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Differential forms in algebraic geometry— A new perspective in the singular case

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(This paper is associated with the plenary talk given by Professor Annette Huber at the 2015 International Meeting AMS-EMS-SPM held in Porto, Portugal.)

Abstract. Differential forms are a rich source of invariants in algebraic geometry. This approach was very successful for smooth varieties, but the singular case is less well-understood. We explain how the use of the h-topology (introduced by Suslin and Voevodsky in order to study motives) gives a very good object also in the singular case, at least in characteristic 0. We also explain problems and solutions in positive characteristic.

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1. Introduction

This note is an extended version of my plenary talk at the AMS-EMS-SPM International Meeting 2015 in Porto. Most of it is aimed at a very general audience. The last sections are more technical and written in a language that assumes a good knowledge of algebraic geometry. We hope that it will be of use for people in the field.

Differential forms originally show up when integrating or differentiating on manifolds. However, the concept also makes perfect sense on algebraic varieties because the derivative of a polynomial is a polynomial.

The object has very many important uses. The one we are concentrating on is as a source of invariants used in order to classify varieties. This approach was very successful for smooth varieties, but the singular case is less well-understood.

We explain how the use of the h-topology (introduced by Suslin and Voevodsky to study motives) gives a very good object also in the singular case, at least in characteristic zero. The approach unifies other ad-hoc notions and simplifies many proofs. We also explain the necessary modifications in positive characteristic and the new problems that show up.

Readers who already know about algebraic differential forms and are convinced that they are important are invited to jump directly to Section 5.

2. Differential forms on algebraic varieties

2.1. Back to calculus. We are all familiar with the notion of a (partial) derivative of a function $f : (a,b) \to \mathbb{R}$ or more generally $f : U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$ open. While there is no denying that this a very useful object—even one of the most importants objects of mathematics—the notion is not very satisfactory from the point of view of the geometric disciplines or physics: it depends on the choice of a coordinate on U!

Differential forms were introduced in order to circumvent this problem. We put

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

A simple computation with the chain rule now shows that $\mathrm{d}f$ is independent of the choice of coordinate system on U. Hence it generalizes to all smooth manifolds: Locally, the differential of a function is given by the above formula. As this is independent of the choice of coordinate, we get a global object. It has a geometric interpretation as a smooth section of the cotangent bundle T^*M of the smooth manifold M.

Definition 2.1. Let M be a smooth manifold. We let Ω^1_M be the space of *smooth differential forms* on M.

The pattern repeats itself when going to higher derivatives. In order to get a coordinate independent derivative of a differential form, we introduce 2-forms.

Definition 2.2. Let M be a smooth manifold, $q \geq 0$ we put

$$\Omega^q_M = \bigwedge^q \Omega^1_M$$

and call its elements *q-forms* on M or *differential forms of degree q*.

In the case $q = 0$ this is read as the algebra of smooth functions on M.

By taking their direct sum we obtain the prototype of a *differential graded algebra*: graded, with a product, and a differential.

(1) The *product* is defined by the wedge product $(\omega, \omega') \mapsto \omega \wedge \omega'$. Its key property is graded commutativity

$$\omega \wedge \omega' = (-1)^{\deg(\omega)\deg(\omega')} \omega' \wedge \omega.$$

(2) The *differential* is a map $d : \Omega_M^q \to \Omega_M^{q+1}$ which agrees with $f \mapsto \mathrm{d}f$ in degree 0 and is uniquely determined by the formula

$$\omega \wedge \omega' \mapsto d\omega \wedge \omega' + (-1)^{\deg(\omega)} \omega \wedge d\omega'$$

in higher degrees. In local coordinates, we get the formula

$$f\, \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q} \mapsto \sum_i \frac{\partial f}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q}$$

with the computation rule $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$. Its key property is

$$d \circ d = 0.$$

In other words, differential forms form a complex.

Observation 2.3. Derivatives of polynomials are polynomials.

This means that the notion of a differential form, its origin in analysis notwithstanding, makes perfect sense in algebraic geometry!

2.2. Algebraic geometry. In order to make this exposition self-contained we make a detour into the basics of algebraic geometry. For the purposes of this note, it suffices to work over an algebraically closed field k, e.g., the complex numbers $\mathbb{C}$. All ideas and problems can be understood in this case.

Approximate Definition 2.4. Let k be an algebraically closed field. A variety consists of a topological space together with a ring of algebraic functions. In detail:

(1) An *affine variety* is given as

$$V = V(f_1, \ldots, f_m) = \{x \in k^n \mid f_i(x) = 0 \text{ for all } i\}$$

for a choice of $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$.

(2) We call

$$k[V] = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_m \rangle$$

the *ring of algebraic functions* on V.

