Periodic homogenization of integral energies under space-dependent differential constraints

Elisa Davoli and Irene Fonseca

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Abstract. A homogenization result for a family of oscillating integral energies

$$u_\varepsilon \mapsto \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) dx, \qquad \varepsilon \to 0^+$$

is presented, where the fields u_ε are subjected to first order linear differential constraints
depending on the space variable x. The work is based on the theory of $\mathscr{A}$-quasiconvexity
with variable coefficients and on two-scale convergence techniques, and generalizes the
previously obtained results in the case in which the differential constraints are imposed by
means of a linear first order differential operator with constant coefficients. The identifica-
tion of the relaxed energy in the framework of $\mathscr{A}$-quasiconvexity with variable coefficients
is also recovered as a corollary of the homogenization result.

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1. Introduction

In this paper we continue the study of the problem of finding an integral represen-
tation for limits of oscillating integral energies

$$u_\varepsilon \mapsto \int_\Omega f\left(x, \frac{x}{\varepsilon^\alpha}, u_\varepsilon(x)\right) dx,$$

where $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^d \to [0, +\infty)$ has standard p-growth, $\Omega \subset \mathbb{R}^N$ is a bounded
open set, $\varepsilon \to 0$, and the fields $u_\varepsilon : \Omega \to \mathbb{R}^d$ are subjected to x-dependent

differential constraints of the type

$$\sum_{i=1}^{N} A^i\left(\frac{x}{\varepsilon^\beta}\right)\frac{\partial u_\varepsilon(x)}{\partial x_i} \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l), \; 1 < p < +\infty, \quad (1.1)$$

or in divergence form

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(A^i\left(\frac{x}{\varepsilon^\beta}\right)u_\varepsilon(x)\right) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l), \; 1 < p < +\infty, \quad (1.2)$$

with $A^i(x) \in \text{Lin}(\mathbb{R}^d;\mathbb{R}^l)$ for every $x \in \mathbb{R}^N$, $i = 1,\ldots,N$, $d,l \geq 1$, and where α, β are two nonnegative parameters. Different regimes are expected to arise, depending on the relation between α and β.

We recently analyzed in [10] the limit case in which $\alpha = 0$, $\beta > 0$, the energy density is independent of the first two variables, and the fields $\{u_\varepsilon\}$ are subjected to (1.2). We will consider here the case in which $\alpha > 0$, $\beta = 0$ and (1.1), i.e., the energy density is oscillating but the differential constraint is fixed and in "nondivergence" form. The situation in which there is an interplay between α and β will be the subject of a forthcoming paper.

The key tool for our study is the notion of $\mathscr{A}$-quasiconvexity with variable coefficients, characterized in [20]. $\mathscr{A}$-quasiconvexity was first investigated by Dacorogna in [8] and then studied by Fonseca and Müller in [12] in the case of constant coefficients (see also [9]). More recently, in [20] Santos extended the analysis of [12] to the case in which the coefficients of the differential operator $\mathscr{A}$ depend on the space variable.

In order to illustrate the main ideas of $\mathscr{A}$-quasiconvexity, we need to introduce some notation. For $i = 1\ldots,N$, consider matrix-valued maps $A^i \in C^\infty(\mathbb{R}^N;\mathbb{M}^{l\times d})$, where for $l,d \in \mathbb{N}$, $\mathbb{M}^{l\times d}$ stands for the linear space of matrices with l rows and d columns, and for every $x \in \mathbb{R}^N$ define $\mathscr{A}$ as the differential operator such that

$$\mathscr{A}u := \sum_{i=1}^{N} A^i(x)\frac{\partial u(x)}{\partial x_i}, \quad x \in \Omega \quad (1.3)$$

for $u \in L^1_{\text{loc}}(\Omega;\mathbb{R}^d)$, where $\frac{\partial u}{\partial x_i}$ is to be interpreted in the sense of distributions. We require that the operator $\mathscr{A}$ satisfies a uniform constant-rank assumption (see [18]), i.e., there exists $r \in \mathbb{N}$ such that

$$\text{rank}\sum_{i=1}^{N} A^i(x)w_i = r \quad \text{for every } w \in \mathbb{S}^{n-1}, \quad (1.4)$$

uniformly with respect to x, where $\mathbb{S}^{N-1}$ is the unit sphere in $\mathbb{R}^N$.

The definitions of $\mathscr{A}$-quasiconvex function and $\mathscr{A}$-quasiconvex envelope in the case of variable coefficients read as follows:

Definition 1.1. Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function, let Q be the unit cube in $\mathbb{R}^N$ centered at the origin,

$$Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^N,$$

and denote by $C_{\mathrm{per}}^\infty(\mathbb{R}^N; \mathbb{R}^d)$ the set of smooth maps which are Q-periodic in $\mathbb{R}^N$. For every $x \in \Omega$ consider the set

$$\mathcal{C}_x := \left\{ w \in C_{\mathrm{per}}^\infty(\mathbb{R}^N; \mathbb{R}^d) : \int_Q w(y)\, dy = 0, \sum_{i=1}^N A^i(x) \frac{\partial w(y)}{\partial y_i} = 0 \right\}. \quad (1.5)$$

For a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^d$, the $\mathscr{A}$-*quasiconvex envelope* of f in $x \in \Omega$ is defined as

$$Q_{\mathscr{A}} f(x, \xi) := \inf \left\{ \int_Q f(x, \xi + w(y))\, dy : w \in \mathcal{C}_x \right\}.$$

f is said to be $\mathscr{A}$-*quasiconvex* if $f(x, \xi) = Q_{\mathscr{A}} f(x, \xi)$ for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^d$.

Denote by $\mathscr{A}^c$ a generic differential operator, defined as in (1.3) and with constant coefficients, i.e. such that

$$A^i(x) \equiv A_c^i \quad \text{for every } x \in \mathbb{R}^N,$$

with $A_c^i \in \mathbb{M}^{l \times d}$, $i = 1, \ldots, N$. We remark that when $\mathscr{A} = \mathscr{A}^c = \mathrm{curl}$, i.e., when $v = \nabla \phi$ for some $\phi \in W_{\mathrm{loc}}^{1,1}(\Omega; \mathbb{R}^m)$, and if Ω is connected, then $d = m \times N$, and $\mathscr{A}$-quasiconvexity reduces to Morrey's notion of quasiconvexity (see [1], [4], [15], [17]).

The first identification of the effective energy associated to periodic integrands evaluated along $\mathscr{A}^c$-free fields was provided in [5], by Braides, Fonseca and Leoni. Their homogenization results were later generalized in [11], where Fonseca and Krömer worked under weaker assumptions on the energy density f. Recently, Matias, Morandotti, and Santos extended the previous results to the case $p = 1$ [16], whereas Kreisbeck and Krömer performed in [13] simultaneous homogenization and dimension reduction in the framework of $\mathscr{A}^c$-quasiconvexity.

This paper is devoted to extending the results in [11] to the framework of $\mathscr{A}$-quasiconvexity with variable coefficients. To be precise, in [11] the authors studied the homogenized energy associated to a family of functionals of the type

$$F_\varepsilon(u_\varepsilon) := \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) dx,$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, $u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega; \mathbb{R}^d)$ and the sequence $\{u_\varepsilon\}$ satisfies a differential constraint of the form $\mathscr{A}^c u_\varepsilon = 0$ for every ε.

We analyze the analogous problem in the case in which $\mathscr{A}$ depends on the space variable and the differential constraint is replaced by the condition

$$\mathscr{A} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l).$$

Our analysis leads to a limit homogenized energy of the form:

$$\mathscr{E}_{\text{hom}}(u) := \begin{cases} \int_\Omega f_{\text{hom}}(x, u(x)) \, dx & \text{if } \mathscr{A}u = 0, \\ +\infty & \text{otherwise in } L^p(\Omega; \mathbb{R}^d), \end{cases}$$

where $f_{\text{hom}} : \Omega \times \mathbb{R}^d \to [0, +\infty)$ is defined as

$$f_{\text{hom}}(x, \xi) := \liminf_{n \to +\infty} \inf_{v \in \mathcal{C}_x} \int_Q f(x, ny, \xi + v(y)) \, dy.$$

Our main result is the following.

Theorem 1.2. *Let $1 < p < +\infty$. Let $A^i \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{M}^{l \times d})$, $i = 1, \ldots, N$, and assume that $\mathscr{A}$ satisfies the constant rank condition (1.4). Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^d \to \mathbb{R}$ be a function satisfying*

$$f(x, \cdot, \xi) \quad \text{is measurable}, \tag{1.6}$$

$$f(\cdot, y, \cdot) \quad \text{is continuous}, \tag{1.7}$$

$$f(x, \cdot, \xi) \quad \text{is Q-periodic}, \tag{1.8}$$

$$0 \le f(x, y, \xi) \le C(1 + |\xi|^p)$$

$$\text{for all } (x, \xi) \in \Omega \times \mathbb{R}^d, \text{ and for a.e. } y \in \mathbb{R}^N. \tag{1.9}$$

Then for every $u \in L^p(\Omega; \mathbb{R}^d)$ there holds

$$\inf \left\{ \liminf_{\varepsilon \to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \right.$$
$$\left. \text{and } \mathscr{A}u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l) \right\}$$

$$= \inf \left\{ \limsup_{\varepsilon \to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \right.$$
$$\left. \text{and } \mathscr{A}u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l) \right\}$$

$$= \mathscr{E}_{\text{hom}}(u).$$

As in [10] and [11], the proof of this result is based on the *unfolding operator*, introduced in [6], [7] (see also [21], [22]). In contrast with [11], Theorem 1.1 (i.e. the case in which $\mathscr{A} = \mathscr{A}^c$), here we are unable to work with exact solutions of the system $\mathscr{A}u_\varepsilon = 0$, but instead we consider sequences of asymptotically $\mathscr{A}$-vanishing fields. This is due to the fact that for $\mathscr{A}$-quasiconvexity with variable coefficients we do not project directly on the kernel of the differential constraint, but construct an "approximate" projection operator P such that for every field $v \in L^p$, the $W^{-1,p}$ norm of $\mathscr{A}Pv$ is controlled by the $W^{-1,p}$ norm of v itself (for a detailed explanation we refer to [20], Subsection 2.1).

In [10] the issue of defining a projection operator was tackled by imposing an additional invertibility assumption on $\mathscr{A}$ and by exploiting the divergence form of the differential constraint. We do not add this invertibility requirement here, instead we use the fact that in our framework the differential operator depends on the "macro" variable x but acts on the "micro" variable y (see (1.5)). Hence it is possible to define a pointwise projection operator $\Pi(x)$ along the argument of [12], Lemma 2.14 (see Lemma 4.1).

As a corollary of our main result we recover an alternative proof of the relaxation theorem [5], Theorem 1.1 in the framework of $\mathscr{A}$-quasiconvexity with variable coefficients, that is we obtain the identification (see Corollary 4.10)

$$\int_D Q_{\mathscr{A}} f\big(x, u(x)\big)\, dx = \mathcal{I}(u, D)$$

for every open subset D of Ω, and for every $u \in L^p(\Omega; \mathbb{R}^d)$ satisfying $\mathscr{A}u = 0$, where the functional $\mathcal{I}$ is defined as

$$\mathcal{I}(u, D) := \inf\Big\{ \liminf_{\varepsilon \to 0} \int_D f\big(x, u_\varepsilon(x)\big) : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^m)$$
$$\text{and } \mathscr{A}u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l)\Big\}.$$
$$(1.10)$$

We point out here that a proof of this relaxation theorem follows directly combining [5], Proof of Theorem 1.1 with the arguments in [20]. The interest in Corollary 4.10 lies in the fact that it is obtained as a by-product of our homogenization result, and thus by adopting a completely different proof strategy.

In analogy to [10] one might expect to be able to apply an approximation argument and extend the results in Theorem 4.3 to the situation in which $A^i \in W^{1,\infty}(\mathbb{R}^N; \mathbb{M}^{l \times d})$, $i = 1 \ldots, N$, which is the least regularity assumption in order for $\mathscr{A}$ to be well defined as a differential operator from L^p to $W^{-1,p}$. We were unable to achieve this generalization, mainly because the projection operator here plays a key role in the proof of both the liminf and the limsup inequalities. In order to work with approximant operators $\mathscr{A}^k$ having smooth coefficients, we

would need to finally construct an "approximate projection operator" P associated to $\mathscr{A}$, whereas the projection argument provided in [20] applies only to the case of smooth differential constraints.

The article is organized as follows. In Section 2 we establish the main assumptions on the differential operator $\mathscr{A}$ and we recall some preliminary results on two-scale convergence. In Section 3 we recall the definition of $\mathscr{A}$-quasiconvex envelope and we construct some examples of $\mathscr{A}$-quasiconvex functions. Section 4 is devoted to the proof of our main result.

