Portugal. Math. (N.S.)
Vol. 73, Fasc. 3, 2016, 207–218
DOI 10.4171/PM/1985

Portugaliae Mathematica

A hyperelliptic Hodge integral

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(Recommended by Rahul Pandharipande)

Abstract. We use orbifold Gromov–Witten theory to evaluate a hyperelliptic Hodge integral that is responsible for the weight with which contracted components attached at Weierstrass points contribute to hyperelliptic Gromov–Witten invariants of surfaces.

Mathematics Subject Classification (primary; secondary): 14N35; 14H10, 14D23

Keywords: Gromov–Witten invariants, Hodge integrals, hyperelliptic curves, orbifolds

1. Introduction

The purpose of this paper is to evaluate a Hodge integral on a compactification of the moduli space of hyperelliptic curves with one marked Weierstrass point.

Theorem 1.1. *Let $\overline{\mathscr{W}}_g$ be the compactification of the moduli space of hyperelliptic curves with a single marked Weierstrass point, described in Section 2. Let $\mathbf{E}$ be the Hodge bundle on $\overline{\mathscr{W}}_g$ and let L be the cotangent line at the marked point, and set $\lambda_i = c_i(\mathbf{E})$ and $\psi = c_1(L)$. Then (1) holds.*

$$\frac{1}{(2g+1)!}\int_{\overline{\mathscr{W}}_g} \frac{\left(1-\lambda_1+\cdots+(-1)^g\lambda_g\right)^2}{1-\psi} = \frac{1}{(2g+1)!}\left(-\frac{1}{4}\right)^g \tag{1}$$

This integral shows up naturally in the enumerative geometry of hyperelliptic curves on surfaces. It measures the weight with which a contracted component of genus g, attached at a Weierstrass point, contributes to hyperelliptic Gromov–Witten invariants (as is visible in Section 5). Our original application of this calculation in [8] was to relate the genus zero Gromov–Witten invariants of $[\mathrm{Sym}^2\,\mathbf{P}^2]$ and the enumerative geometry of hyperelliptic curves in $\mathbf{P}^2$. It has since been used in the study of hyperelliptic curves on abelian surfaces [4], [7].

Expanded as a power series in ψ, the leading term of the integrand in (1) is a multiple of $\lambda_g\lambda_{g-1}$, whose integral on the hyperelliptic locus was calculated by

*This work was partly supported by an NSA Young Investigator's Grant, Award #H98230-14-1-0107.

Faber and Pandharipande [6], Corollary to Proposition 3. Cavalieri and Ross computed integrals of the form $\int \lambda_g \lambda_{g-i} \psi_1^{m_1} \dots \psi_k^{m_k}$ using localization [5], Theorem 2.3, but without the additional constraint imposed here that the marked points be Weierstrass points.

We prove Theorem 1.1 by reinterpreting $\overline{\mathscr{W}_g}$ as a moduli space of stable maps from *orbifold curves* to the stack $B(\mathbf{Z}/2\mathbf{Z})$. The oribfold curves in question are the quotient stacks obtained by dividing hyperelliptic curves by their hyperelliptic involutions. The integral (1) measures the virtual contributions of contracted components to orbifold Gromov–Witten invariants, so we find a particularly simple example of such an invariant to evaluate (see Section 5). As the quotient of a hyperelliptic curve by its hyperelliptic involution is rational, this is a genus 0 Gromov–Witten invariant, and it succumbs easily to an application of the WDVV equations (see Section 4).

Studying the stack quotient of a hyperelliptic curve by the hyperelliptic involution amounts to studying the hyperelliptic curve equivariantly, so it is in principle possible to perform the same calculations without reliance on orbifolds. However, orbifold Gromov–Witten theory (introduced by Chen and Ruan [5] and developed algebraically by Abramovich, Graber, and Vistoli [1], [2], [3]) confers an advantage in that it allows us to use a particularly convenient target, $\mathbf{P}(1,1,2)$, that is not a global quotient by a $\mathbf{Z}/2\mathbf{Z}$-action.

I am indebted to the anonymous referees for their help in improving the exposition of this paper.