(3) The variety V is *non-singular of dimension $n - m$* if the matrix

$$\left(\frac{\partial f_i}{\partial X_j}(x)\right)_{i,j} \in M_{n\times m}(k)$$

has rank m for all $x \in V$.

(4) A subset $U \subset V$ is called *open* if $V \backslash U$ is itself an affine variety.

(5) A *variety* X is a topological space together with an open covering $X = U_1 \cup \cdots \cup U_N$ and for every i a choice of homeomorphisms of U_i to an affine algebraic variety V_i and a compatible choice of algebraic transition functions.

(6) A general variety X is *non-singular* if all V_i are non-singular.

- Polynomials define functions on k^n, hence also on V. All elements of the ideal $(f_1, \ldots, f_n)$ vanish on V, hence elements of $k[V]$ induce set-theoretic functions on V with values in k, and this is how we think of them. Note, however, that the element of $k[V]$ is *not* fully determined by this function.
- The case $m = 0$ is allowed. We call this variety *affine n-space* $\mathbb{A}^n$. It is non-singular.
- The non-singularity condition asks for the rank to be the maximum of what is possible. This condition is the same that (over $\mathbb{R}$ or $\mathbb{C}$) guarantees that $V(f_1, \ldots, f_n)$ is a submanifold.
- It is not difficult to see that we have defined a topology on V.
- The prototype of a non-affine variety is *projective space* $\mathbb{P}^n = (k^{n+1} - \{0\})/\sim$ where $x \sim y$ if $x = \lambda y$ for some $\lambda \in k^*$. Its standard cover by the sets U_i of points whose i-coordinate does not vanish is the required cover by affine varieties, in this case by copies of $\mathbb{A}^n$. Hence it is non-singular.

We hope that the above is enough to help people very far from algebraic geometry to understand what we want to say. At the same time, this may be confusing for those actually do know the correct definitions.

Remark 2.5. We are cheating here in a couple of ways.

(1) Different choices of $f_1, \ldots, f_m$ can define the same subset of k^n. The way we have set things up, they define, however, a different ring of algebraic functions and hence a different variety. This is not the standard approach in textbooks on algebraic geometry, but saves us from discussing vanishing ideals and Hilbert's Nullstellensatz. It means that $k[V]$ may have nilpotent elements, something we are not used to from analysis. Actually, more advanced algebraic geometry is set up to handle such rings.

(2) By *variety* we really mean a separated scheme of finite type over an algebraically closed field.

(3) Our approximate definition is very close to the standard definition of a manifold by an atlas, avoiding sheaves. It is not complete because we have not discussed how open subsets of affine varieties are given the structure of an affine variety. This can be fixed with a bit of effort. The real gap is that a separatedness condition is missing. (It replaces the Hausdorff property in the analytic setting.) One often avoids this problem by restricting to *quasi-projective* varieties: those which are open subsets of closed subsets of $\mathbb{P}^n$.

We are now all set in order to define differential forms.

Definition 2.6. Let $V = V(f_1, \dots, f_m) \subset \mathbb{A}^n$ be an affine variety. The set of *algebraic differential forms* on V is the $k[V]$-module generated by symbols $dX_1, \dots, dX_n$ with relations $df_1, \dots df_m$

$$\Omega^1_V = \langle dX_1, \dots, dX_n \rangle_{k[V]} / \langle df_1, \dots, df_m \rangle.$$

For $q \geq 0$ we define

$$\Omega^q_V = \bigwedge^q \Omega^1_V.$$

Its elements are called *q-forms* or (*algebraic*) *differential forms of degree q*.

It comes with a derivation (a k-linear map satisfying the Leibniz rule)

$$d : k[V] \to \Omega^1_V, \qquad f \mapsto df = \sum_i \frac{\partial f}{\partial X_i} dX_i.$$

In fact, it can be characterized by a universal property: d is the universal k-linear derivation of $k[V]$. From the universal property or by direct computation, we see that the module Ω^1_V and the derivation d are independent of the choice of coordinates on V.

Example 2.7 (affine plane). Consider the affine plane $\mathbb{A}^2 = k^2$ with coordinates X, Y. Then:

$$\Omega^1_{\mathbb{A}^2} = \langle dX, dY \rangle_{k[X,Y]}.$$

This is a free module of rank 2.

Example 2.8 (hyperbola). Let $G = V(XY - 1) \subset \mathbb{A}^2$. We have

$$G = \{(x, y) \in k^2 \mid xy = 1\},$$

hence it can be identified with k^* via projection to the first coordinate. We have

$$k[G] = k[X, Y]/(XY - 1) = k[X, X^{-1}] \quad \text{with } X^{-1} = Y.$$

We work out the module of differential forms. We have $d(XY - 1) = X\,dY + Y\,dX$ and hence by definition

$$\Omega^1_G = \langle dX, dY \rangle_{k[X, X^{-1}]} / \langle Y\,dX + X\,dY \rangle.$$

We rewrite using $0 = X\,dY + Y\,dX$ in Ω^1_G:

$$dY = -X^{-1} Y\,dX = -X^{-2}\,dX.$$

This means that the generator dY is not needed and we get

$$\Omega^1_G = k[X, X^{-1}]\,dX.$$

This is a free module of rank 1.