Notation. Throughout this paper, $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\mathcal{O}(\Omega)$ is the set of open subsets of Ω, Q denotes the unit cube in $\mathbb{R}^N$ centered at the origin and with normals to its faces parallel to the vectors in the standard orthonormal basis of $\mathbb{R}^N$, $\{e_1, \ldots, e_N\}$, i.e.,

$$Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^N.$$

Given $1 < p < +\infty$, we denote by p' its conjugate exponent, that is

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Whenever a map $u \in L^p, C^\infty, \ldots$, is Q-periodic, that is

$$u(x + e_i) = u(x) \qquad i = 1, \ldots, N$$

for a.e. $x \in \mathbb{R}^N$, we write $u \in L^p_{\mathrm{per}}, C^\infty_{\mathrm{per}}, \ldots$, respectively. We will implicitly identify the spaces $L^p(Q)$ and $L^p_{\mathrm{per}}(\mathbb{R}^N)$. We will designate by $\langle \cdot, \cdot \rangle$ the duality product between $W^{-1,p}$ and $W^{1,p'}_0$.

We adopt the convention that C will denote a generic constant, whose value may change from expression to expression in the same formula.

2. Preliminary results

In this section we introduce the main assumptions on the differential operator $\mathscr{A}$ and we recall some preliminary results about $\mathscr{A}$-quasiconvexity and two-scale convergence.

2.1. Preliminaries. For $i = 1, \ldots, N$, consider the matrix-valued functions $A^i \in C^\infty(\mathbb{R}^N; \mathbb{M}^{l \times d})$. For $1 < p < +\infty$ and $u \in L^p(\Omega; \mathbb{R}^d)$, we set

$$\mathscr{A}u := \sum_{i=1}^{N} A^i(x) \frac{\partial u(x)}{\partial x_i} \in W^{-1,p}(\Omega; \mathbb{R}^l).$$

For every $x_0 \in \Omega$ and $u \in L^p(\Omega; \mathbb{R}^d)$ we define

$$\mathscr{A}(x_0)u := \sum_{i=1}^{N} A^i(x_0) \frac{\partial u(x)}{\partial x_i} \in W^{-1,p}(\Omega; \mathbb{R}^l).$$

We will also consider the operators

$$\mathscr{A}_x w := \sum_{i=1}^{N} A^i(x) \frac{\partial w(x, y)}{\partial x_i}$$

and

$$\mathscr{A}_y w := \sum_{i=1}^{N} A^i(x) \frac{\partial w(x, y)}{\partial y_i}$$

for every $w \in L^p(\Omega \times Q; \mathbb{R}^d)$. Finally, for every $x_0 \in \Omega$ and for $w \in L^p(\Omega \times Q; \mathbb{R}^d)$, we set

$$\mathscr{A}_x(x_0)w := \sum_{i=1}^{N} A^i(x_0) \frac{\partial w(x, y)}{\partial x_i}$$

and

$$\mathscr{A}_y(x_0)w := \sum_{i=1}^{N} A^i(x_0) \frac{\partial w(x, y)}{\partial y_i}.$$

For every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \backslash \{0\}$, let $\mathbb{A}(x, \lambda)$ be the linear operator

$$\mathbb{A}(x, \lambda) := \sum_{i=1}^{N} A^i(x)\lambda_i \in \mathbb{M}^{l \times d}.$$

We assume that $\mathscr{A}$ satisfies the following *constant rank condition*:

$$\text{rank}\left(\sum_{i=1}^{N} A^i(x)\lambda_i\right) = r \quad \text{for some } r \in N \text{ and for all } x \in \mathbb{R}^N, \lambda \in \mathbb{R}^N \backslash \{0\}. \quad (2.1)$$

For every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \backslash \{0\}$, let $\mathbb{P}(x, \lambda) : \mathbb{R}^d \to \mathbb{R}^d$ be the linear projection on Ker $\mathbb{A}(x, \lambda)$, and let $\mathbb{Q}(x, \lambda) : \mathbb{R}^l \to \mathbb{R}^d$ be the linear operator given by

$$\mathbb{Q}(x, \lambda)\mathbb{A}(x, \lambda)\xi := \xi - \mathbb{P}(x, \lambda)\xi \quad \text{for all } \xi \in \mathbb{R}^d,$$

$$\mathbb{Q}(x, \lambda)\xi = 0 \qquad\qquad\qquad \text{if } \xi \notin \text{Range } \mathbb{A}(x, \lambda).$$

The main properties of $\mathbb{P}(\cdot, \cdot)$ and $\mathbb{Q}(\cdot, \cdot)$ are stated in the following proposition (see [20], Subsection 2.1).

Proposition 2.1. *Under the constant rank condition* (2.1), *for every $x \in \mathbb{R}^N$ the operators $\mathbb{P}(x, \cdot)$ and $\mathbb{Q}(x, \cdot)$ are, respectively, 0-homogeneous and (-1)-homogeneous. Moreover, $\mathbb{P} \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \backslash \{0\}; \mathbb{M}^{d \times d})$ and $\mathbb{Q} \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \backslash \{0\}; \mathbb{M}^{d \times l})$.*

2.2. Two-scale convergence. We recall here the definition and some properties of two-scale convergence. For a detailed treatment of the topic we refer to, e.g., [2], [14], [19]. Throughout this subsection $1 < p < +\infty$.

Definition 2.2. If $v \in L^p(\Omega \times Q; \mathbb{R}^d)$ and $\{u_\varepsilon\} \in L^p(\Omega; \mathbb{R}^d)$, we say that $\{u_\varepsilon\}$ *weakly two-scale converges to v in* $L^p(\Omega \times Q; \mathbb{R}^d)$, $u_\varepsilon \overset{2-s}{\rightharpoonup} v$, if

$$\int_\Omega u_\varepsilon(x) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Q v(x, y) \cdot \varphi(x, y) \, dy \, dx$$

for every $\varphi \in L^{p'}(\Omega; C_{\text{per}}^\infty(\mathbb{R}^N; \mathbb{R}^d))$.

We say that $\{u_\varepsilon\}$ *strongly two-scale converges to v in* $L^p(\Omega \times Q; \mathbb{R}^d)$, $u_\varepsilon \overset{2-s}{\longrightarrow} v$, if $u_\varepsilon \overset{2-s}{\rightharpoonup} v$ and

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega; \mathbb{R}^d)} = \|v\|_{L^p(\Omega \times Q; \mathbb{R}^d)}.$$

Bounded sequences in $L^p(\Omega; \mathbb{R}^d)$ are pre-compact with respect to weak two-scale convergence. To be precise (see [2], Theorem 1.2),

Proposition 2.3. *Let $\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^d)$ be bounded. Then, there exists $v \in L^p(\Omega \times Q; \mathbb{R}^d)$ such that, up to the extraction of a (non relabeled) subsequence, $u_\varepsilon \overset{2-s}{\rightharpoonup} v$ weakly two-scale in $L^p(\Omega \times Q; \mathbb{R}^d)$, and, in particular*

$$u_\varepsilon \rightharpoonup \int_Q v(x, y) \, dy \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d).$$

The following result will play a key role in the proof of the limsup inequality (see [11], Proposition 2.4, Lemma 2.5 and Remark 2.6).

Proposition 2.4. *Let $v \in L^p(\Omega; C_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d))$ or $v \in L_{\text{per}}^p(\mathbb{R}^N; C(\overline{\Omega}; \mathbb{R}^d))$. Then, the sequence $\{v_\varepsilon\}$ defined as*

$$v_\varepsilon(x) := v\left(x, \frac{x}{\varepsilon}\right)$$

is p-equiintegrable, and

$$v_\varepsilon \xrightarrow{2-s} v \qquad \text{strongly two-scale in } L^p(\Omega; \mathbb{R}^d).$$

2.3. The unfolding operator. We collect here the definition and some properties of the *unfolding operator* (see e.g. [7], [6], [21], [22]).

Definition 2.5. Let $u \in L^p(\Omega; \mathbb{R}^d)$. For every $\varepsilon > 0$, the unfolding operator $T_\varepsilon : L^p(\Omega; \mathbb{R}^d) \to L^p\big(\mathbb{R}^N; L^p_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)\big)$ is defined componentwise as

$$T_\varepsilon(u)(x, y) := u\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon(y - \lfloor y\rfloor)\right) \qquad \text{for a.e. } x \in \Omega \text{ and } y \in \mathbb{R}^N, \quad (2.2)$$

where u is extended by zero outside Ω and $\lfloor\cdot\rfloor$ denotes the least integer part.

The next proposition and the subsequent theorem allow to express the notion of two-scale convergence in terms of L^p convergence of the unfolding operator.

Proposition 2.6 (see [7], [22]). *T_ε is a nonsurjective linear isometry from $L^p(\Omega; \mathbb{R}^d)$ to $L^p(\mathbb{R}^N \times Q; \mathbb{R}^d)$.*

The following theorem provides an equivalent characterization of two-scale convergence in our framework (see [22], Proposition 2.5 and Proposition 2.7, [14], Theorem 10).

Theorem 2.7. *Let Ω be an open bounded domain and let $v \in L^p(\Omega \times Q; \mathbb{R}^d)$. Assume that v is extended to be 0 outside Ω. Then the following conditions are equivalent:*

(i) *$u_\varepsilon \xrightarrow{2-s} v$ weakly two scale in $L^p(\Omega \times Q; \mathbb{R}^d)$,*

(ii) *$T_\varepsilon u_\varepsilon \rightharpoonup v$ weakly in $L^p(\mathbb{R}^N \times Q; \mathbb{R}^d)$.*

Moreover,

$$u^\varepsilon \xrightarrow{2-s} v \qquad \text{strongly two scale in } L^p(\Omega \times Q; \mathbb{R}^d)$$

if and only if

$$T_\varepsilon u_\varepsilon \to v \qquad \text{strongly in } L^p(\mathbb{R}^N \times Q; \mathbb{R}^d).$$

The following proposition is proved in [11], Proposition A.1.

Proposition 2.8. *For every* $u \in L^p(\Omega; \mathbb{R}^d)$ *(extended by 0 outside* Ω*),*

$$\|u - T_\varepsilon u\|_{L^p(\mathbb{R}^N \times Q; \mathbb{R}^d)} \to 0$$

as $\varepsilon \to 0$.

3. $\mathscr{A}$-quasiconvex functions

In this section we recall the notion of $\mathscr{A}$-quasiconvexity and $\mathscr{A}$-quasiconvex envelope, and we provide some examples of $\mathscr{A}$-quasiconvex functions in the case in which $\mathscr{A}$ has variable coefficients.

We start by recalling the main definitions when $\mathscr{A} = \mathscr{A}^c$, where $\mathscr{A}^c$ is a first order differential operator with constant coefficients, that is, for every $u \in L^p(\Omega; \mathbb{R}^d)$,

$$\mathscr{A}^c u(x) := \sum_{i=1}^N A_c^i \frac{\partial u(x)}{\partial x_i} \in W^{-1,p}(\Omega; \mathbb{R}^l),$$

with $A_c^i \in \mathbb{M}^{l \times d}$ for $i = 1, \ldots, N$.

Definition 3.1. Let $f : \Omega \times \mathbb{R}^d \to [0, +\infty)$ be a Carathéodory function, let $\mathscr{A}^c$ be a first order differential operator with constant coefficients, and consider the set

$$\mathcal{C}_{\text{const}} := \left\{ w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) : \int_Q w(y)\, dy = 0 \text{ and } \sum_{i=1}^N A_c^i \frac{\partial w(y)}{\partial y_i} = 0 \right\}.$$

The $\mathscr{A}^c$-*quasiconvex envelope* of f is the function $Q^{\mathscr{A}^c} f : \Omega \times \mathbb{R}^d \to [0, +\infty)$, given by

$$Q^{\mathscr{A}^c} f(x, \xi) := \inf \left\{ \int_Q f(x, \xi + w(y))\, dy : w \in \mathcal{C}_{\text{const}} \right\}, \qquad (3.1)$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^d$.

We say that f is $\mathscr{A}^c$-*quasiconvex* if

$$f(x, \xi) = Q^{\mathscr{A}^c} f(x, \xi) \qquad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^d.$$

Similarly, in the case in which $\mathscr{A}$ depends on the space variable, the definitions of $\mathscr{A}$-quasiconvex envelope and $\mathscr{A}$-quasiconvex function read as in Definition 1.1.

We stress that $\mathscr{A}$-quasiconvexity and pointwise $\mathscr{A}(x)$-quasiconvexity are related by the following "fixed point" relation:

$$Q_{\mathscr{A}}f(x,\xi) = Q^{\mathscr{A}(x)}f(x,\xi) \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^d. \tag{3.2}$$

The remaining part of this section is devoted to illustrating these concepts with some explicit examples of $\mathscr{A}$-quasiconvex functions. We first exhibit an example where $\mathscr{A}$-quasiconvexity reduces to $\mathscr{A}^c$-quasiconvexity for a suitable operator $\mathscr{A}^c$ with constant coefficients.