Conventions. We work throughout over the complex numbers. The letter C is used for orbifold curves of genus 0 and $\tilde{C}$ is used for a double cover of C by a hyperelliptic curve. The coarse moduli space of C is denoted $\overline{C}$. We reserve G for the group $\mathbf{Z}/2\mathbf{Z}$. The only nontrivial stabilizer group that appears on orbifold curves will be $\mathbf{Z}/2\mathbf{Z}$, so we will abbreviate 'orbifold curve with $\mathbf{Z}/2\mathbf{Z}$ stabilizers' to 'orbifold curve'.

For orbifold Gromov–Witten theory, we will follow the definitions and notation of [2], [3].

2. Hyperelliptic curves

We begin by explaining the notation used in Theorem 1.1. Suppose that $g \geq 2$. Let $\mathscr{H}_g \subset \mathscr{M}_g$ be the locus of hyperelliptic curves and let $\overline{\mathscr{H}_g} \subset \overline{M}_g$ be its closure. If $[\tilde{C}] \in \mathscr{H}_g$ then $\tilde{C}$ is canonically equipped with an action of G. In fact, the same holds when $[\tilde{C}] \in \overline{\mathscr{H}_g}$:

Lemma 2.1. *Let $\tilde{C}$ be a stable curve that is the stable limit of a family of hyperelliptic curves. Then the hyperelliptic involution extends canonically over $\tilde{C}$.*

Proof. Indeed, suppose that $[\tilde{C}] \in \overline{\mathscr{H}}_g$ and let $\tilde{C}'$ be a versal deformation of $\tilde{C}$ over a base S'. Then $\underline{\mathrm{Aut}}(\tilde{C}')$ is finite over S'. The automorphism described above yields a rational section of $\underline{\mathrm{Aut}}(\tilde{C}')$. The closure of the image of this section is a subscheme of $\underline{\mathrm{Aut}}(\tilde{C}')$ that is finite over and birational to, hence equal to, S'. The section therefore extends canonically over S'. □

Now consider a family $S \to \overline{\mathscr{H}}_g$ corresponding to a family of stable curves $\tilde{C}$ over S with an automorphism σ. Let $W \subset \tilde{C}$ be the intersection of the fixed locus of σ with the smooth locus of $\tilde{C}$ over S.

The projection $W \to S$ is étale, and, if S is a versal family, generically of degree $2g+2$. It may be extended to a finite, but not necessarily étale, cover of S, because of monodromy when exactly two Weierstrass points collide. We eliminate this monodromy in a universal way.

There is a divisor $\Xi_{\mathrm{irr}} \subset \overline{\mathscr{H}}_g$ corresponding to the locus where two Weierstrass points collide; generically, points of Ξ_{irr} correspond to irreducible curves with a single node whose brances are points that are conjugate under σ. The locus Ξ_{irr} is a normal crossings divisor, so one may perform a root stack construction of order two along the components of Ξ_{irr} to yield a finite, birational map of Deligne–Mumford stacks:

$$\tilde{\mathscr{H}}_g \to \overline{\mathscr{H}}_g$$

Taking $S = \tilde{\mathscr{H}}_g$, above, the locus W constructed above has trivial monodromy, hence extends to an étale cover of $\tilde{\mathscr{H}}_g$, which we denote $\overline{\mathscr{W}}_g$.

By blowing up the nodes parameterized by Ξ_{irr}, we can regard $\overline{\mathscr{W}}_g$ as the moduli space of pairs $[\tilde{C}, p]$ where $\tilde{C}$ is a curve with an involution and p is a marked fixed point of that involution that does not lie in the singular locus of $\tilde{C}$.

Lemma 2.2. *The projection $\overline{\mathscr{W}}_g \to \tilde{\mathscr{H}}_g$ is finite and étale, of degree $2g+2$.*

Let $\mathscr{L} = N_{\overline{\mathscr{W}}_g/\bar{\mathscr{C}}_g}$ be the normal bundle to $\overline{\mathscr{W}}_g$ inside the universal curve over $\tilde{\mathscr{H}}_g$, and let $\mathbf{E}$ be the Hodge bundle (whose fiber over $[\tilde{C}, p] \in \overline{\mathscr{W}}_g$ is $H^0(\tilde{C}, \omega_{\tilde{C}})$). We write $\psi = c_1(\mathscr{L})$ and $\lambda_i = c_i(\mathbf{E})$.