Note that in these two examples the rank of Ω^1_V is equal to the dimension. This is no accident, but the correct general statement is more complicated.

Remark 2.9. Again we are cheating, this time by not discussing the non-affine case, even though we want to consider it later on. The correct point of view is to see Ω^q_V as a quasi-coherent sheaf on V rather than simply a module. Evaluating on affine open subvarieties brings us in the special case disussed before. Over non-affine varieties we need to distinguish between the sheaf Ω^q_V and its global sections $\Omega^q_V(V)$. The problem does not appear in the analytic situation because the sheaf of smooth differential forms on a paracompact manifold is determined by its global sections.

Differential forms play a key role in algebraic geometry, in particular in the quest to classify algebraic varieties.

(1) They are a rich source of *discrete invariants*. We will discuss this in more detail in the next section.
(2) Once the discrete invariants are fixed, we need to understand how algebraic varieties vary in families. This is addressed by *deformation theory*. It turns out that it also uses differential forms and their cohomology.
(3) In good cases, sets of isomorphism classes of algebraic varieties with fixed discrete invariants are themselves paramentrized by algebraic varieties. Following Griffiths we study these *moduli spaces* via *period maps* to generalized Grassmanians. Rougly these period maps are defined by integrating differential forms.

3. Application: source of invariants

We now want to discuss three examples of discrete invariants of algebraic varietes defined using differential forms.

3.1. Genus. As before let k be an algebraically closed field, e.g., $\mathbb{C}$. Let C be a non-singular projective curve over k.

Example 3.1 (planar curves). Let $f \in k[X, Y, Z]$ be a non-constant homogeneous polynomial, $C = V(f) \subset \mathbb{P}^2$ the set of zeroes of f. This defines a projective curve. It is non-singular if the gradient $\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right)$ does not vanish on C.

In the case of the complex numbers, a non-singular projective curve defines a projective 1-dimensional complex manifold, i.e., a compact Riemann surface. Recall that orientable compact surfaces are classified (as topological spaces) by their genus, the number of "holes".

Definition 3.2. Let C be as above. Then

$$g = \dim_k \Omega^1_C(C)$$

is called the *genus* of C.

Example 3.3. Let C be projective planar curve given by the equation $Y^4 = X^3Z + XZ^3 + Z^4$. It is non-singular and has genus 3.

In the case $k = \mathbb{C}$ we recover the topological notion. Hence we have found a completely algebraic definition of the genus. In can be used over all fields, including those of positive characteristic.

Example 3.4. Let $C = \mathbb{P}^1$. It has an affine cover by $U_0 = \mathbb{A}^1$ with coordinate X and $U_1 = \mathbb{P}^1 \backslash \{0\} \cong \mathbb{A}^1$ with coordinate $Y = X^{-1}$. Differential forms on U_0 are given by the module $k[X]\,dX$, differential forms on U_1 by $k[Y]\,dY$. We have $dY = dX^{-1} = -X^{-1}\,dX$ on the intersection. A global differential form is given by a pair $P(X)\,dX$, $Q(Y)\,dY$ (with polynomials P, Q) such that $P(X) = Q(X^{-1})X^{-1}$. The only solution is $P = Q = 0$. We have $g = 0$. Indeed, over the complex numbers $\mathbb{P}^1 = \hat{\mathbb{C}}$ is a sphere and hence has topological genus 0.

Remark 3.5. The genus is well-defined. By this we mean that it is always finite. This is a key step in the proof of the famous Theorem of Riemann-Roch. Note that this contrasts with the examples given in the last section, where the spaces of differential forms were of finite rank over $k[V]$ and hence infinite dimensional as k-vector spaces. Both these examples were affine, whereas the above definition is in the projective case!

3.2. De Rham cohomology. We now restrict to base fields k of characteristic 0 and non-singular k-varieties X. Recall that we have a complex

$$\Omega_X^0 \xrightarrow{d^0} \Omega_X^1 \xrightarrow{d^1} \Omega_X^2 \xrightarrow{d^2} \cdots$$

i.e., $d^i \circ d^{i-1} = 0$. Hence we can define its cohomology:

Definition 3.6. Let X be non-singular. We define *algebraic de Rham cohomology* of X as hypercohomology of the above complex of sheaves of vector spaces

$$H^i_{\mathrm{dR}}(X) = \mathbb{H}^i(X, \Omega_X^*).$$

Example 3.7. Assume X is in addition affine. Then the definition simplifies to

$$H^i_{\mathrm{dR}}(X) = \ker(d^i)/\mathrm{im}(d^{i-1}).$$

We get numbers from these vector spaces by taking dimensions.