Example 3.2. Let $1 < p < +\infty$ and define

$$f(x,\xi) := a(x)b(\xi) \quad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^d,$$

with $a \in L^p(\Omega)$ and $b \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$. In order for f to be $\mathscr{A}$-quasiconvex, the function b must satisfy

$$b(\xi) = \inf\left\{ \int_Q b(\xi + w(y))\, dy : w \in \bigcup_{x \in \Omega} C_x \right\}.$$

Consider the case in which C_x is the same for every x, for example when the differential constraint is provided by the operator:

$$\mathscr{A}w(x) := \sum_{i=1}^{N} M(x)A_c^i \frac{\partial w(x)}{\partial x_i},$$

where $M : \Omega \to \mathbb{M}^{l \times l}$ and $\det M(x) > 0$ for every $x \in \Omega$. In this case,

$$C_x = \left\{ w \in C_{\text{per}}^\infty(\mathbb{R}^N; \mathbb{R}^d) : \int_Q w(y)\, dy = 0 \text{ and } \mathscr{A}^c w = 0 \right\}$$

for every x, where

$$\mathscr{A}^c w(x) := \sum_{i=1}^{N} A_c^i \frac{\partial w(x)}{\partial x_i}.$$

Hence, f is $\mathscr{A}$-quasiconvex if and only if b is $\mathscr{A}^c$-quasiconvex.

In the previous example, $\mathscr{A}$-quasiconvexity could be reduced to $\mathscr{A}^c$-quasiconvexity owing to the fact that the class C_x was constant in x. We provide now an example where an analogous phenomenon occurs, despite the fact that C_x varies with respect to x. To be precise, we consider the case in which $\mathscr{A}$ is

a smooth perturbation of the divergence operator. In this situation, the $\mathscr{A}$-quasiconvex envelope of f coincides with its convex envelope.

Example 3.3. We consider a smooth perturbation of the divergence operator in a set $\Omega \subset \mathbb{R}^2$, that is

$$\mathscr{A}u := \begin{pmatrix} a(x) & 0 \\ 0 & 1 \end{pmatrix} \nabla u \quad \text{for every } u \in L^p(\Omega; \mathbb{R}^2), \ 1 < p < +\infty,$$

with $a \in C(\overline{\Omega})$ and

$$\frac{1}{2} \leq a(x) < 1 \quad \text{for every } x \in \Omega.$$

We notice that

$$\ker \mathbb{A}(x, \lambda) = \{\xi \in \mathbb{R}^2 : a(x)\lambda_1\xi_1 + \lambda_2\xi_2 = 0\},$$

and therefore

$$\text{rank } \mathbb{A}(x, \lambda) = 1 \quad \text{for every } x \in \Omega \text{ and } \lambda \in \mathbb{R}^2\backslash\{0\}.$$

In this situation, the class $\mathcal{C}_x$ depends on x, since we have

$$\mathcal{C}_x = \left\{ w \in C_{\text{per}}^\infty(\mathbb{R}^2; \mathbb{R}^2) : \int_Q w(y)\,dy = 0 \text{ and } a(x)\frac{\partial w_1(y)}{\partial y_1} + \frac{\partial w_2(y)}{\partial y_2} = 0 \right\},$$

although

$$\bigcup_{\lambda \in \mathbb{S}^1} \ker \mathbb{A}(x, \lambda) = \mathbb{R}^2.$$

Let now $f : \mathbb{R}^d \to [0, +\infty)$ be a continuous map. By [12], Proposition 3.4, it follows that $Q_{\mathscr{A}}f(\xi)$ coincides with the convex envelope of f evaluated at ξ, exactly as in the case of the divergence operator (see [12], Remark 3.5 (iv)).

We conclude this section with an example in which the notion of $\mathscr{A}$-quasiconvexity can not be reduced to $\mathscr{A}^c$-quasiconvexity with respect to a constant operator $\mathscr{A}^c$.

Example 3.4. Here we consider a smooth perturbation of the curl operator in a set $\Omega \subset \mathbb{R}^2$. Let $\mathscr{A} : L^p(\Omega; \mathbb{R}^2) \to W^{-1,p}(\Omega; \mathbb{R}^4)$ be given by

$$\mathscr{A}u = -a_1(x)\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2},$$

where $a_1 \in C(\overline{\Omega})$ is not constant, and satisfies

$$\frac{1}{2} \le a_1(x) \le 1 \quad \text{for every } x \in \Omega.$$

We first notice that

$$\ker \mathbb{A}(x, \lambda) := \{\xi \in \mathbb{R}^2 : \lambda_1 a_1(x)\xi_2 = \lambda_2 \xi_1\}$$
$$= \{\xi \in \mathbb{R}^2 : \xi_2 = \alpha \lambda_2 \text{ and } \xi_1 = \alpha a_1(x)\lambda_1, \alpha \in \mathbb{R}\},$$

hence

$$\operatorname{rank} \mathbb{A}(x, \lambda) = 3 \quad \text{for every } x \in \Omega, \lambda \in \mathbb{S}^1.$$

The class $\mathcal{C}_x$ depends on x and there holds

$$\mathcal{C}_x = \left\{ w \in C^\infty_{\text{per}}(\mathbb{R}^2; \mathbb{R}^2) : \int_Q w(y)\, dy = 0 \text{ and } a_1(x)\frac{\partial w_2(y)}{\partial y_1} = \frac{\partial w_1(y)}{\partial y_2} \right\}$$
$$= \left\{ w \in C^\infty_{\text{per}}(\mathbb{R}^2; \mathbb{R}^2) : \int_Q w(y)\, dy = 0, w_1(y) = a_1(x)\frac{\partial \varphi(y)}{\partial y_1}, \right.$$
$$\left. \text{and } w_2(y) = \frac{\partial \varphi(y)}{\partial y_2}, \text{where } \varphi \in C^\infty_{\text{per}}(\mathbb{R}^2) \right\}. \tag{3.3}$$

Let now $g : \Omega \times \mathbb{R}^2 \to [0, +\infty)$ be a quasiconvex function and let $f : \Omega \times \mathbb{R}^2 \to [0, +\infty)$ be defined as

$$f(x, \xi) := g\left(x, \left(\frac{\xi_1}{a_1(x)}, \xi_2\right)\right) \quad \text{for a.e. } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^2.$$

We claim that

$$Q_{\mathscr{A}} f(x, \xi) = f(x, \xi) \quad \text{for a.e. } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^2.$$

Indeed, by (3.3) there holds

$$\inf_{w \in \mathcal{C}_x} \int_Q f(x, \xi + w(y))\, dy$$
$$= \inf\left\{ \int_Q f\left(x, \left(\xi_1 + a_1(x)\frac{\partial \varphi(y)}{\partial y_1}, \xi_2 + \frac{\partial \varphi(y)}{\partial y_2}\right)\right) dy : \varphi \in C^\infty_{\text{per}}(\mathbb{R}^2) \right\}$$

$$= \inf\left\{\int_Q g\left(x, \left(\frac{\xi_1}{a_1(x)} + \frac{\partial\varphi(y)}{\partial y_1}, \xi_2 + \frac{\partial\varphi(y)}{\partial y_2}\right)\right) dy : \varphi \in C^\infty_{per}(\mathbb{R}^2)\right\}$$

$$= Qg\left(x, \left(\frac{\xi_1}{a_1(x)}, \xi_2\right)\right)$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^2$, where Qg denotes the quasiconvex envelope of the function g. The claim follows by the definition of f and the quasiconvexity of g.

4. A homogenization result for $\mathscr{A}$-free fields

In this section we prove a homogenization result for oscillating integral energies under weak L^p convergence of $\mathscr{A}$-vanishing maps. Fix $1 < p < +\infty$ and consider a function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^d \to [0, +\infty)$ satisfying (1.6)–(1.9).

In analogy with the case of constant coefficients (see [11], Definition 2.9), we define the class of $\mathscr{A}$-free fields as the set

$$\mathscr{F} := \left\{v \in L^p(\Omega \times Q; \mathbb{R}^d) : \mathscr{A}_y v = 0 \text{ and } \mathscr{A}_x \int_Q v(x, y)\, dy = 0\right\}, \qquad (4.1)$$

where both the previous differential conditions are in the sense of $W^{-1,p}$.

We aim at obtaining a characterization of the homogenized energy

$$\inf\left\{\liminf_{\varepsilon\to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d)\right.$$
$$\left. \text{and } \mathscr{A}u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l)\right\}. \qquad (4.2)$$

We start with a preliminary lemma, which will allow us to define a pointwise projection operator. We will be using the notation introduced in Sections 2 and 3.

Lemma 4.1. *Let $1 < p < +\infty$. Let $A^i \in C^\infty_{per}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, $i = 1\ldots, N$, and assume that the associated first order differential operator $\mathscr{A}$ satisfies (2.1). Then, for every $x \in \Omega$ there exists a projection operator*

$$\Pi(x) : L^p(Q; \mathbb{R}^d) \to L^p(Q; \mathbb{R}^d)$$

such that

(P1) $\Pi(x)$ *is linear and bounded, and vanishes on constant maps,*

(P2) $\Pi(x) \circ \Pi(x)\psi(y) = \Pi(x)\psi(y)$ *and* $\mathscr{A}_y(x)\big(\Pi(x)\psi(y)\big) = 0$ *in* $W^{-1,p}(Q;\mathbb{R}^l)$
  for a.e. $x \in \Omega$, *for every* $\psi \in L^p(Q;\mathbb{R}^d)$,

(P3) *there exists a constant* $C = C(p) > 0$, *independent of* x, *such that*

$$\|\psi(y) - \Pi(x)\psi(y)\|_{L^p(Q;\mathbb{R}^d)} \leq C\|\mathscr{A}_y(x)\psi(y)\|_{W^{-1,p}(Q;\mathbb{R}^l)}$$

  for a.e. $x \in \Omega$, *for every* $\psi \in L^p(Q;\mathbb{R}^d)$ *with* $\int_Q \psi(y)\,dy = 0$,

(P4) *if* $\{\psi_n\}$ *is a bounded p-equiintegrable sequence in* $L^p(Q;\mathbb{R}^d)$, *then* $\{\Pi(x)\psi_n(y)\}$ *is a p-equiintegrable sequence in* $L^p(\Omega \times Q;\mathbb{R}^d)$,

(P5) *if* $\varphi \in C^1\big(\Omega; C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d)\big)$ *then the map* φ_Π, *defined by*

$$\varphi_\Pi(x,y) := \Pi(x)\varphi(x,y) \qquad \text{for every } x \in \Omega \text{ and } y \in \mathbb{R}^N,$$

  satisfies $\varphi_\Pi \in C^1\big(\Omega; C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d)\big)$.

Proof. For every $x \in \Omega$, let $\Pi(x)$ be the projection operator provided by [12], Lemma 2.14. Properties (P1) and (P2) follow from [12], Lemma 2.14.

In order to prove (P3), fix $x \in \Omega$ and $\psi \in C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d)$. Let $\mathbb{A}$, $\mathbb{P}$ and $\mathbb{Q}$ be the operators defined in Subsection 2.1. Writing the operator $\Pi(x)$ explicitly, we have

$$\Pi(x)\psi(y) := \sum_{\lambda \in \mathbb{Z}^N\setminus\{0\}} \mathbb{P}(x,\lambda)\hat{\psi}(\lambda)e^{2\pi i y \cdot \lambda},$$

where

$$\hat{\psi}(\lambda) := \int_Q \psi(y)e^{-2\pi i y \cdot \lambda}\,dy, \qquad \text{for every } \lambda \in \mathbb{Z}^N\setminus\{0\}$$

are the Fourier coefficients associated to ψ.

By the (-1)-homogeneity of the operator $\mathbb{Q}$ (see Proposition 2.1) we deduce

$$|\psi(y) - \Pi(x)\psi(y)|^p = \Big|\sum_{\lambda \in \mathbb{Z}^N\setminus\{0\}} \mathbb{Q}(x,\lambda)\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)e^{2\pi i y \cdot \lambda}\Big|^p$$

$$= \Big|\sum_{\lambda \in \mathbb{Z}^N\setminus\{0\}} \frac{1}{|\lambda|}\mathbb{Q}\Big(x,\frac{\lambda}{|\lambda|}\Big)\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)e^{2\pi i y \cdot \lambda}\Big|^p. \qquad (4.3)$$

For $1 < p < 2$, by the smoothness of $\mathbb{Q}$ and by applying first Hölder's inequality and then Hausdorff–Young inequality, we obtain the estimate

$$|\psi(y) - \Pi(x)\psi(y)|^p$$

$$\leq \Big(\max_{\lambda \in \mathbb{Z}^N, |\lambda|=1} \|\mathbb{Q}(x,\lambda)\| \Big)^p \Big(\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \frac{1}{|\lambda|^p} \Big) \Big(\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} |\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)|^{p'} \Big)^{p/p'}$$

$$\leq C \Big(\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} |\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)|^{p'} \Big)^{p/p'} \leq C \|\mathscr{A}_y(x)\psi(y)\|^p_{L^p(Q;\mathbb{R}^d)}, \tag{4.4}$$

where we used the fact that

$$|\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)| \leq C |\widehat{\mathscr{A}_y(x)\psi(y)}(\lambda)| \tag{4.5}$$

for every $x \in \Omega$ and $\lambda \in \mathbb{Z}^N \setminus \{0\}$, by the definition of the Fourier coefficients, and where both constants in (4.4) and (4.5) are independent of λ and x.