3. Orbifold maps

We will use the theories of orbifold stable maps and orbifold Gromov–Witten theory as developed in [1], [2], [3]. Our notation for the moduli space of genus 0, degree β orbifold stable maps with n_1 ordinary marked points and n_2 orbifold points will be

$$\bar{\mathscr{M}}(X; n_1, n_2; \beta),$$

which is an open substack of the corresponding Artin stack of pre-stable maps, $\mathfrak{M}(X; n_1, n_2; \beta)$. When the target has dimension zero, we drop β from the notation. We reserve the letter G for the group $\mathbf{Z}/2\mathbf{Z}$.

Lemma 3.1. *There is a finite, étale cover*

$$\bar{\mathscr{M}}(BG; 0, 2g+2) \to \overline{\mathscr{W}}_g \tag{3}$$

of degree $(2g+1)!$.

Proof. To give a point of $\bar{\mathscr{M}}(BG; 0, 2g+2)$ means to give a rational, nodal, orbifold curve C with $2g+2$ marked points with $\mathbf{Z}/2\mathbf{Z}$-stabilizers and a representable map $C \to BG$. This map entails a 2-to-1 cover $\tilde{C} \to C$ where $\tilde{C}$ is an ordinary nodal curve. Sending $[C]$ to $[\tilde{C}]$ gives a map (4).

$$\bar{\mathscr{M}}(BG; 0, 2g+2) \to \bar{\mathscr{M}}_g \tag{4}$$

It is possible to deform C to a smooth rational curve with $2g+2$ marked orbifold points; the maps $C \to BG$ deform as well, since BG is étale over a point (alternatively, the double covers $\tilde{C} \to C$ deform because they are étale). Therefore the curve $\tilde{C}$ deforms to a smooth hyperelliptic curve, which shows that our map takes values in $\overline{\mathscr{H}}_g$.

In fact, the $2g+2$ marked orbifold points of C lift to $2g+2$ marked points of $\tilde{C}$. Forgetting all but the first of these, we get a lift of this map to one taking values in $\overline{\mathscr{W}}_g$. The whole construction can be run in reverse: given a curve $[\tilde{C}, p] \in \overline{\mathscr{W}}_g$, where $[\tilde{C}] \in \overline{\mathscr{H}}_g$ and p is a marked Weierstrass point of $\tilde{C}$, we can recover C and its $2g+2$ marked orbifold points by taking the stack quotient of C by the hyperelliptic involution and choosing an ordering on the remaining Weierstrass points. This proves that $\bar{\mathscr{M}}(BG; 0, 2g+2)$ is a $(2g+1)!$-fold étale cover of $\overline{\mathscr{W}}_g$. □

Let $\pi : C \to S$ be the total space of a family of rational orbifold curves with a map $C \to BG$, and let $\tilde{C}$ be the corresponding family of stable hyperelliptic curves. For each $i = 1, \ldots, 2g+2$, there is a closed substack D_i, the i-th marked G-gerbe. The preimage of D_i in $\tilde{C}$ will be denoted $\tilde{D}_i$ and is a family of marked Weierstrass points. The map $D_i \subset C \to BG$ trivializes the gerbe D_i, yielding an isomorphism $D_i \simeq \tilde{D}_i \times BG$. Since the hyperelliptic involution acts nontrivially on the cotangent space at a Weierstrass point, we may use this isomorphism to identify

$$N^{\vee}_{D_i/C} = N_{\tilde{D}_i/\tilde{C}} \boxtimes \rho_1$$

where ρ_1 is the non-trivial representation of G, viewed as a line bundle on D_i pulled back from BG. Therefore the i-th cotangent line bundle of the hyperelliptic curve may be constructed as

$$L_i = \pi_*(N^\vee_{D_i/C} \otimes \rho_1) = (\pi \circ \tau)_* N^\vee_{\tilde{D}_i/\tilde{C}}. \tag{5}$$

Applying this construction universally yields line bundles $\mathscr{L}_i$ on $\bar{\mathscr{M}}(BG; 0, 2g+2)$. In particular, $\mathscr{L}_1$ is the pullback of $\mathscr{L}$ under the projection $\bar{\mathscr{M}}(BG; 0, 2g+2) \to \overline{\mathscr{W}}_g$.