Example 3.8. Let $X = C$ be a non-singular projective curve. Then

$$\dim_k H^1_{\mathrm{dR}}(C) = 2g$$

with g the genus defined above. Hence we have found another, completely algebraic way of defining the genus of a curve.

Example 3.9 (hyperbola). Let $G = V(XY - 1) \subset \mathbb{A}^2$ be again the hyperbola. Recall that we are working in characteristic 0. The de Rham complex has the shape

$$\Omega_G^* = [k[X, X^{-1}] \xrightarrow{d} k[X, X^{-1}]\, dX].$$

The only functions with derivative zero are the constant ones, hence

$$H^0_{\mathrm{dR}}(G) = \ker(d) = k.$$

For $i \neq -1$ the monome $X^i\, dX$ has a preimage under d. It is given by $\frac{1}{i+1} X^{i+1}$. (Note that this is where we need characteristic 0.) For $i = -1$ the preimage would be $\log(X)$, but this is not algebraic. Hence

$$H^1_{\mathrm{dR}}(G) = k[X, X^{-1}]\, dX/\mathrm{im}(d) = k\frac{dX}{X}.$$

Let us now specialise to $k = \mathbb{C}$. Then $G = \mathbb{C}^*$ is homotopy equivalent to the unit circle 1, hence its singular cohomology is concentrated in degrees 0 and 1. Again de Rham cohomology reproduces the topological invariant.

These examples of curves fit into a much more general pattern.

Theorem 3.10. *Let $k = \mathbb{C}$, X a non-singular algebraic variety. Then there is a canonical isomorphism*

$$H^i_{\mathrm{dR}}(X) \cong H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C})$$

where X^{an} is the complex manifold defined by X.

This isomorphism is known as *period isomorphism*. It has an explicit description by integrating differential forms over simplices.

Again this is a completely algebraic way of defining these topological invariants. The story continues with the construction of a mixed Hodge structure on singular cohomology of X by Deligne.

Remark 3.11. If $k = \mathbb{Q}$, the period isomorphism contains a lot of very interesting arithmetic information. By base change to $\mathbb{C}$, it induces

$$H^i_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The *period matrix* is defined as the matrix of this isomorphism in a rational basis on both sides. Its entries depend on the choice of bases, but their $\mathbb{Q}$-linear span does not. These are the *period numbers* of $H^i(X)$. The set of all periods contains many interesting numbers like $\sqrt{2}$, π, $\zeta(3)$, but is still countable. There are deep and long standing conjectures on their transcendence properties. Researchers in mathematical physics are also interested in period numbers because values of Feynman integrals are also period numbers. The formal properties of the algebra of period numbers are best understood in the light of the theory of motives. The book [HMS] aimed at a better understanding of this relation was actually the starting point of the present project on differential forms.

3.3. Kodaira dimension. We are back to the case of a ground field of any characteristic. Let X be a non-singular projective algebraic variety of dimension d.

Definition 3.12. Let $\omega = \Omega^d_X$ be the *canonical sheaf* on X.

We are going to give the definition of the Kodaira dimension of X in terms of ω. The definition is probably not immediately accessible without some knowledge of algebraic geometry. The outcome is an element in the set

$$\{-\infty, 0, \dots, d\}.$$

It is instructive to see what happens in the case $X = C$ a non-singular projective curve. We have three cases:

- $\kappa(C) = -\infty \Leftrightarrow g = 0$;
- $\kappa(C) = 0 \Leftrightarrow g = 1$;
- $\kappa(C) = 1 \Leftrightarrow g \geq 2$.

This is precisely the classification of surfaces into parabolic (or positive curvature), elliptic (or flat) and hyperbolic (or negative curvature) ones. Again an important property from differential geometry has found a completely algebraic definition.

To conclude, we give the formal definition: For $n \geq 1$ consider the finite dimensional vector space $\omega^{\otimes n}(X)$. Let $s_0, \dots, s_N$ be a basis. This defines a rational map

$$\pi_n : X \to \mathbb{P}^N, \qquad x \mapsto [s_0(x) : \cdots : s_N(x)].$$

More abstractly, π_n is the rational map to projective space defined by the line bundle $\omega^{\otimes n}$. It is regular outside the vanishing set of $s_0, \dots, s_N$, possibly nowhere.

Definition 3.13. The *Kodaira dimension* $\kappa(X)$ is defined as the maximum of $\dim \pi_n(X)$ for $n \geq 1$.

The maximum exists because the dimension of $\pi_n(X)$ is bounded by $\dim X = d$.

Example 3.14. Let again C be a non-singular projective curve.