Consider now the case in which $p \geq 2$. By (4.3) we have

$$|\psi(y) - \Pi(x)\psi(y)|^p$$

$$\leq \Big(\max_{\lambda \in \mathbb{Z}^N, |\lambda|=1} \|Q(x,\lambda)\| \Big)^p \Big(\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \frac{1}{|\lambda|^{p'}} \Big)^{p/p'} \Big(\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} |\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)|^p \Big)$$

$$\leq C \Big(\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} |\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)|^p \Big)$$

$$\leq C \Big(\sup_{\lambda \in \mathbb{Z}^N \setminus \{0\}} |\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)|^{p-2} \Big) \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} |\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)|^2. \tag{4.6}$$

By the definition of Fourier coefficients and by Hölder's inequality we have

$$|\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)| \leq C \|\mathscr{A}_y(x)\psi(y)\|_{L^p(Q;\mathbb{R}^l)}$$

for every $x \in \Omega$ and $\lambda \in \mathbb{Z}^N \setminus \{0\}$. In view of [12], Theorem 2.9,

$$\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} |\mathbb{A}(x,\lambda)\hat{\psi}(\lambda)|^2 = \|\mathscr{A}_y(x)\psi(y)\|^2_{L^2(Q;\mathbb{R}^l)}.$$

Therefore by (4.6), applying again Hölder's inequality,

$$|\psi(y) - \Pi(x)\psi(y)|^p$$

$$\leq C \|\mathscr{A}_y(x)\psi(y)\|^{p-2}_{L^p(Q;\mathbb{R}^l)} \|\mathscr{A}_y(x)\psi(y)\|^2_{L^2(Q;\mathbb{R}^l)} \leq C \|\mathscr{A}_y(x)\psi(y)\|^p_{L^p(Q;\mathbb{R}^l)} \tag{4.7}$$

where the constant C is independent of x and y. Property (P3) follows by (4.4) and (4.7) via a density argument.

(P4) follows directly from (P3), arguing as in the proof of [12], Lemma 2.14 (iv).

Let now $\varphi \in C^1\big(\Omega; C_{\mathrm{per}}^\infty(\mathbb{R}^N; \mathbb{R}^d)\big)$. The regularity of the map φ_Π is a direct consequence of Proposition 2.1, the definition of Π and the regularity of $\mathscr{A}$. Indeed,

$$\varphi_\Pi(x, y) := \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \mathbb{P}(x, \lambda) \hat{\varphi}(x, \lambda) e^{2\pi i y \cdot \lambda}, \tag{4.8}$$

for every $x \in \Omega$ and $y \in \mathbb{R}^N$, where

$$\hat{\varphi}(x, \lambda) := \int_Q \varphi(x, \xi) e^{-2\pi i \xi \cdot \lambda} \, d\xi$$

for every $x \in \Omega$ and $\lambda \in \mathbb{Z}^N \setminus \{0\}$. By the regularity of φ and by [12], Theorem 2.9 we obtain the estimate

$$\Big(4\pi^2 \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} |\lambda|^2 |\hat{\varphi}(x, \lambda)|^2\Big)^{1/2} \leq \sum_{i=1}^N \Big\|\frac{\partial \varphi}{\partial y_i}(x, y)\Big\|_{L^2(Q; \mathbb{R}^d)} \leq C$$

for every $x \in \Omega$, hence by Proposition 2.1 and Cauchy-Schwartz inequality there holds

$$\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}, |\lambda| \geq n} |\mathbb{P}(x, \lambda) \hat{\varphi}(x, \lambda) e^{2\pi i y \cdot \lambda}|$$

$$\leq C \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}, |\lambda| \geq n} |\hat{\varphi}(x, \lambda)|$$

$$\leq C \Big(\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}, |\lambda| \geq n} |\hat{\varphi}(x, \lambda)|^2 |\lambda|^2\Big)^{1/2} \Big(\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}, |\lambda| \geq n} \frac{1}{|\lambda|^2}\Big)^{1/2}$$

$$\leq C \Big(\sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}, |\lambda| \geq n} \frac{1}{|\lambda|^2}\Big)^{1/2}. \tag{4.9}$$

By (4.9) the series in (4.8) is uniformly convergent, and hence φ_Π is continuous. The differentiability of φ_Π follows from an analogous argument. $\quad\square$

For every $v \in L^p(\Omega \times Q; \mathbb{R}^d)$, let

$$\mathcal{S}_v := \{\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^d) : u_\varepsilon \xrightarrow{2-s} v \quad \text{weakly two-scale in } L^p(\Omega \times Q; \mathbb{R}^d)$$

$$\text{and } \mathscr{A} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l)\}. \tag{4.10}$$

Let also

$$\mathcal{S} := \{\bigcup \mathcal{S}_v : v \in L^p(\Omega \times Q; \mathbb{R}^d)\}. \qquad (4.11)$$

We provide a characterization of the set $\mathcal{S}$.

Lemma 4.2. *Let $v \in L^p(\Omega \times Q; \mathbb{R}^d)$. Let $\mathscr{A}$ be a first order differential operator with variable coefficients, satisfying (2.1). The following conditions are equivalent:*

(C1) $v \in \mathscr{F}$ *(see (4.1))*;

(C2) $\mathcal{S}_v$ *is nonempty.*

Proof. We first show that (C2) implies (C1). Let $v \in L^p(\Omega \times Q; \mathbb{R}^d)$ and let $\{u_\varepsilon\} \subset \mathcal{S}_v$. Consider a test function $\psi \in W_0^{1,p}(\Omega; \mathbb{R}^l)$. Then

$$\langle \mathscr{A}u_\varepsilon, \psi \rangle \to 0 \qquad \text{as } \varepsilon \to 0.$$

On the other hand,

$$\langle \mathscr{A}u_\varepsilon, \psi \rangle := -\sum_{i=1}^N \int_\Omega \left(\frac{\partial A^i(x)}{\partial x_i} u_\varepsilon(x) \cdot \psi(x) + A^i(x)u_\varepsilon(x) \cdot \frac{\partial \psi(x)}{\partial x_i} \right) dx \qquad (4.12)$$

and by Proposition 2.3, up to the extraction of a (not relabeled) subsequence,

$$u_\varepsilon \rightharpoonup \int_Q v(x, y) \, dy \qquad \text{weakly in } L^p(\Omega; \mathbb{R}^d). \qquad (4.13)$$

Passing to the limit in (4.12) yields

$$\mathscr{A}_x \int_Q v(x, y) \, dy = 0 \qquad \text{in } W^{-1,p}(\Omega; \mathbb{R}^l).$$

In order to deduce the second condition in the definition of $\mathscr{F}$, we consider a sequence of test functions $\{\varepsilon\varphi\left(\frac{x}{\varepsilon}\right)\psi(x)\}$, where $\varphi \in W_0^{1,p'}(Q; \mathbb{R}^d)$ and $\psi \in C_c^\infty(\Omega)$. Since this sequence is uniformly bounded in $W_0^{1,p'}(\Omega; \mathbb{R}^d)$, we have

$$\left\langle \mathscr{A}u_\varepsilon, \varepsilon\varphi\left(\frac{\cdot}{\varepsilon}\right)\psi \right\rangle \to 0$$

as $\varepsilon \to 0$, where

$$\left\langle \mathscr{A}u_\varepsilon, \varepsilon\varphi\left(\frac{\cdot}{\varepsilon}\right)\psi \right\rangle$$

$$= -\varepsilon \sum_{i=1}^{N} \int_{\Omega} \left(\frac{\partial A^i(x)}{\partial x_i} u_\varepsilon(x) \cdot \varphi\left(\frac{x}{\varepsilon}\right)\psi(x) + A^i(x)u_\varepsilon(x) \cdot \varphi\left(\frac{x}{\varepsilon}\right)\frac{\partial \psi(x)}{\partial x_i} \right) dx$$

$$- \sum_{i=1}^{N} \int_{\Omega} A^i(x)u_\varepsilon(x) \cdot \frac{\partial \varphi}{\partial x_i}\left(\frac{x}{\varepsilon}\right)\psi(x)\, dx.$$

Passing to the subsequence of $\{u_\varepsilon\}$ extracted in (4.13), the first line of the previous expression converges to zero. By the definition of two-scale convergence, the second line converges to

$$- \sum_{i=1}^{N} \int_{\Omega} \int_{Q} A^i(x)v(x, y) \cdot \frac{\partial \varphi(y)}{\partial y_i}\psi(x)\, dy\, dx,$$

and thus

$$\left\langle \sum_{i=1}^{N} A^i(x)\frac{\partial v(x, \cdot)}{\partial y_i}, \varphi \right\rangle_{W^{-1,p}(Q;\mathbb{R}^l),\, W_0^{1,p'}(Q;\mathbb{R}^l)} = 0$$

for a.e. $x \in \Omega$, that is

$$\mathscr{A}_y v = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x \in \Omega.$$

This completes the proof of (C1).

Assume now that (C1) holds true, i.e., $v \in \mathscr{F}$. In order to construct the sequence $\{u_\varepsilon\}$, set

$$v_1(x, y) = v(x, y) - \int_Q v(x, z)\, dz.$$

We first assume that $v_1 \in C^1\left(\Omega; C_{\text{per}}^1(\mathbb{R}^N; \mathbb{R}^d)\right)$. Defining

$$u_\varepsilon(x) := \int_Q v(x, y)\, dy + v_1\left(x, \frac{x}{\varepsilon}\right) \quad \text{for a.e. } x \in \Omega,$$

by Proposition 2.4 we have $u_\varepsilon \xrightarrow{2-s} v$ strongly two-scale in $L^p(\Omega \times Q; \mathbb{R}^d)$. Moreover, by the definition of $\mathscr{F}$ and Propositions 2.3 and 2.4,

$$\sum_{i=1}^{N} A^i(x) \frac{\partial u_\varepsilon(x)}{\partial x_i} = \sum_{i=1}^{N} A^i(x) \frac{\partial}{\partial x_i} \left(\int_Q v(x, y) \, dy + v_1 \left(x, \frac{x}{\varepsilon} \right) \right)$$

$$= \sum_{i=1}^{N} A^i(x) \frac{\partial v}{\partial x_i} \left(x, \frac{x}{\varepsilon} \right) \rightharpoonup \sum_{i=1}^{N} A^i(x) \frac{\partial}{\partial x_i} \int_Q v(x, y) \, dy = 0$$

weakly in $L^p(\Omega; \mathbb{R}^l)$. Hence v satisfies (C2).

In the general case in which $v_1 \in L^p(\Omega \times Q; \mathbb{R}^d)$, we first need to approximate v_1 in order to keep the periodicity condition during the subsequent regularization. To this purpose, we extend v_1 to 0 outside $\Omega \times Q$, we consider a sequence $\{\varphi_j\} \in C_c^\infty(Q)$ such that $0 \le \varphi_j \le 1$ and $\varphi_j \to 1$ pointwise, and we define the maps

$$v_1^j(x, y) := \varphi_j(y) v_1(x, y) \qquad \text{for a.e. } x \in \Omega \text{ and } y \in Q.$$

Extend these maps to $\Omega \times \mathbb{R}^N$ by periodicity. It is straightforward to see that

$$v_1^j \to v_1 \qquad \text{strongly in } L^p\big(\Omega; L_{\text{per}}^p(\mathbb{R}^N; \mathbb{R}^d)\big) \tag{4.14}$$

by the dominated convergence theorem. Moreover

$$\|\mathscr{A}_y v_1^j\|_{W^{-1,p}(Q; \mathbb{R}^l)} \to 0 \qquad \text{strongly in } L^p(\Omega). \tag{4.15}$$

Indeed, by (4.14), and since $\mathscr{A}_y v = 0$,

$$\|\mathscr{A}_y(x) v_1^j(x, \cdot)\|_{W^{-1,p}(Q; \mathbb{R}^l)} \to 0 \qquad \text{for a.e. } x \in \Omega,$$

and

$$\|\mathscr{A}_y(x) v_1^j(x, \cdot)\|_{W^{-1,p}(Q; \mathbb{R}^l)}^p \le C \|\varphi_j\|_{L^\infty(Q)}^p \|v_1(x, y)\|_{L^p(Q; \mathbb{R}^d)}^p \le C \|v_1(x, y)\|_{L^p(Q; \mathbb{R}^d)}^p$$

for a.e. $x \in \Omega$. Thus (4.15) follows by the dominated convergence theorem.