We also have the hyperelliptic Hodge bundle:

$$\mathbf{E} = (\pi \circ \tau)_* \omega_{\tilde{C}/S} = \pi_*(\omega_{C/S} \otimes \rho_1)$$

The second equality holds because

$$\tau_* \omega_{\tilde{C}/S} = \tau_* \tau^* \omega_{C/S} = \omega_{C/S} \otimes \tau_* \tau^* \mathcal{O}_C = \omega_{C/S} \otimes (\rho_0 \oplus \rho_1),$$

where we have written ρ_0 and ρ_1 for the trivial and nontrivial representations of G, respectively. The first summand contributes nothing to the Hodge bundle, since $\omega_{C/S} = \omega_{C/S} \otimes \rho_0$ has no nonzero global sections (because C has genus 0), so $(\pi \circ \tau)_* \omega_{\tilde{C}/S} = \pi_*(\omega_{C/S} \otimes \rho_1)$.

By Serre duality, $\mathbf{E}$ is dual to

$$\mathbf{E}^\vee = R^1(\pi \circ \tau)_*(\mathcal{O}_S) = R^1\pi_*(\rho_0 \oplus \rho_1) = R^1\pi_*\rho_1. \tag{6}$$

The second equality comes from $R^1\pi_*\rho_0 = R^1\pi_*\mathcal{O}_C = 0$, which is because C has genus 0.

Applying this construction universally, we obtain the Hodge bundle over $\bar{\mathscr{M}}(BG; 0, 2g+2)$, which is pulled back from the Hodge bundle over $\overline{\mathscr{W}}_g$. We may now restate Theorem 1.1 using $\bar{\mathscr{M}}(BG; 0, 2g+2)$:

Theorem 3.2. *With definitions as above,*

$$\int_{\bar{\mathscr{M}}(BG; 0, 2g+2)} \frac{c(\mathbf{E}^\vee)^2}{c(L_1^\vee)} = (2g-1)! \int_{\overline{\mathscr{W}}_g} \frac{c(\mathbf{E}^\vee)^2}{c(\psi)} = \left(-\frac{1}{4}\right)^g.$$

The first equality is the combination of Lemma 3.1 and the discussion preceding the statement of the theorem. The second equality will be proved by interpreting the integral as a Gromov–Witten invariant on a weighted projective space (Section 5) and evaluating it recursively using the WDVV equations (Section 4).

4. A Gromov–Witten invariant of P(1, 1, 2)

Let $X = \mathbf{P}(1,1,2)$ be the weighted projective space $[(\mathbf{A}^3 \backslash \{0\})/\mathbf{G}_m]$, acting with weights $1, 1, 2$. Let $\bar{\mathscr{M}}(X; n_1, n_2; \beta)$ be the moduli space of genus zero orbifold stable maps to X with n_1 ordinary marked points and n_2 orbifold marked points and degree β. The degree is evaluated by integrating $c_1(\mathcal{O}(1))$ over the curve and so is an element of $\frac{1}{2}\mathbf{Z}$.

The virtual dimension is given by the formula

$$\text{v.dim}\, \bar{\mathcal{M}}(X; n_1, n_2; \beta) = \dim X - 3 + \int_\beta c_1(TX) + n_1 + n_2 - \sum_{i=1}^{n_2} \text{age}(x_i)$$

where x_i, $i = 1, \ldots, n_2$ is the set of orbifold marked points and $\text{age}(x_i)$ is the sum of the t_j such that the eigenvalues of the action of the stabilizer of x_i acting on TX are $e^{2\pi i t_j}$, $j = 1, \ldots, n_2$, listed with multiplicity. If $f : C \to X$ is a representable map then any orbifold point of C must be carried by f to the unique stacky point of X, which is represented by $(0, 0, 1)$. The automorphisms act with eigenvalues $-1, -1$ on the fiber of the tangent bundle at this point, so the age is 1.

The Euler sequence here is

$$0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2) \to TX \to 0$$

so $c_1(TX) = 4c_1(\mathcal{O}(1))$. Thus,

$$\text{v.dim}\, \bar{\mathcal{M}}(X; n_1, n_2; \beta) = 4d - 1 + n_1$$

where $d = \int_\beta c_1(\mathcal{O}(1))$.

