weaker than associativity, these kinds of structures may be useful to the random generation of finite abelian groups. The procedure is very simple: randomly generate a ccm-magma $M$ with at least one idempotent, say $e$ (although this is only important if we are interested in internal structures), and then define

```
A(i,j)=find(M(:,e)==M(i,j))
```

for every $i$ and $j$, in order to obtain a matrix $A$ with the multiplication table for an abelian group with $e$ as unit element.

The notion of ccm-magma may also be defined internally in every category with binary products (as it is done in [8] for midpoint algebras) and so, some interesting interactions at this level are also expected, especially for the case of topological ccm-magmas.

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