Convolving first with respect to y and then with respect to x we construct a sequence $\{v_1^{\delta, j}\} \in C^\infty\big(\Omega; C_{\text{per}}^\infty(\mathbb{R}^N; \mathbb{R}^d)\big)$ such that

$$v_1^{\delta, j} \to v_1^j \qquad \text{strongly in } L^p\big(\Omega; L_{\text{per}}^p(\mathbb{R}^N; \mathbb{R}^d)\big), \tag{4.16}$$

and

$$\|\mathscr{A}_y v_1^{\delta, j}\|_{W^{-1,p}(Q; \mathbb{R}^l)} \to 0 \qquad \text{strongly in } L^p(\Omega), \tag{4.17}$$

as $\delta \to 0$. In view of (4.14)–(4.17), a diagonal argument provides a subsequence $\{\delta(j)\}$ such that $\{v_1^{\delta(j), j}\}$ satisfies

$$v_1^{\delta(j), j} \to v_1 \qquad \text{strongly in } L^p\big(\Omega; L_{\text{per}}^p(\mathbb{R}^N; \mathbb{R}^d)\big) \tag{4.18}$$

and

$$\|\mathscr{A}_y v_1^{\delta(j),j}\|_{W^{-1,p}(Q;\mathbb{R}^l)} \to 0 \qquad \text{strongly in } L^p(\Omega). \tag{4.19}$$

Set

$$w^j(x,y) := \Pi(x)\left(v_1^{\delta(j),j}(x,y) - \int_Q v_1^{\delta(j),j}(x,y)\,dy\right)$$

for a.e. $x \in \Omega$ and $y \in Q$. By Lemma 4.1 we have $w^j \in C^\infty\left(\Omega; C_{\text{per}}^\infty(\mathbb{R}^N;\mathbb{R}^d)\right)$,

$$\mathscr{A}_y w^j = 0 \qquad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x \in \Omega, \tag{4.20}$$

and

$$\begin{aligned}
\|w^j - v_1\|_{L^p(\Omega\times Q;\mathbb{R}^d)}^p &\le C\Big(\Big\|w^j - v_1^{\delta(j),j} + \int_Q v_1^{\delta(j),j}(x,y)\,dy\Big\|_{L^p(\Omega\times Q;\mathbb{R}^d)}^p \\
&\qquad + \Big\|v_1^{\delta(j),j} - v_1 - \int_Q v_1^{\delta(j),j}(x,y)\,dy\Big\|_{L^p(\Omega\times Q;\mathbb{R}^d)}^p\Big) \\
&\le C\Big(\int_\Omega \|\mathscr{A}_y(x)v_1^{\delta(j),j}(x,y)\|_{W^{-1,p}(Q;\mathbb{R}^l)}^p\,dx \\
&\qquad + \Big\|v_1^{\delta(j),j} - v_1 - \int_Q v_1^{\delta(j),j}(x,y)\,dy\Big\|_{L^p(\Omega\times Q;\mathbb{R}^d)}^p\Big).
\end{aligned}$$

Therefore, in view of (4.18) and (4.19),

$$w^j \to v_1 \qquad \text{strongly in } L^p(\Omega \times Q;\mathbb{R}^d). \tag{4.21}$$

We set

$$u_\varepsilon^j(x) := \int_Q v(x,y)\,dy + w^j\left(x,\frac{x}{\varepsilon}\right) \qquad \text{for a.e. } x \in \Omega.$$

By Proposition 2.4 and (4.21),

$$u_\varepsilon^j \xrightarrow{2-s} v \qquad \text{strongly two-scale in } L^p(\Omega \times Q;\mathbb{R}^d) \tag{4.22}$$

as $\varepsilon \to 0$ and $j \to +\infty$, in this order. Moreover, by (4.20) and since $v \in \mathscr{F}$,

$$\mathscr{A}u_\varepsilon^j = \sum_{i=1}^N A^i(x)\frac{\partial w^j}{\partial x_i}\left(x,\frac{x}{\varepsilon}\right).$$

By (4.21), Proposition 2.4, and the compact embedding of $W^{-1,p}$ into L^p, we conclude that

$$\mathscr{A}u_\varepsilon^j \to \mathscr{A}\int_Q v_1(x,y)\,dy = 0 \tag{4.23}$$

strongly in $W^{-1,p}(\Omega;\mathbb{R}^l)$, as $\varepsilon \to 0$ and $j \to +\infty$, in this order. By (4.22), (4.23), and Theorem 2.7 it follows in particular that

$$\lim_{j\to+\infty}\lim_{\varepsilon\to 0}(\|T_\varepsilon u_\varepsilon^j - v\|_{L^p(\Omega\times Q;\mathbb{R}^d)} + \|\mathscr{A}u_\varepsilon^j\|_{W^{-1,p}(\Omega;\mathbb{R}^l)}) = 0.$$

Attouch's diagonalization lemma [3], Lemma 1.15 and Corollary 1.16 provide us with a subsequence $\{j(\varepsilon)\}$ such that, setting $u_\varepsilon := u_\varepsilon^{j(\varepsilon)}$, there holds

$$T_\varepsilon u_\varepsilon \xrightarrow{2-s} v \quad \text{strongly two-scale in } L^p(\Omega \times Q;\mathbb{R}^d)$$

and

$$\mathscr{A}u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l).$$

The thesis follows applying again Theorem 2.7. □

In order to state the main result of this section we introduce the classes

$$\mathscr{U} := \{u \in L^p(\Omega;\mathbb{R}^d) : \mathscr{A}u = 0\} \tag{4.24}$$

and

$$\mathscr{W} := \left\{w \in L^p(\Omega \times Q;\mathbb{R}^d) : \int_Q w(x,y)\,dy = 0 \text{ and } \mathscr{A}_y w = 0\right\}. \tag{4.25}$$

It is clear that $v \in \mathscr{F}$ if and only if

$$\int_Q v(x,y)\,dy \in \mathscr{U} \quad \text{and} \quad v - \int_Q v(x,y)\,dy \in \mathscr{W}.$$

Let $\tilde{\mathscr{E}}_{\text{hom}} : L^p(\Omega;\mathbb{R}^d) \to L^p(\Omega;\mathbb{R}^d)$ be the functional

$$\tilde{\mathscr{E}}_{\text{hom}}(u) := \begin{cases} \liminf_{n\to+\infty}\inf_{w\in\mathscr{W}}\int_\Omega\int_Q f(x,ny,u(x)+w(x,y))\,dy\,dx \\ \qquad\qquad\qquad\qquad\qquad \text{if } u \in \mathscr{U}, \\ +\infty \qquad\qquad\qquad\qquad\quad \text{otherwise in } L^p(\Omega;\mathbb{R}^d). \end{cases} \tag{4.26}$$

We now provide a first characterization of (4.2).

Theorem 4.3. *Under the assumptions of Theorem 1.2, for every* $u \in L^p(\Omega; \mathbb{R}^d)$ *there holds*

$$
\inf \left\{ \liminf_{\varepsilon \to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \right.
$$
$$
\left. \text{and } \mathscr{A}u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l) \right\}
$$
$$
= \inf \left\{ \limsup_{\varepsilon \to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \right.
$$
$$
\left. \text{and } \mathscr{A}u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l) \right\}
$$
$$
= \tilde{\mathscr{E}}_{\text{hom}}(u). \tag{4.27}
$$

We subdivide the proof of Theorem 4.3 into the proof of a limsup inequality (Corollary 4.5) and a liminf inequality (Propositions 4.6 and 4.7).

We first show how an adaptation of the construction in Lemma 4.2 yields an outline for proving the limsup inequality in (4.27).

Proposition 4.4. *Under the assumptions of Theorem 1.2, for every* $n \in \mathbb{N}$, $u \in \mathscr{U}$ *and* $w \in \mathscr{W}$ *there exists a sequence* $\{u_\varepsilon\} \in \mathcal{S}_{u+w}$ *(see (4.10)) such that*

$$
u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d), \tag{4.28}
$$
$$
\limsup_{\varepsilon \to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) dx \le \int_\Omega \int_Q f(x, ny, u(x) + w(x, y)) \, dy \, dx. \tag{4.29}
$$

Proof. Step 1: We first assume that $u \in C(\Omega; \mathbb{R}^d)$ and $w \in C^1(\Omega; C^1_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d))$. Arguing as in [11], Proof of Proposition 2.7 we introduce the auxiliary function

$$
g(x, y) := f(x, ny, u(x) + w(x, y))
$$

for every $x \in \Omega$ and for a.e. $y \in \mathbb{R}^N$. By definition, $g \in C(\Omega; L^p_{\text{per}}(\mathbb{R}^N))$. Hence, setting

$$
g_\varepsilon(x) := g\left(x, \frac{x}{n\varepsilon}\right) \quad \text{for a.e. } x \in \Omega,
$$

Proposition 2.4 yields

$$
\lim_{\varepsilon \to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u(x) + w\left(x, \frac{x}{n\varepsilon}\right)\right) dx = \lim_{\varepsilon \to 0} \int_\Omega g_\varepsilon(x) \, dx = \int_\Omega \int_Q g(x, y) \, dy \, dx
$$
$$
= \int_\Omega \int_Q f(x, ny, u(x) + w(x, y)) \, dy \, dx.
$$

Define

$$u_\varepsilon(x) := u(x) + w\left(x, \frac{x}{n\varepsilon}\right) \quad \text{for a.e. } x \in \Omega.$$

By the periodicity of w in the second variable and by the definition of $\mathscr{W}$,

$$u_\varepsilon \rightharpoonup u + \int_Q w(x, y)\, dy = u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d).$$

By Proposition 2.4, $u_\varepsilon \xrightarrow{2-s} u + w$ strongly two-scale in $L^p(\Omega \times Q; \mathbb{R}^d)$. Finally (recalling the definitions of the classes $\mathscr{U}$ and $\mathscr{W}$) by the regularity of w and by Proposition 2.4,

$$\mathscr{A} u_\varepsilon = \sum_{i=1}^N A^i(x) \frac{\partial w}{\partial x_i}\left(x, \frac{x}{n\varepsilon}\right) \rightharpoonup \sum_{i=1}^N A^i(x) \frac{\partial}{\partial x_i} \int_Q w(x, y)\, dy = 0$$

weakly in $L^p(\Omega; \mathbb{R}^l)$ and hence strongly in $W^{-1,p}(\Omega; \mathbb{R}^l)$, due to the compact embedding of L^p into $W^{-1,p}$.

Step 2: Consider the general case in which $u \in \mathscr{U}$ and $w \in \mathscr{W}$. Arguing as in the second part of the proof of Lemma 4.2 (up to (4.21)), we construct a sequence $\{w^j\} \in C^1(\Omega; C^1_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d))$ such that

$$w^j \to u + w \quad \text{strongly in } L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)), \tag{4.30}$$

and

$$\mathscr{A}_y w^j = 0 \quad \text{in } W^{-1,p}(\Omega; \mathbb{R}^l) \text{ for a.e. } x \in \Omega, \text{ for every } j.$$

Set

$$u^j_\varepsilon(x) := w^j\left(x, \frac{x}{n\varepsilon}\right) \quad \text{for a.e. } x \in \Omega.$$

By Proposition 2.4, there holds

$$u^j_\varepsilon \xrightarrow{2-s} u + w \quad \text{strongly two-scale in } L^p(\Omega \times Q; \mathbb{R}^d), \tag{4.31}$$

as $\varepsilon \to 0$ and $j \to +\infty$, in this order. In addition, arguing as in the proof of (4.23), we have

$$\mathscr{A} u^j_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l) \tag{4.32}$$

as $\varepsilon \to 0$ and $j \to +\infty$, in this order.

To conclude, it remains to study the asymptotic behavior of the energies associated to the sequence $\{u_\varepsilon^j\}$. Consider the functions

$$g^j(x, y) := f\left(x, ny, w^j(x, y)\right)$$

and

$$g_\varepsilon^j(x) := g^j\left(x, \frac{x}{n\varepsilon}\right).$$

Arguing as in Step 1, we obtain

$$\lim_{j\to+\infty} \lim_{\varepsilon\to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon^j(x)\right) dx$$
$$= \lim_{j\to+\infty} \lim_{\varepsilon\to 0} \int_\Omega g_\varepsilon^j(x)\, dx = \lim_{j\to+\infty} \int_\Omega \int_Q g^j(x, y)\, dy\, dx$$
$$= \int_\Omega \int_Q f\left(x, ny, u(x) + w(x, y)\right) dy\, dx \qquad (4.33)$$

where we used the periodicity of g_ε^j, together with (1.9) and (4.30). In view of (4.31)–(4.33), Attouch's diagonalization lemma [3], Lemma 1.15 and Corollary 1.16, and Theorem 2.7, we obtain a subsequence $\{j(\varepsilon)\}$ such that $u_\varepsilon := u_\varepsilon^{j(\varepsilon)}$ satisfies both (4.28) and (4.29). $\square$

Proposition 4.4 yields the following limsup inequality.