We will follow the conventions of [3] concerning Gromov–Witten invariants. In particular, if $\alpha_1, \ldots, \alpha_n$ are classes on the rigidified inertia stack of X, with n_1 coming from the untwisted sector and n_2 from the twisted sector, and $d \in \frac{1}{2}\mathbf{Z}$, then

$$\langle \alpha_1, \ldots, \alpha_n \rangle_d = \int_{\mathcal{M}(X; n_1, n_2; d)^{\text{vir}}} e_1^*(\alpha_1) \cap \cdots \cap e_n^*(\alpha_n)$$

and $\langle \alpha_1, \ldots, \alpha_n, * \rangle_d$ will denote the class in the orbifold Chow ring of X such that

$$\langle \alpha_1, \ldots, \alpha_n, \alpha' \rangle_d = \langle \alpha_1, \ldots, \alpha_n, * \rangle_d \cap \alpha'.$$

See [2], Section 4.7 for more details about this notation.

The inertia stack of X is $X \amalg BG$ and the rigidified inertia stack is $X \amalg$ (point). Let γ be the fundamental class of the second component. Let p be the class of an ordinary point in X. We will spend the rest of this section computing the following invariant:

Lemma 4.1. *With notation as introduced above,*

$$\langle p, \gamma, \ldots, \gamma \rangle_{1/2} = \left(-\frac{1}{4}\right)^g \tag{7}$$

Let's put $h = c_1(\mathcal{O}(1))$. Then $p = 2h^2$.

A schematic of the orbifold Chow ring of X with its structure as a graded vector space is shown below.

$$\begin{array}{rc|c|c} & 0 & 1 & 2 \\ X & \mathbf{Q} & \mathbf{Q}h & \mathbf{Q}p \\ \text{(point)} & & \mathbf{Q}\gamma & \end{array}$$

It is easy to see that $\bar{\mathcal{M}}(X;1,2;0) \cong BG$ and therefore that $\gamma^2 = \frac{1}{2}p = h^2$. Therefore a presentation of the orbifold Chow ring is $\mathbf{Q}[h,\gamma]/(h^2 - \gamma^2)$. Note in particular that this satisfies Poincaré duality.

Lemma 4.2. *The Gromov–Witten invariants of X have the following properties.*

(a) *If* $\langle \gamma^{\otimes n}, \alpha \rangle_0 \neq 0$ *then* $n = 2$ *and* $\alpha = 1$.

(b) *The invariant* $\langle \gamma^{\otimes n}, h, * \rangle_0$ *is zero for all n.*

Proof. For (a), if α comes from the untwisted sector then the invariant is computed on $\bar{\mathcal{M}}(X;1,n;0)$, which has virtual dimension 0, so the invariant will be zero unless $\alpha = 1$. But then it will vanish by the unit axiom unless $n = 2$. If α comes from the twisted sector then it is computed on $\bar{\mathcal{M}}(X;0,n+1;0)$, which has virtual dimension -1, so the invariant vanishes.

For (b), it is sufficient by linearity to show that $\langle \gamma^{\otimes n}, 2h, * \rangle_0 = 0$. But the Chow class $2h$ can be represented by a line that doesn't pass through the uniqe orbifold point $(0,0,1)$. Since this is a degree zero invariant, this means it is computed on an empty moduli space, i.e., it is zero. □

The WDVV equations (see [3], Theorem 6.2.1) give

$$\sum_{\substack{a+b=2g-1 \\ d_1+d_2=1/2}} \langle \langle h, h, \gamma^{\otimes a}, * \rangle_{d_1}, \gamma, \gamma, \gamma^{\otimes b} \rangle_{d_2} = \sum_{\substack{a+b=2g-1 \\ d_1+d_2=1/2}} \langle \langle h, \gamma, \gamma^{\otimes a}, * \rangle_{d_1}, h, \gamma, \gamma^{\otimes b} \rangle_{d_2}. \quad (8)$$

where d_1 and d_2 can take the values 0 and $\frac{1}{2}$ in the sums.

Consider first the right side of the equality. One of the d_i must be zero, so consider the invariant $\langle h, \gamma, \ldots, \gamma, * \rangle_0$. This is zero by Lemma 4.2 (b). Therefore the right side is zero.