(1) If the genus is 0, then C is the projective line. We have $\omega = \mathcal{O}(-2)$ by Riemann-Roch and hence $\omega^{\otimes n} = \mathcal{O}(-2n)$ does not have non-trivial global sections for any n. This gives Kodaira dimension $-\infty$.

(2) If the genus is 1, then C is an elliptic curve. The canonical bundle is trivial, i.e., $\omega = \mathcal{O}$. The space $\omega^{\otimes n}(C) = \mathcal{O}(C) = k$ is 1-dimensional for all n. Hence π_n is the constant map to a point and $\kappa(C) = 0$.

(3) If $g \geq 2$, then the degree of ω is positive, again by Riemann-Roch. From this we work out that for high enough n, the map π_n becomes injective and $\kappa(C) = 1$.

4. Problems and solutions in the singular case

4.1. Differential forms in the singular case. We now turn to the case of *singular* varieties. Note that the definitions we gave in earlier sections are also possible in the singular case. This is true for the notion of algebraic differential forms (Definition 2.6) as well as the different discrete invariants we deduced from it in

Section 3. However, these invariants do no longer capture the properties we want to capture. At the root of all these problems is a simple fact: Ω^1_V is no longer a vector bundle in the singular case.

Remark 4.1. We are not entitled to be surprised by this fact. It is true by definition: Definition 2.4 is basically saying that a variety is non-singular if Ω^1_V is a vector bundle.

We consider the simplest example of a singular variety.

Example 4.2 (coordinate cross). Let V be given by the equation $XY = 0$ in $\mathbb{A}^2$. Its Jacobi matrix

$$\left(\frac{\partial XY}{\partial X}, \frac{\partial XY}{\partial Y}\right) = (Y, X)$$

vanishes in $(0,0) \in V$. Hence the variety is singular. By Definition 2.6 we have

$$\Omega^1_V = \langle dX, dY \rangle_{k[X,Y]/XY} / \langle X\,dY + Y\,dX \rangle.$$

In other words

$$\omega = X\,dY = -Y\,dY$$

is a non-zero differential form on V which vanishes when restricted to the complement of $(0,0)$. This is true because outside of the origin either X or Y is invertible, but multiplication by X or Y kills ω because of the relation $XY = 0$.

Moreover, $dX \wedge dY$ is a non-vanishing 2-form on V. This is something we do not expect to happen on a 1-dimensional variety.

We need to fix a bit terminology here.

Definition 4.3 ([HKK] Definition 2.3). A class ω in Ω^q_V is called *torsion*, if there is a dense open $U \subset V$ such that $\omega|_U = 0$. The module Ω^q_V is called *torsion free*, if the only torsion class is $\omega = 0$.

Example 4.4. $\omega = X\,dY$ is a non-trivial torsion class in $\Omega^1_{V(XY)}$.

Remark 4.5. If V is affine and integral, then a class ω is torsion in the sense of the above definition if there is an $f \in k[V]$ such that $f\omega = 0$, i.e., it is a torsion element in the module theoretic sense. There a competing candidates for the generalisation to general varieties, see the discussion in [HKK], Warning 2.5. The above is most useful for our needs.

Depending on the setting and the intended application, different replacements of differential forms are in use. Among them are the following examples:

(1) torsion free differentials: $\Omega^q_X/\text{torsion}$;

(2) reflexive differentials: the $\mathcal{O}_X$-double dual of Ω^q_X;

(3) the Du Bois complex.

All are useful in certain applications, e.g, when classifying algebraic varieties up to isomorphism or up to birational equivalence (minimal model program) or in studying certain types of singularities. We are going to explain these notions in more detail. Readers outside of algebraic geometry are invited to jump directly to Section 5.

4.2. Torsion free differentials.

Definition 4.6. Let k be a field, X a separated scheme of finite type over X. Let T^q be the $\mathcal{O}_X$-sub module of torsion elements of Ω^q_X. By abuse of terminology we call

$$\Omega^q_X/T^q$$

the module of *torsion free differentials*.

Note that the module is obviously torsion free. However, it is still not a vector bundle.

Example 4.7. Let $k = \mathbb{C}$. Consider the variety $X = X_{1,2} = \mathbb{A}^2/\pm 1$, the quotient of $\mathbb{A}^2$ under the operation of $(x, y) \mapsto (-x, -y)$. It is explicitly given as the affine variety with $k[X] = \{f \in k[T_1, T_2] \mid f(T_1, T_2) = f(-T_1, -T_2)\}$. It has torsion free Ω^1_X, but it is not reflexive (isomorphic to its $\mathcal{O}_X$-double dual) by [GR] Proposition 4.1 and 4.3.

One of the reasons this notion is useful is that it is well-behaved under morphisms. This was shown in the context of analytic spaces by Ferrari ([Fer70]). The algebraic argument can be found in [Keb13b]. The proof is based on resolution of singularities.