Corollary 4.5. *Under the assumptions of Theorem* 1.2, *for every* $u \in L^p(\Omega; \mathbb{R}^d)$

$$\inf\left\{ \limsup_{\varepsilon\to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \right.$$
$$\left. \text{and } \mathscr{A}u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l) \right\}$$

$$\leq \tilde{\mathscr{E}}_{\mathrm{hom}}(u).$$

We now turn to the proof of the liminf inequality in Theorem 4.3. For simplicity, we subdivide it into two intermediate results.

Proposition 4.6. *Under the assumptions of Theorem* 1.2, *for every sequence* $\varepsilon_n \to 0^+$, $u \in \mathscr{U}$ *and* $\{u_n\} \in L^p(\Omega; \mathbb{R}^d)$ *with* $u_n \rightharpoonup 0$ *weakly in* $L^p(\Omega; \mathbb{R}^d)$ *and* $\mathscr{A}u_n \to 0$, *there exists a* p-*equiintegrable family of functions*

$$\mathscr{V} := \{v_{v,n} : v, n \in \mathbb{N}\}$$

such that $\mathcal{V}$ is a bounded subset of $L^p(\Omega; \mathbb{R}^d)$, for every $v \in \mathbb{N}$ and as $n \to +\infty$

$$v_{v,n} \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d),$$

$$\mathcal{A}v_{v,n} \to 0 \quad \text{strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \text{ for every } 1 < q < p.$$

Furthermore,

$$\liminf_{n \to +\infty} \int_\Omega f\left(x, \frac{x}{\varepsilon_n}, u(x) + u_n(x)\right) dx$$

$$\geq \sup_{v \in \mathbb{N}} \left\{ \liminf_{n \to +\infty} \int_\Omega f\left(x, vnx, u(x) + v_{v,n}(x)\right) dx \right\}.$$

Proof. The proof follows the argument of [11], Proof of Proposition 3.8. We sketch the main steps for the convenience of the reader.

Step 1: We first truncate our sequence in order to achieve p-equiintegrability. Without loss of generality, up to translations and dilations we can assume that $\Omega \subset Q$. Arguing as in [12], Proof of Lemma 2.15, we construct a p-equiintegrable sequence $\{\tilde{u}_n\} \in L^p(\Omega; \mathbb{R}^d)$ such that

$$\tilde{u}_n - u_n \to 0 \quad \text{strongly in } L^q(\Omega; \mathbb{R}^d) \text{ for every } 1 < q < p,$$

$$\tilde{u}_n \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d),$$

$$\mathcal{A}\tilde{u}_n \to 0 \quad \text{strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \text{ for every } 1 < q < p,$$

and

$$\liminf_{n \to +\infty} \int_\Omega f\left(x, \frac{x}{\varepsilon_n}, u(x) + u_n(x)\right) dx \geq \liminf_{n \to +\infty} \int_\Omega f\left(x, \frac{x}{\varepsilon_n}, u(x) + \tilde{u}_n(x)\right) dx.$$

Step 2: we consider the sequence

$$\tilde{k}_{v,n} := \frac{1}{v\varepsilon_n}.$$

If $\{\tilde{k}_{v,n}\}$ is a sequence of integers (without loss of generality we can assume that it is increasing as n increases), then there is nothing to prove and we simply set

$$v_{v,k} := \begin{cases} \tilde{u}_n & \text{if } k = \tilde{k}_{v,n}, \\ 0 & \text{otherwise.} \end{cases}$$

In the case in which $\{\tilde{k}_{v,n}\}$ is not a sequence of integers, we define

$$k_{v,n} := \frac{\theta_{v,n}}{v\varepsilon_n},$$

where

$$\theta_{v,n} := v\varepsilon_n \left\lfloor \frac{1}{v\varepsilon_n} \right\rfloor.$$

In particular

$$\theta_{v,n} \to 1 \qquad \text{as } n \to +\infty. \tag{4.34}$$

An adaptation of [11], Lemma 2.8 applied to $\{\tilde{u}_n\}$ yields a p-equiintegrable sequence $\{\bar{u}_n\} \in L^p(Q; \mathbb{R}^d)$ such that

$$\tilde{u}_n - \bar{u}_n \to 0 \qquad \text{strongly in } L^p(\Omega; \mathbb{R}^d),$$

$$\tilde{u}_n \rightharpoonup 0 \qquad \text{weakly in } L^p(Q \backslash \Omega; \mathbb{R}^d),$$

$$\mathscr{A}\bar{u}_n \to 0 \qquad \text{strongly in } W^{-1,q}(Q; \mathbb{R}^l) \text{ for every } 1 < q < p. \tag{4.35}$$

Arguing as in [11], Proof of Proposition 3.8 we obtain

$$\liminf_{n\to+\infty} \int_\Omega f\left(x, \frac{x}{\varepsilon_n}, u + u_n(x)\right) dx \geq \liminf_{n\to+\infty} \int_\Omega f\left(x, vk_{v,n}x, u(x) + v_{v,k_{v,n}}(x)\right) dx,$$

where

$$v_{v,k_{v,n}}(x) := \bar{u}_n(\theta_{v,n}x) \qquad \text{for a.e. } x \in \Omega,$$

$v \in \mathbb{N}$ and $n \in \mathbb{N}$ are large enough so that $\theta_{v,n}\Omega \subset Q$. Setting

$$v_{v,n} := \begin{cases} v_{v,k_{v,n}} & \text{if } n = k_{v,n}, \\ 0 & \text{otherwise,} \end{cases}$$

the sequence $\{v_{v,n}\}$ is uniformly bounded in $L^p(\Omega; \mathbb{R}^d)$, p-equiintegrable, and satisfies

$$v_{v,n} \rightharpoonup 0 \qquad \text{weakly in } L^p(\Omega; \mathbb{R}^d)$$

as $n \to +\infty$. To conclude, it remains only to show that

$$\mathscr{A}v_{v,n} \to 0 \qquad \text{strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \text{ for every } 1 < q < p \tag{4.36}$$

as $n \to +\infty$.

Let q as above be fixed, and let $\varphi \in W_0^{1,q'}(\Omega; \mathbb{R}^l)$. A change of variables yields

$$|\langle \mathscr{A} v_{v,k_{v,n}}, \varphi \rangle|$$

$$= \left| \int_\Omega \left(\sum_{i=1}^N A^i(x) \bar{u}_n(\theta_{v,n}x) \cdot \frac{\partial \varphi(x)}{\partial x_i} + \sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i} \bar{u}_n(\theta_{v,n}x) \cdot \varphi(x) \right) dx \right|$$

$$= \frac{1}{\theta_{v,n}^N} \left| \int_{\theta_{v,n}\Omega} \left(\sum_{i=1}^N A^i\left(\frac{y}{\theta_{v,n}}\right) \bar{u}_n(y) \cdot \frac{\partial \varphi}{\partial x_i}\left(\frac{y}{\theta_{v,n}}\right) \right. \right.$$

$$\left. \left. + \sum_{i=1}^N \frac{\partial A^i}{\partial x_i}\left(\frac{y}{\theta_{v,n}}\right) \bar{u}_n(y) \cdot \varphi\left(\frac{y}{\theta_{v,n}}\right) \right) dy \right|.$$

For n big enough $\theta_{v,n}\Omega \subset Q$. Hence, by (4.34), adding and subtracting the quantity

$$\frac{1}{\theta_{v,n}^N} \int_{\theta_{v,n}\Omega} \left(\sum_{i=1}^N A^i(y) \bar{u}_n(y) \cdot \frac{\partial \varphi}{\partial x_i}\left(\frac{y}{\theta_{v,n}}\right) + \sum_{i=1}^N \frac{\partial A^i}{\partial x_i}(y) \bar{u}_n(y) \cdot \varphi\left(\frac{y}{\theta_{v,n}}\right) \right) dy,$$

we deduce the upper bound

$$|\langle \mathscr{A} v_{v,n}, \varphi \rangle|$$

$$\leq C \sum_{i=1}^N \left\| A^i(y) - A^i\left(\frac{y}{\theta_{v,n}}\right) \right\|_{C^0(\bar{Q}; \mathbb{M}^{l\times d})} \|\bar{u}_n\|_{L^q(Q;\mathbb{R}^d)} \|\varphi\|_{W_0^{1,q'}(\Omega;\mathbb{R}^l)}$$

$$+ C \sum_{i=1}^N \left\| \frac{\partial A^i}{\partial y_i}(y) - \frac{\partial A^i}{\partial x_i}\left(\frac{y}{\theta_{v,n}}\right) \right\|_{C^0(\bar{Q}; \mathbb{M}^{l\times d})} \|\bar{u}_n\|_{L^q(Q;\mathbb{R}^d)} \|\varphi\|_{W_0^{1,q'}(\Omega;\mathbb{R}^l)}$$

$$+ C \|\mathscr{A} \bar{u}_n\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} \|\varphi\|_{W_0^{1,q'}(\Omega;\mathbb{R}^l)}.$$

Property (4.36) follows now by (4.34) and (4.35). □

To complete the proof of the liminf inequality in (4.27) we apply the *unfolding operator* (see Subsection 2.3) to the set $\mathscr{V}$ constructed in Proposition 4.6.

Proposition 4.7. *Under the assumptions of Theorem* 1.2, *for every* $u \in \mathscr{U}$ *and every family* $\mathscr{V} = \{v_{v,n} : v, n \in \mathbb{N}\}$ *as in Proposition* 4.6 *there holds*

$$\liminf_{v \to +\infty} \liminf_{n \to +\infty} \int_\Omega f(x, vnx, u(x) + v_{v,n}(x)) \, dx \geq \tilde{\mathscr{E}}_{\hom}(u).$$

Proof. Fix $u \in \mathcal{U}$ and let $\{v_{\nu,n} : \nu, n \in \mathbb{N}\}$ be p-equiintegrable and bounded in $L^p(\Omega; \mathbb{R}^d)$, with

$$v_{\nu,n} \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \tag{4.37}$$

and

$$\mathscr{A}v_{\nu,n} \to 0 \quad \text{strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \text{ for every } 1 < q < p, \tag{4.38}$$

as $n \to +\infty$, for every $\nu \in \mathbb{N}$. Fix $\Omega' \subset\subset \Omega$ and for $z \in \mathbb{Z}^N$ and $n \in \mathbb{N}$, define

$$Q_{\nu,z} := \frac{z}{\nu} + \frac{1}{\nu}Q,$$

and

$$Z^\nu := \{z \in \mathbb{Z}^N : Q_{\nu,z} \cap \Omega' \neq \emptyset\}.$$

We consider the maps

$$T_{1/\nu}v_{\nu,n}(x, y) := v_{\nu,n}\left(\frac{1}{\nu}\lfloor \nu x \rfloor + \frac{1}{\nu}y\right) \quad \text{for a.e. } x \in \Omega, \ y \in Q,$$

where we have extended the sequence $\{v_{\nu,n}\}$ to zero outside Ω. A change of variables yields

$$\int_\Omega f\big(x, \nu n x, u(x) + v_{\nu,n}(x)\big)\, dx$$

$$\geq \sum_{z \in Z^\nu} \int_{Q_{\nu,z}} f\big(x, \nu n x, u(x) + v_{\nu,n}(x)\big)\, dx$$

$$= \nu^N \sum_{z \in Z^\nu} \int_Q f\left(\frac{z}{\nu} + \frac{y}{\nu}, ny, u\left(\frac{z}{\nu} + \frac{y}{\nu}\right) + v_{\nu,n}\left(\frac{z}{\nu} + \frac{y}{\nu}\right)\right) dy$$

$$= \sum_{z \in Z^\nu} \int_{Q_{\nu,z}} \int_Q f\left(\frac{\lfloor \nu x \rfloor}{\nu} + \frac{y}{\nu}, ny, T_{1/\nu}u(x, y) + T_{1/\nu}v_{\nu,n}(x, y)\right) dy\, dx$$

$$\geq \sum_{z \in Z^\nu} \int_{Q_{\nu,z} \cap \Omega'} \int_Q f\left(\frac{\lfloor \nu x \rfloor}{\nu} + \frac{y}{\nu}, ny, T_{1/\nu}u(x, y) + T_{1/\nu}v_{\nu,n}(x, y)\right) dy\, dx,$$

where the last inequality is due to (1.9). By [11], Proposition 3.6 (i) and Proposition 2.8 we conclude that

$$\int_\Omega f\big(x, vnx, u(x) + v_{v,n}(x)\big)\, dx$$

$$\geq \sigma_v + \sum_{z \in Z^v} \int_{Q_{v,z} \cap \Omega'} \int_Q f\big(x, ny, u(x) + \hat{v}_{v,z,n}(y)\big)\, dy\, dx, \qquad (4.39)$$

where

$$\hat{v}_{v,z,n}(y) := T_{1/v} v_{v,n}\left(\frac{z}{v}, y\right)$$

for a.e. $y \in Q$, and $\sigma_v \to 0$ as $v \to +\infty$. The sequence $\{\hat{v}_{v,z,n}\}$ is p-equiintegrable by [11], Proposition A.2, and is uniformly bounded by (4.37) and Proposition 2.6, since

$$\int_Q |\hat{v}_{v,z,n}(y)|^p\, dy = \frac{1}{v^N} \int_{Q_{v,z}} |v_{v,n}(x)|^p\, dx.$$