On the left side, note that if $d_1 = 0$ then the corresponding term of the sum will be zero by the divisor axiom unless $a = 0$ also. In that case we get

$$\langle \langle h, h, * \rangle_0, \gamma, \gamma, \gamma^{\otimes(2g-1)} \rangle_{1/2} = \langle h^2, \gamma^{\otimes(2g+1)} \rangle_{1/2}$$

The remaining terms on the left of (8) have the form

$$\langle\langle h,h,\gamma^{\otimes a},*\rangle_{1/2},\gamma^{\otimes(b+2)}\rangle_0=\frac{1}{4}\langle\langle\gamma^{\otimes a},*\rangle_{1/2},\gamma^{\otimes(b+2)}\rangle_0=\frac{1}{4}\langle\gamma^{\otimes a},\langle\gamma^{\otimes(b+2)},*\rangle_0\rangle_{1/2}$$

(the first equality is by two applications of the divisor axiom). By Lemma 4.2 (a), these invariants vanish for $b>0$, so we are left with

$$\frac{1}{4}\langle\gamma^{\otimes(2g-1)},\langle\gamma,\gamma,*\rangle_0\rangle_{1/2}=\frac{1}{4}\langle\gamma^{\otimes(2g-1)},\gamma^2\rangle_{1/2}.$$

Thus (8) reduces to

$$\langle h^2,\gamma^{\otimes(2g+1)}\rangle_{1/2}+\frac{1}{4}\langle\gamma^{\otimes(2g-1)},\gamma^2\rangle_{1/2}=0.$$

Since $h^2=\gamma^2=\frac{1}{2}p$ we get

$$\langle p,\gamma^{\otimes(2g+1)}\rangle_{1/2}=\left(-\frac{1}{4}\right)^g\langle p,\gamma\rangle_{1/2}$$

by induction. The invariant on the right side of this equality is easily seen to be 1. Indeed, $\bar{\mathscr{M}}(X;1,1;\frac{1}{2})$ may be identified with

$$\mathbf{P}(\Gamma(X,\mathcal{O}(1)))\cong\mathbf{P}^1.$$

The virtual dimension of $\bar{\mathscr{M}}(X;0,1;\frac{1}{2})$ is also 1, so we only need to solve the enumerative problem to compute $\langle p,\gamma\rangle_{1/2}$. If $(u,v)\in\mathbf{P}^1$ is a point, then the condition that the corresponding curve interpolate the point $(x,y,z)\in X$ is $ux+vy=0$. This has exactly one solution if $(x,y)\neq(0,0)$ so we conclude that $\langle p,\gamma\rangle_{1/2}=1$. This completes the proof of Lemma 4.1.

5. The virtual fundamental class

Lemma 5.1. *Let C be a smooth orbifold curve. Suppose there is a representable map $f:C\to X$ of degree $\frac{1}{2}$. Then C has at most 1 orbifold point.*

Proof. Let L be a line through the orbifold point of X (the vanishing locus of a section of $\mathcal{O}(1)$). The preimage of L in C is an effective divisor of degree $\frac{1}{2}$, hence is one of the orbifold points P of C. Then $C\backslash P\to X\backslash L$ is a representable map, and $\mathbf{P}(1,1,2)\backslash L$ is representable by a scheme, so $C\backslash P$ is representable by a scheme as well. □

Proposition 5.2. *There are isomorphisms*

$$\bar{\mathcal{M}}\left(X;0,2g+1;\frac{1}{2}\right) \cong \bar{\mathcal{M}}\left(X;0,1;\frac{1}{2}\right) \times \bar{\mathcal{M}}(BG;0,2g+2)$$

Proof. If $(f,C) \in \bar{\mathcal{M}}(X;0,2g+1;\frac{1}{2})$, then C has a unique irreducible component C_0 with $\deg f|_{C_0} = \frac{1}{2}$; all other components have degree 0. By the lemma, C_0 has exactly 1 orbifold point. The remaining orbifold points must lie on a union of components that is attached at the unique orbifold point of C. Thus every point of $\bar{\mathcal{M}}(X;0,2g+1;\frac{1}{2})$ lies in the image of the gluing map

$$\iota : \bar{\mathcal{M}}\left(X;0,1;\frac{1}{2}\right) \times \bar{\mathcal{M}}(BG;0,2g+2) \to \overline{M}(X;0,2g+1)$$

that attaches the marked point from the first component to the first marked point from the second component.