Proposition 4.8 ([Keb13b] Corollary 2.7). *Let k be a field of characteristic* 0 *and $f : X \to Y$ a morphisms of reduced k-varieties. Then f induces a natural pull-back*

$$f^* : \Omega^q_Y/T_q \to \Omega^q_X/T_q.$$

Remark 4.9. This is false if k has positive characteristic [HKK] Example 3.6. The issue is not so much resolution of singularities, but the existence of inseparable morphisms!

The notion was used by Kebekus in [Keb13b] in order to study differential forms on rationally chain connected varieties. An earlier application is in Namikawa's study of deformations of singular holomorphic symplectic varieties, see [N01].

4.3. Reflexive differentials.

Definition 4.10. Let k be a field and X a k-variety. We define the sheaf of *reflexive differentials* as

$$\Omega_X^{[q]} = \mathscr{H}om_{\mathcal{O}_X}\big(\mathscr{H}om_{\mathcal{O}_X}(\Omega_X^q, \mathcal{O}_X), \mathcal{O}_X\big).$$

The sheaf of reflexive differentials is reflexive, i.e., isomorphic to its double dual. However, it still does not form a vector bundle in general.

Example 4.11. An explicit example is again the variety $X = \mathbb{A}^2/\pm 1$. More generally, this is the case for any singular variety which is klt (this stands for Kawamata log terminal): if $\Omega_X^{[1]}$ is a vector bundle, then so is the tangent bundle. By the Zariski–Lipman conjecture proved in [GKKP11], Theorem 6.1 for klt singularities, this implies that the variety is smooth.

Remark 4.12. (1) If X is non-singular, then $\Omega_X^{[q]} = \Omega_X^q$.

(2) If X is normal, let $j : X^{sm} \to X$ be the inclusion of the non-singular locus. Its complement has codimension at least 2. Then

$$\Omega_X^{[q]} = j_*\Omega_{X^{sm}}^q.$$

This means that reflexive differentials on normal varieties are particularly simple to handle. Proving assertions on reflexive differentials is equivalent to proving assertions on the smooth locus.

(3) On the downside, reflexive differentials are not functorial.

The objects were first introduced by Knighten ([Kni73]) under the name of Zariski differentials. The above terminology was coined by Kebekus and Kovács ([KeK08]). See the survey [Keb13b] on their use in modern birational geometry, and how their use is implicit in the work of Viehweg.

4.4. Du Bois complex. The definition of the Du Bois complex is inspired by Deligne's construction of the mixed Hodge structure on singular cohomology in [Del74].

Definition 4.13. Let k be a field of characteristic 0. Let X be k-variety. Let $\pi : X_\bullet \to X$ be proper hypercover with all X_i smooth. We put

$$\underline{\Omega}^q_X = R\pi_*\Omega^q_{X_\bullet}$$

as an object in the derived category of coherent sheaves on X.

Du Bois showed in [DuB81] the well-definedness of this object. The notion was coined by Steenbrink [S83].

Remark 4.14. (1) If X is smooth, then π can be chosen as the identity and $\underline{\Omega}^q_X = \Omega^q_X[0]$ viewed as complex concentrated in degree 0.

(2) If X is proper, then by construction $\mathbb{H}^i(X, \underline{\Omega}^q_X)$ computes the q-th step of the Hodge filtration on $H^{i+q}_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C})$.

Definition 4.15 ([S83] (3.5)). A variety X is called *Du Bois* if $\underline{\Omega}^0_X = \mathcal{O}_X$.

This is a fairly general class of singularities containing all singularities of the minimal model program, e.g., rational ([Kov99]), log-canonical ([KK10]) singularites. For a thorough discussion of these singularities and more, see [Kol13]. The Du Bois hypothesis is used to deduce vanishing theorems, a key tool in birational geometry, in particular the minimal model program.

5. A new approach in the singular case

Key Idea 5.1. We propose to address the singular case by changing the topology.

This is meant in the sense of a Grothendieck topology. Note that all algebraic varieties are topological spaces (see Definition 2.4), but the topology is very weak. E.g., all non-empty open subsets of $\mathbb{A}^n$ are dense. The intersection of two such is again dense. Ca. 1960 Grothendieck suggested to generalize the notion of topology. We replace the system of open *subsets* by a broader class of *morphisms* $V \to X$.

Example 5.2. Let X be a topological manifold. We obtain a Grothendieck topology by allowing all $f : V \to X$ which are local homeomorphisms.