By the boundedness of $\{v_{v,n} : v, n \in \mathbb{N}\}$ in $L^p(\Omega; \mathbb{R}^d)$, and by (4.37) there holds

$$\hat{v}_{v,z,n} \rightharpoonup 0 \qquad \text{weakly in } L^p(Q; \mathbb{R}^d) \qquad (4.40)$$

as $n \to +\infty$, for every $z \in Z^v$, $v \in \mathbb{N}$. Denoting by $\chi_{Q_{v,z} \cap \Omega'}$ the characteristic functions of the sets $Q_{v,z} \cap \Omega'$, we claim that

$$\limsup_{v \to +\infty} \limsup_{n \to +\infty} \left\| \left\| \mathscr{A}_y(x) \sum_{v \in \mathbb{Z}^v} \chi_{Q_{v,z} \cap \Omega'}(x) \hat{v}_{v,z,n}(y) \right\|_{W^{-1,q}(Q; \mathbb{R}^l)} \right\|_{L^q(\Omega)} = 0 \qquad (4.41)$$

for every $1 < q < p$. Indeed, fix $1 < q < p$, and let $\psi \in W_0^{1,q'}(Q; \mathbb{R}^l)$. Then

$$\left| \left\langle \mathscr{A}_y\left(\frac{z}{v}\right) \hat{v}_{v,z,n}, \psi \right\rangle \right| = \left| \int_Q \sum_{i=1}^N A^i\left(\frac{z}{v}\right) v_{v,n}\left(\frac{z}{v} + \frac{y}{v}\right) \cdot \frac{\partial \psi(y)}{\partial y_i}\, dy \right|$$

$$= v^N \left| \int_{Q_{v,z}} \sum_{i=1}^N A^i\left(\frac{z}{v}\right) v_{v,n}(x) \cdot \frac{\partial \psi}{\partial y_i}(vx - z)\, dx \right|.$$

Adding and subtracting to the previous expression the quantity

$$v^N \int_{Q_{v,z}} \sum_{i=1}^N A^i(x) v_{v,n}(x) \cdot \frac{\partial \psi}{\partial y_i}(vx - z)\, dx,$$

and setting $\phi_z^v(x) := \psi(vx - z)$ for a.e. $x \in \Omega$, we obtain the estimate

$$\left| \left\langle \mathscr{A}_y \left(\frac{z}{\nu} \right) \hat{v}_{\nu,z,n}, \psi \right\rangle \right|$$

$$\leq \nu^N \left\| \sum_{i=1}^N \left(A^i \left(\frac{z}{\nu} \right) - A^i(x) \right) v_{\nu,n}(x) \right\|_{L^q(Q_{\nu,z}; \mathbb{R}^l)} \left\| \frac{\partial \psi}{\partial y_i} (\nu x - z) \right\|_{L^{q'}(Q_{\nu,z}; \mathbb{R}^l)}$$

$$+ \nu^{N-1} \left\| \frac{\partial}{\partial x_i} \left(A^i(x) v_{\nu,n}(x) \right) \right\|_{W^{-1,q}(Q_{\nu,z}; \mathbb{R}^l)} \| \phi_z^\nu \|_{W_0^{1,q'}(Q_{\nu,z}; \mathbb{R}^l)}.$$

A change of variables yields the upper bound

$$\left\| \frac{\partial \psi}{\partial y_i} (\nu x - z) \right\|_{L^{q'}(Q_{\nu,z}; \mathbb{R}^l)} + \frac{\| \phi_z^\nu \|_{W_0^{1,q'}(Q_{\nu,z}; \mathbb{R}^l)}}{\nu} \leq \frac{C}{\nu^{N/q'}} \| \psi \|_{W_0^{1,q'}(Q; \mathbb{R}^l)}.$$

Thus, by the regularity of the operators A^i,

$$\left| \left\langle \mathscr{A}_y \left(\frac{z}{\nu} \right) \hat{v}_{\nu,z,n}, \psi \right\rangle \right|$$

$$\leq C \nu^{N/q-1} \left\| \sum_{i=1}^N \frac{\partial A^i}{\partial x_i} \right\|_{L^\infty(Q; \mathbb{M}^{l \times d})} \| v_{\nu,n} \|_{L^q(Q_{\nu,z}; \mathbb{R}^d)} \| \psi \|_{W_0^{1,q'}(Q; \mathbb{R}^l)}$$

$$+ C \nu^{N/q} \left\| \frac{\partial}{\partial x_i} \left(A^i(x) v_{\nu,n}(x) \right) \right\|_{W^{-1,q}(Q_{\nu,z}; \mathbb{R}^l)} \| \psi \|_{W_0^{1,q'}(Q; \mathbb{R}^l)}. \qquad (4.42)$$

Using again the Lipschitz regularity of the operators A^i, $i = 1, \ldots, N$, we deduce

$$\| \mathscr{A}_y(x) \hat{v}_{\nu,z,n}(y) \|_{W^{-1,q}(Q; \mathbb{R}^l)} \leq \sum_{i=1}^N \left\| A^i(x) - A^i \left(\frac{z}{\nu} \right) \right\|_{L^\infty(Q; \mathbb{M}^{l \times d})} \| \hat{v}_{\nu,z,n} \|_{L^q(Q; \mathbb{R}^l)}$$

$$+ \left\| \mathscr{A}_y \left(\frac{z}{\nu} \right) \hat{v}_{\nu,z,n}(y) \right\|_{W^{-1,q}(Q; \mathbb{R}^l)}$$

$$\leq \frac{C}{\nu} \left\| \sum_{i=1}^N \frac{\partial A^i}{\partial x_i} \right\|_{L^\infty(Q; \mathbb{M}^{l \times d})} \| \hat{v}_{\nu,z,n} \|_{L^q(Q; \mathbb{R}^l)}$$

$$+ \left\| \mathscr{A}_y \left(\frac{z}{\nu} \right) \hat{v}_{\nu,z,n}(y) \right\|_{W^{-1,q}(Q; \mathbb{R}^l)} \qquad (4.43)$$

for a.e. $x \in Q_{\nu,z}$. Hence, by (4.42) and (4.43), we obtain

$$\sum_{z\in\mathbb{Z}^\nu}\int_{Q_{\nu,z}\cap\Omega'}\left\|\mathscr{A}_y(x)\hat{v}_{\nu,z,n}(y)\right\|_{W^{-1,q}(Q;\mathbb{R}^l)}^q\,dx$$

$$\leq\sum_{z\in\mathbb{Z}^\nu}\frac{C}{\nu^q}\int_{Q_{\nu,z}\cap\Omega'}\left\|\sum_{i=1}^N\frac{\partial A^i}{\partial x_i}\right\|_{L^\infty(Q;\mathbb{M}^{l\times d})}^q\|\hat{v}_{\nu,z,n}\|_{L^q(Q;\mathbb{R}^d)}^q\,dx$$

$$+\sum_{z\in\mathbb{Z}^\nu}\int_{Q_{\nu,z}\cap\Omega'}\left\|\mathscr{A}_y\left(\frac{z}{\nu}\right)\hat{v}_{\nu,z,n}(y)\right\|_{W^{-1,q}(Q;\mathbb{R}^l)}^q\,dx$$

$$\leq\frac{C}{\nu^q}\left\|\sum_{z\in\mathbb{Z}^\nu}\chi_{Q_{\nu,z}\cap\Omega'}(x)\hat{v}_{\nu,z,n}(y)\right\|_{L^q(\Omega\times Q;\mathbb{R}^d)}^q+\frac{C}{\nu^q}\|v_{\nu,n}\|_{L^q(\Omega;\mathbb{R}^d)}$$

$$+\frac{C}{\nu^q}\sum_{z\in\mathbb{Z}^\nu}\int_{Q_{\nu,z}\cap\Omega'}(x)\hat{v}_{\nu,z,n}\left\|\sum_{i=1}^N\frac{\partial A^iv_{\nu,n}}{\partial x_i}\right\|_{W^{-1,q}(Q_{\nu,z};\mathbb{R}^l)}$$

$$\leq\frac{C}{\nu^q}\left\|\sum_{z\in\mathbb{Z}^\nu}\chi_{Q_{\nu,z}\cap\Omega'}(x)\hat{v}_{\nu,z,n}(y)\right\|_{L^q(\Omega\times Q;\mathbb{R}^d)}^q+\frac{C}{\nu^q}\|v_{\nu,n}\|_{L^q(\Omega;\mathbb{R}^d)}$$

$$+C\nu^{N/q}\left\|\sum_{i=1}^N\frac{\partial A^iv_{\nu,n}}{\partial x_i}\right\|_{W^{-1,q}(\Omega;\mathbb{R}^l)}$$

Property (4.41) follows now by (4.37) and (4.38), and by the compact embedding of L^p into $W^{-1,p}$.

Consider the maps

$$w_{\nu,n}(x,y):=\begin{cases}\Pi(x)\big(\hat{v}_{\nu,z,n}(y)-\int_Q\hat{v}_{\nu,z,n}(\xi)\,d\xi\big) & \text{for }x\in Q_{\nu,z}\cap\Omega',\\ -\int_Q\Pi(x)\big(\hat{v}_{\nu,z,n}(y)-\int_Q\hat{v}_{\nu,z,n}(\xi)\,d\xi\big)\,dy & z\in Z^\nu,\ y\in Q,\\ 0 & \text{otherwise in }\Omega.\end{cases}$$

By Lemma 4.1 the sequence $\{w_{\nu,n}\}$ is p-equiintegrable, and

$$\mathscr{A}_yw_{\nu,n}=0\quad\text{in }W^{-1,p}(Q;\mathbb{R}^l)\text{ for a.e. }x\in\Omega,$$

for all $\nu,n\in N$. In particular, $\{w_{\nu,n}\}\subset\mathscr{W}$. We claim that

$$\left\|w_{\nu,n}(x,y)-\sum_{z\in Z^\nu}\chi_{Q_{\nu,z}\cap\Omega'}(x)\hat{v}_{\nu,z,n}(y)\right\|_{L^q(\Omega\times Q;\mathbb{R}^d)}\to 0\tag{4.44}$$

as $n\to+\infty$, $\nu\to+\infty$, for every $1<q<p$.

In fact, by Lemma 4.1 there holds

$$\left\| \Pi(x)\left(\hat{v}_{v,z,n}(y) - \int_Q \hat{v}_{v,z,n}(\xi)\, d\xi\right) - \hat{v}_{v,z,n}(y) \right\|^q_{L^q(Q;\mathbb{R}^d)}$$

$$\leq C\left(\|\mathscr{A}_y(x)\hat{v}_{v,z,n}(y)\|^q_{W^{-1,q}(Q;\mathbb{R}^l)} + \left| \int_Q \hat{v}_{v,z,n}(y)\, dy \right|^q \right).$$

Therefore

$$\left\| \sum_{z \in \mathbb{Z}^v} \chi_{Q_{v,z} \cap \Omega'}(x)\left(\Pi(x)\left(\hat{v}_{v,z,n}(y) - \int_Q \hat{v}_{v,z,n}(\xi)\, d\xi\right) - \hat{v}_{v,z,n}(y) \right) \right\|^q_{L^q(\Omega \times Q;\mathbb{R}^d)}$$

$$\leq C\left(\sum_{z \in \mathbb{Z}^v} \int_{Q_{v,z} \cap \Omega'} \|\mathscr{A}_y(x)\hat{v}_{v,z,n}(y)\|^q_{W^{-1,q}(Q;\mathbb{R}^l)}\, dx \right.$$

$$\left. + \sum_{z \in \mathbb{Z}^v} \int_{Q_{v,z} \cap \Omega'} \left| \int_Q \hat{v}_{v,z,n}(y)\, dy \right|^q dx \right). \tag{4.45}$$

The first term in the right-hand side of (4.45) converges to zero as $n \to +\infty$ and $v \to +\infty$, in this order, owing to (4.41). The second term in the right-hand side of (4.45) converges to zero as $n \to +\infty$ and $v \to +\infty$, in this order, by the dominated convergence theorem, owing to (4.40) and the uniform boundedness in L^p of $\{\hat{v}_{v,z,n}\}$. Hence, both the left-hand side of (4.45) and the quantity

$$\int_Q \left\{ \sum_{z \in \mathbb{Z}^v} \chi_{Q_{v,z} \cap \Omega'}(x)\Pi(x)\left(\hat{v}_{v,z,n}(y) - \int_Q \hat{v}_{v,z,n}(\xi)\, d\xi\right) \right\} dy \to 0,$$

converge to zero as $n \to +\infty$ and $v \to +\infty$, and we obtain (4.44).