This is a closed embedding, so to complete the proof, we must show that the image of this map is open in $\bar{\mathcal{M}}(X;0,2g+1;\frac{1}{2})$. Consider a first-order deformation (C',f') of (C,f). Let C_1 be the contracted component of C and let C_0 be the component of positive degree; write Q for the unique orbifold point of X, which has coordinates $(0,0,1)$. If (C',f') were not in the image of ι, then C' would be a first-order smoothing of C. But then consider the map $N_{C_1/C'} \to (f|_{C_1})^* T_Q X$. If P is the point of attachment between C_0 and C_1, then $N_{C_1/C'}|_P$ is spanned by $T_P C_0$. Moreover, C_0 meets Q transversally (since $f|_{C_0}$ has degree $\frac{1}{2}$), which implies that the map $N_{C_1/C'} \to (f|_{C_1})^* T_Q X$ is nonzero at P.

On the other hand, $N_{C_1/C'} \cong \mathcal{O}_{C_1}(-P)$, and $(f|_{C_1})^* T_Q X \cong (f|_{C_1})^*(\rho_1 \oplus \rho_1)$ because f contracts C_1 onto Q and $T_Q X \cong \rho_1 \oplus \rho_1$. Thus we obtain a pair of sections of $\rho_1 \otimes \mathcal{O}_{C_1}(P)$, at least one of which does not vanish at P.

Let $\pi : C_1 \to \bar{C}_1$ be the coarse moduli space. Then we get a section of $\pi_*(\rho_1 \otimes \mathcal{O}_{C_1}(P))$ that is not everywhere zero. But $\pi_*(\rho_1 \otimes \mathcal{O}_{C_1}(P)) = \mathcal{O}_{\bar{C}_1}(-2g-1)$ where $2g+2$ is the number of orbifold points on C_1. In particular, all sections of $\pi_*(\rho_1 \otimes \mathcal{O}_{C_1}(P))$ vanish. This contradicts the nonvanishing of the section at P. □

Now that we know how the moduli space looks, we must determine the virtual fundamental class. We use the deformation-obstruction sequence,

$$\mathrm{Def}(C) \to \mathrm{Obs}(f) \to \mathrm{Obs}(C,f) \to \mathrm{Obs}(C) = 0.$$

We know that $\mathrm{Obs}(C,f)$ is a vector bundle because $\bar{\mathcal{M}}(X;0,2g+1;\frac{1}{2})$ is smooth. The virtual fundamental class is the top Chern class of this vector bundle.

Obs(f) is the relative obstruction space for the map

$$\bar{\mathcal{M}}\left(X; 0, 2g+1; \frac{1}{2}\right) \to \mathfrak{M}(BG; 0, 2g+1),$$

where $\mathfrak{M}(BG; 0, 2g+1)$ is the Artin stack of pre-stable maps to BG. If (C, f) is a curve in $\bar{\mathcal{M}}(X; 0, 2g+1; \frac{1}{2})$ then we have just seen that C is the union of two curves, C_0 and C_1, along an orbifold point, with $\deg(f|_{C_0}) = \frac{1}{2}$ and $\deg(f|_{C_1}) = 0$. It is clear that any deformation of C that is trivial near the node will extend to a deformation of (C, f): indeed, C_0 is rigid and C_1 is contracted by f. Thus, the image of $\mathrm{Def}(C) \to \mathrm{Obs}(f)$ is the space of deformations of the node. If we name the nodal point P, then the deformations of the node are parameterized by $\pi_*(T_P C_0 \otimes T_P C_1)$, so we have an exact sequence on $\bar{\mathcal{M}}(X; 0, 2g+1; \frac{1}{2})$,

$$0 \to \pi_*(T_P C_0 \otimes T_P C_1) \to \mathrm{Obs}(f) \to \mathrm{Obs}(C, f) \to 0.$$

Explicitly, $\mathrm{Obs}(f) = R^1\pi_* f^* TX$, where $f : C \to X$ is the universal map. Tensoring the normalization sequence for the node P with f^*TX and taking cohomology, we obtain

$$H^0(T|_P) \to H^1(T) \to H^1(T|_{C_0}) \oplus H^1(T|_{C_1}) \to H^1(T|_P) = 0,$$

writing $T = f^*TX$. Note that $H^0(T|_P) = 0$ since P is an orbifold point and $T|_P \cong \rho_1 \oplus \rho_1$ has no invariant sections.