Definition 5.3. Let $\mathscr{C}$ be a category. A basis for a *Grothendieck topology* on $\mathscr{C}$ is given by *covering families*, i.e., collections of morphisms in $\mathscr{C}$

$$(\varphi_i : V_i \to U)_{i \in I},$$

satisfying the following axioms:

- An isomorphism $\varphi : V \to U$ is a covering family with an index set containing only one element.
- If $(\varphi_i : V_i \to U)_{i \in I}$ is a covering family, and $f : V \to U$ a morphism in $\mathscr{C}$, then for each $i \in I$ there exists the pullback diagram

$$\begin{array}{ccc} V \times_U V_i & \xrightarrow{F_i} & V_i \\ \Phi_i \downarrow & & \downarrow \varphi_i \\ V & \xrightarrow{f} & U \end{array}$$

  in $\mathscr{C}$, and $(\Phi_i : V \times_U V_i \to V)_{i \in I}$ is a covering family of V.
- If $(\varphi_i : V_i \to U)_{i \in I}$ is a covering family of U, and for each V_i there is given a covering family $(\varphi_j^i : V_j^i \to V_i)_{j \in J(i)}$, then

$$(\varphi_i \circ \varphi_j^i : V_j^i \to U)_{i \in I, j \in J(i)}$$

  is a covering family of U.

After fixing a Grothendieck topology, we have given a new meaning to the word "local". That some statement holds *locally in the Grothendieck topology* means that it holds after passing to a covering family. Once we have a Grothendieck topology, standard concepts for topological spaces like sheaves make sense. The idea was extremely successful in algebraic geometry.

The topology that we are going to use was introduced by Voevodsky in 1996 ([Voe96]). He called it the h-topology where h seems to stand for *homology* because it is very natural from the point of view singular homology.

Approximate Definition 5.4. Consider the category Sch_k of k-varieties. The h-topology on Sch_k is generated by

- open covers;
- $\tilde{X} \to X$ proper and surjective.

Properness is a strong compactness condition. In the analytic setting it means that preimages of compact subsets are compact.

Here is the original version:

Definition 5.5 ([Voe96] Section 3.1). A morphism of schemes $p : X \to Y$ is called *topological epimorphism* if Y has the quotient topology of X. It is a *universal topological epimorphism* if any base change of p is a topological epimorphism.

The h-topology on the category $(\mathrm{Sch}/X)_{\mathrm{h}}$ of separated schemes of finite type over X is the Grothendieck topology with coverings finite families $\{p_i : U_i \to Y\}$ such that $\bigcup_i U_i \to Y$ is a universal topological epimorphism.

By [Voe96] Theorem 3.1.9 the approximate definition is also correct, if understood correctly.

What makes the h-topology so nice, is the fact that every variety is h-locally non-singular. This is the famous resolution of singularities.

Theorem 5.6 (Hironaka 1964 [Hi64]). *Let k be a field of characteristic 0 and X a k-variety. Then there is $\pi : \tilde{X} \to X$ proper surjective with X non-singular.*

Remark 5.7. In positive characteristic, the above statement is still true by de Jong's [dJ96]. However, looking in more detail the map π can be chosen birational in Hironaka's case, but not in de Jong's. We will see later that this makes a significant difference in our application to differential forms.

We can now introduce our main player: h-differentials. They are h-locally given by algebraic differentials.

Definition 5.8. Let Ω^q_{h} be the sheafification of

$$X \mapsto \Omega^q(X)$$

in the h-topology.

This means concretely:

$$\Omega^q_{\mathrm{h}}(X) = \ker\bigl(\Omega^1(X_0) \xrightarrow{p_1^* - p_2^*} \Omega^1(X_1)\bigr)$$

where

- $X_0 \to X$ is an h-cover with X_0 non-singular;
- $X_1 \to X_0 \times_X X_0$ is an h-cover with X_1 non-singular.

Here the fibre product $X_0 \times_X X_0$ replaces the intersections of open sets that we would use in the ordinary topology. An h-differential form is h-locally given by a differential form on a non-singular variety such that the data is compatible on "intersections", at least h-locally.

We are going to describe the results on h-differentials in more detail in the next section. Let us sum up the main points now.

Remark 5.9 (characteristic 0). Let k be a field of characteristic 0.

- Neither the module of differential forms nor their cohomology and other related invariants change in the non-singular case.
- They are always torsion free.
- h-differentials agree with reflexive differentials in the case of certain mild singularities and with torsion free differentials in certain other cases.
- They are always related to the complex of Du Bois differentials.
- They can be used to define algebraic de Rham cohomology in the singular case.

It thus turns out that the notion unifies the ad hoc definitions given before and simplifies proofs. It also also very natural, once one is familiar with the notion of a Grothendieck topology.

Remark 5.10. So why have they now been used before? One answer is that they have, see Section 8 on earlier work. The other is that the h-topology is a very unusual topology. In contrast to the étale or the flat topology it really changes the topology. In the language of algebraic geometry: it is not sub-canonical. Working h-locally, we can not longer distinguish between the affine line and the standard cuspidal singularity given by $Y^2 = X^3$. Depending on the intended application this will be a feature (making life easier) or a bug (destroying what we want to study).