Up to the extraction of a (not relabeled) subsequence, we can assume that

$$\liminf_{v \to +\infty} \liminf_{n \to +\infty} \sum_{z \in Z^v} \int_{Q_{v,z} \cap \Omega'} \int_Q f\left(x, ny, u(x) + \hat{v}_{v,z,n}(y)\right) dy\, dx$$

$$= \lim_{v \to +\infty} \liminf_{n \to +\infty} \sum_{z \in Z^v} \int_{Q_{v,z} \cap \Omega'} \int_Q f\left(x, ny, u(x) + \hat{v}_{v,z,n}(y)\right) dy\, dx. \tag{4.46}$$

Hence, in view of (4.44) and (4.46) we can extract a subsequence $\{n(v)\}$ such that

$$\lim_{v \to +\infty} \liminf_{n \to +\infty} \sum_{z \in Z^v} \int_{Q_{v,z} \cap \Omega'} \int_Q f\left(x, ny, u(x) + \hat{v}_{v,z,n}(y)\right) dy\, dx \tag{4.47}$$

$$= \lim_{v \to +\infty} \sum_{z \in Z^v} \int_{Q_{v,z} \cap \Omega'} \int_Q f\left(x, n(v)y, u(x) + \hat{v}_{v,z,n(v)}(y)\right) dy\, dx, \tag{4.48}$$

and

$$w_{v,n(v)}(x,y) - \sum_{z \in Z^v} \chi_{Q_{v,z} \cap \Omega'}(x)\hat{v}_{v,z,n(v)}(y) \to 0 \qquad \text{strongly in } L^q(\Omega \times Q; \mathbb{R}^d),$$

$$(4.49)$$

for every $1 < q < p$. Going back to (4.39), by [11], Proposition 3.5 (ii), (4.47) and (4.49),

$$\liminf_{v \to +\infty} \liminf_{n \to +\infty} \int_\Omega f(x, vnx, u(x) + v_{v,n}(x)) \, dx$$

$$\geq \liminf_{v \to +\infty} \int_{\Omega'} \int_Q f(x, n(v)y, u(x) + w_{v,n(v)}(x,y)) \, dy \, dx.$$

By the p-equiintegrability of $\{w_{v,n(v)}\}$ and by (1.9), letting $|\Omega \setminus \Omega'|$ tend to zero, we conclude

$$\liminf_{v \to +\infty} \liminf_{n \to +\infty} \int_\Omega f(x, vnx, u(x) + v_{v,n}(x)) \, dx$$

$$\geq \liminf_{v \to +\infty} \int_\Omega \int_Q f(x, n(v)y, u(x) + w_{v,n(v)}(x,y)) \, dy \, dx$$

$$\geq \liminf_{v \to +\infty} \inf_{w \in \mathcal{W}} \int_\Omega \int_Q f(x, n(v)y, w(x,y)) \, dy \, dx$$

$$\geq \liminf_{n \to +\infty} \inf_{w \in \mathcal{W}} \int_\Omega \int_Q f(x, ny, w(x,y)) \, dy \, dx = \mathcal{E}_{\text{hom}}(u). \qquad \square$$

Proof of Theorem 4.3. The proof follows by combining Corollary 4.5 with Propositions 4.6 and 4.7. $\qquad \square$

Corollary 4.8. *Under the same assumptions of Theorem 4.3, for every $u \in \mathcal{U}_F$*

$$\tilde{\mathcal{E}}_{\text{hom}}(u) = \mathcal{E}_{\text{hom}}(u) := \int_\Omega f_{\text{hom}}(x, u(x)) \, dx,$$

where

$$f_{\text{hom}}(x, u(x)) = \liminf_{n \to +\infty} \inf_{v \in \mathcal{C}_x} \int_Q f(x, ny, u(x) + v(y)) \, dy,$$

and $\mathcal{C}_x$ is the class defined in (1.5).

Proof. We omit the proof of this corollary as it follows from [11], Remark 3.3 (ii) and by adapting the arguments in [11], Corollary 3.2 and Lemma 4.9 below. $\qquad \square$

Proof of Theorem 1.2. The thesis results from Theorem 4.3 and Corollary 4.8. $\qquad \square$

We conclude this section by showing that Theorem 4.3 yields a relaxation result in the framework of $\mathscr{A}$-quasiconvexity with variable coefficients. Before stating the corollary, we prove a preliminary lemma which guarantees the measurability of the function $x \mapsto Q_\mathscr{A} f(x, u(x))$ for every $u \in L^p(\Omega; \mathbb{R}^d)$.

Lemma 4.9. *Let $1 < p < +\infty$, $u \in L^p(\Omega; \mathbb{R}^d)$, let $\mathscr{A}$ be as in Theorem 1.2, and let $f : \Omega \times \mathbb{R}^d \to [0, +\infty)$ be a Carathéodory function satisfying*

$$0 \leq f(x, \xi) \leq C(1 + |\xi|^p) \quad \text{for a.e. } x \in \Omega \times \mathbb{R}^d, \text{ and for all } \xi \in \mathbb{R}^d.$$

Then the map

$$x \mapsto Q_\mathscr{A} f(x, u(x))$$

is measurable in Ω.

Proof. We first remark that

$$Q_\mathscr{A} f(x, u(x)) = \inf_{r \in (0, +\infty)} Q_\mathscr{A}^r f(x, u(x)) \quad \text{for a.e. } x \in \Omega, \tag{4.50}$$

where

$$Q_\mathscr{A}^r f(x, u(x)) := \inf\left\{ \int_Q f(x, u(x) + w(y)) \, dy : w \in \mathcal{C}_x \text{ and } \|w\|_{L^p(Q; \mathbb{R}^d)} \leq r \right\},$$

and $\mathcal{C}_x$ is the class defined in (1.5). Clearly

$$Q_\mathscr{A}^r f(x, u(x)) \geq Q_\mathscr{A} f(x, u(x))$$

for a.e. $x \in \Omega$, for every $r \in \mathbb{N}$. Moreover, for every $\varepsilon > 0$ there exists $w_\varepsilon \in \mathcal{C}_x$ such that

$$Q_\mathscr{A} f(x, u(x)) \geq \int_Q f(x, u(x) + w_\varepsilon(y)) \, dy - \varepsilon$$

$$\geq Q_\mathscr{A}^{\|w_\varepsilon\|_{L^p(Q; \mathbb{R}^d)}} f(x, u(x)) - \varepsilon \geq \inf_{r \in \mathbb{N}} Q_\mathscr{A}^r f(x, u(x)) - \varepsilon,$$

which in turn implies the second inequality in (4.50).

By (4.50) it is enough to show that $x \mapsto Q_\mathscr{A}^r f(x, u(x))$ is measurable for every $r \in \mathbb{N}$. We claim that

$$Q_\mathscr{A}^r f(x, u(x)) = \sup_{n \in \mathbb{N}} Q_\mathscr{A}^{r,n} f(x, u(x)) \quad \text{for a.e. } x \in \Omega, \tag{4.51}$$

where

$$Q_{\mathscr{A}}^{r,n} f(x, u(x)) = \inf\Big\{ \int_Q f(x, u(x) + w(y))\, dy + n\|\mathscr{A}_y(x)w(x, \cdot)\|_{W^{-1,p}(Q;\mathbb{R}^l)} :$$

$$w \in L^p(Q; \mathbb{R}^d), \int_Q w(y)\, dy = 0 \text{ and } \|w\|_{L^p(Q;\mathbb{R}^d)} \le r\Big\}.$$

Clearly,

$$Q_{\mathscr{A}}^{r,n} f(x, u(x)) \le Q_{\mathscr{A}}^r f(x, u(x))$$

for a.e. $x \in \Omega$, for all $n \in \mathbb{N}$. To prove the opposite inequality, fix $x \in \Omega$, and for every $n \in \mathbb{N}$, let $w_n \in L^p(Q; \mathbb{R}^d)$, with $\int_Q w_n(y)\, dy = 0$ and $\|w_n\|_{L^p(Q;\mathbb{R}^d)} \le r$, be such that

$$\int_Q f(x, u(x) + w_n(y))\, dy + n\|\mathscr{A}_y(x)w_n\|_{W^{-1,p}(Q;\mathbb{R}^l)}$$

$$\le Q_{\mathscr{A}}^{r,n} f(x, u(x)) + \frac{1}{n} \le f(x, u(x)) + \frac{1}{n} \qquad (4.52)$$

(the last inequality holds because $0 \in C_x$ for every $x \in \Omega$). Since $\{w_n\}$ is uniformly bounded in $L^p(Q; \mathbb{R}^d)$ and by (4.52)

$$\|\mathscr{A}_y(x)w_n\|_{W^{-1,p}(Q;\mathbb{R}^l)} \to 0$$

as $n \to +\infty$, there exists a map $w \in L^p(Q; \mathbb{R}^d)$, with $\int_Q w(y)\, dy = 0$, $\|w\|_{L^p(Q;\mathbb{R}^d)} \le r$ and $\mathscr{A}_y(x)w = 0$ such that

$$w_n \rightharpoonup w \qquad \text{weakly in } L^p(Q; \mathbb{R}^d).$$

By [5], Lemma 3.1 we can construct a sequence $\{\tilde{w}_n\}$ such that $\int_Q \tilde{w}_n(y)\, dy = 0$, $\mathscr{A}_y(x)\tilde{w}_n = 0$ for every $n \in \mathbb{N}$, and

$$\liminf_{n \to +\infty} \int_Q f(x, u(x) + \tilde{w}_n(y))\, dy \le \liminf_{n \to +\infty} \int_Q f(x, u(x) + w_n(y))\, dy$$

$$\le \sup_{n \in \mathbb{N}} Q_{\mathscr{A}}^{r,n} f(x, u(x)).$$

In view of (4.50) we have

$$Q_{\mathscr{A}}^r f(x, u(x)) \le Q_{\mathscr{A}} f(x, u(x)) \le \int_Q f(x, u(x) + \tilde{w}_n(y))\, dy \qquad \text{for every } n \in \mathbb{N},$$

and we obtain the second inequality in (4.51). By the measurability of

$$x \mapsto Q_{\mathscr{A}}^{r,n} f\big(x, u(x)\big)$$

for every $r, n \in \mathbb{N}$ (we can reduce it to a countable pointwise infimum of measurable functions), we deduce the measurability of

$$x \mapsto Q_{\mathscr{A}}^{r} f\big(x, u(x)\big)$$

for every $r \in \mathbb{N}$, which in turn implies the thesis. $\qquad\square$

For every $D \in \mathcal{O}(\Omega)$ and $u \in L^{p}(\Omega; \mathbb{R}^{d})$, define

$$
\mathcal{I}(u, D) := \inf \Big\{ \liminf_{n \to +\infty} \int_{D} f\big(x, u_n(x)\big) : u_n \rightharpoonup u \quad \text{weakly in } L^{p}(\Omega; \mathbb{R}^{m})
$$
$$
\text{and } \mathscr{A} u_n \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^{l}) \Big\}.
$$
(4.53)

Corollary 4.8 provides us with the following integral representation of $\mathcal{I}$.

Corollary 4.10. *Let $1 < p < +\infty$ and let $\mathscr{A}$ be as in Theorem 1.2. Let $f : \Omega \times \mathbb{R}^{d} \to [0, +\infty)$ be a Carathéodory function satisfying*

$$0 \leq f(x, \xi) \leq C(1 + |\xi|^{p}) \quad \text{for a.e. } x \in \Omega, \text{ and for all } \xi \in \mathbb{R}^{d}.$$

Then

$$\int_{D} Q_{\mathscr{A}} f\big(x, u(x)\big)\, dx = \mathcal{I}(u, D)$$

for all $D \in \mathcal{O}(\Omega)$ and $u \in L^{p}(\Omega; \mathbb{R}^{d})$ with $\mathscr{A} u = 0$.

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E. Davoli, Department of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

E-mail: elisa.davoli@univie.ac.at

I. Fonseca, Department of Mathematics, Carnegie Mellon University, Forbes Avenue, Pittsburgh, PA 15213, USA

E-mail: fonseca@andrew.cmu.edu