We can also calculate $H^1(T|_{C_0}) = 0$ using the Euler sequence, which pulls back to

$$0 \to \mathcal{O} \to \mathcal{O}(P) \oplus \mathcal{O}(P) \oplus \mathcal{O}(2P) \to T|_{C_0} \to 0$$

since $f|_{C_0}$ has degree $\frac{1}{2}$ and $f^*\mathcal{O}(1) = \mathcal{O}(P)$. Pushing this sequence forward to the coarse moduli space via $q : C_0 \to \bar{C}_0$ (note q_* is exact) gives

$$0 \to \mathcal{O}_{\bar{C}_0} \to \mathcal{O}_{\bar{C}_0} \oplus \mathcal{O}_{\bar{C}_0} \oplus \mathcal{O}_{\bar{C}_0}(q(P)) \to q_* T\big|_{C_0} \to 0.$$

Now taking cohomology and noting that $H^1(\mathcal{O}_{\bar{C}_0}) = H^1(\mathcal{O}_{\bar{C}_0}(q(P))) = H^2(\mathcal{O}_{\bar{C}_0}) = 0$, we deduce that $H^1(T|_{C_0}) = 0$ from the long exact sequence.

It now follows that $\mathrm{Obs}(f) = H^1(T|_{C_1})$. But, as already remarked, $f|_{C_1}$ factors through the orbifold point of X, so $T|_{C_1}$ is the pullback of the tangent bundle at this point, which is $\rho_1 \oplus \rho_1$. Thus,

$$\mathrm{Obs}(f) = R^1\pi_*(\rho_1 \oplus \rho_1) \cong \mathbf{E}^\vee \oplus \mathbf{E}^\vee$$

where $\mathbf{E}$ is the Hodge bundle (see (6)).

We therefore have an exact sequence,

$$0 \to \pi_*(T_P C_0 \otimes T_P C_1) \to \mathbf{E}^\vee \oplus \mathbf{E}^\vee \to \mathrm{Obs}(C, f) \to 0.$$

We have $T_P C_0 \cong \rho_1$, so we get the exact sequence,

$$0 \to \mathscr{L}_1^\vee \to \mathbf{E}^\vee \oplus \mathbf{E}^\vee \to \mathrm{Obs}(C, f) \to 0$$

where $\mathscr{L}_1$ was defined in (5).

Now, we evaluate the Gromov–Witten invariant (7). Consider the cartesian diagram

$$\begin{array}{ccc} e^{-1}(p) & \longrightarrow & \bar{\mathscr{M}}\left(X; 1, 2g+1; \frac{1}{2}\right) \\ \big\downarrow & & \big\downarrow{\scriptstyle e} \\ p & \xrightarrow{\quad i \quad} & X \end{array}$$

where e is evaluation at the ordinary marked point. Under the identification of Proposition 5.2, e factors through the evaluation map on $\bar{\mathscr{M}}\left(X; 1, 1; \frac{1}{2}\right)$. Thus, $e^{-1}(p)$ may be identified with $\bar{\mathscr{M}}(BG; 2g+2)$. Now,

$$\langle p, \gamma^{\otimes(2g+1)} \rangle_{1/2} = \int i^! \left[\bar{\mathscr{M}}\left(X; 1, 2g+1; \frac{1}{2} \right) \right]^{\mathrm{vir}} = \int_{\bar{\mathscr{M}}(BG; 2g+2)} \frac{c(\mathbf{E}^\vee)^2}{c(L_1^\vee)}.$$

But we have also seen in Section 4 that

$$\langle p, \gamma^{\otimes(2g+1)} \rangle_{1/2} = \left(-\frac{1}{4} \right)^g$$

and this completes the proof of Theorem 3.2.

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Received November 4, 2015; revised June 8, 2016

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