The situation is more complicated in positive characteristic p. The new problem is introduced by the *Frobenius morphism* $F : x \mapsto x^p$. Its differential is

$$dF(x) = dx^p = px^{p-1} = 0.$$

Corollary 5.11 ([HKK] Lemma 6.1). $\Omega^q_{\mathrm{h}} = 0$.

We can get around this by replacing the h-topology by weaker versions like the cdh- or the eh-topology.

- good news: $\Omega^q_X(X) = \Omega^q_{\mathrm{cdh}}(X) = \Omega^q_{\mathrm{eh}}(X)$ if X is non-singular ([HKK] Theorem 5.11).
- bad news: torsion still exists and it is not functorial ([HKK] Corollary 5.16, 5.17)

Algebraic geometry in positive characteristic is much less well understood than in characteristic 0. We hope that cdh-differentials and their variations will turn out to be a valuable tool.

6. Results in characteristic 0

We explain the main results of [HJ] obtained with C. Jörder. Throughout this section, k is a field of characteristic 0.

6.1. Descent. Our first main result establishes that h-differentials behave like ordinary differentials in the non-singular case. This is a minimal requirement for the theoy that generalizes a well-established and successful tool.

The category of h-sheaves is abelian. We can derive the left exact functor $\Gamma(X, \cdot)$ of sections on X and obtain h-*cohomology*. We denote it $H^i_{\mathrm{h}}(X, \cdot)$. Ordinary sheaf cohomology on the topological space X is denoted $H^i(X, \cdot)$.

Theorem 6.1 ([HJ] Theorem 3.6, Corollary 6.5). *Let X be a non-singular k-variety. Then*

$$H^i_{\mathrm{h}}(X, \Omega^q_{\mathrm{h}}) = H^i(X, \Omega^q_X).$$

In particular (the case $i = 0$), we have $\Omega^q_{\mathrm{h}}(X) = \Omega^q_X(X)$ in the non-singular case; a fact that was mentioned before.

The main tool in working in the h-topology is the *blow-up sequence*. Consider a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

with $\tilde{X} \to X$ proper birational, i.e., proper surjective and an isomorphism over a dense open subset of X, $Z \to X$ a closed immersion, E the preimage of Z; subject to the condition that $\tilde{X} \backslash E \to X \backslash Z$ is an isomorphism. Following Voevodsky, we call this a *a blow-up square*.

For h-cohomology, the blow-up square induces a long exact sequence. Indeed, $\{Z \to X, Y \to X\}$ is an h-cover, hence it induces the standard Čech spectral sequence. In this particular case, it degenerates into a single exact sequence

$$\cdots \to H^i_{\mathrm{h}}(X, \Omega^q_{\mathrm{h}}) \to H^i_{\mathrm{h}}(Z, \Omega^q_{\mathrm{h}}) \oplus H^i_{\mathrm{h}}(\tilde{X}, \Omega^q_{\mathrm{h}}) \to H^i_{\mathrm{h}}(Z, \Omega^q_{\mathrm{h}}) \to H^{i+1}_{\mathrm{h}}(X, \Omega^q_{\mathrm{h}}) \to \cdots$$

Combining this sequence with the Descent Theorem 6.1 and standard properties of differential forms and their cohomology on non-singular varieties, we can deduce many simple properties, e.g., (see [HJ] Proposition 4.2, Corollary 6.8):

$$H^i_{\mathrm{h}}(X, \Omega^q_{\mathrm{h}}) = 0 \qquad \text{for } i > \dim X \text{ or } q > \dim X$$

A particularly nice application, showing the strength of these techniques, is well-definedness of *rational singularities*. Recall that a variety X has *rational singularities*, if there is a proper birational morphism $\pi : \tilde{X} \to X$ with $\tilde{X}$ non-singular such that

$$R\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X[0].$$

We want to show independence of the choice of resolution. This means that we have shown that the condition is satisfied in the special case where X is also non-singular and connected. Let d be its dimension. By Serre duality, this is equivalent to showing

$$R\pi_* \Omega^d_{\tilde{X}} = \Omega^d_X[0].$$

This is equivalent to

$$H^i(\tilde{X}, \Omega^d_{\tilde{X}}) = H^i(X, \Omega^d_X)$$

for all such X and π. Rewriting this as h-cohomology, the equality is immediate from the blow-up sequence with Z the exceptional locus of π.

Remark 6.2. Looking more closely, this is not a new proof. One can use strong resolution of singularities to reduce to the case where π is the blow-up in a non-singular center. Then everything can be done by explicit computation. The same argument goes into the proof of the Descent Theorem 6.1. The same pattern repeats itself in other results. While the argument is probably not new, its presentation is streamlined.

6.2. An alternative description. The descent theorem allows us to give a topology free description of h-differentials, which is very intersting from a conceptual point of view.

Let Sm_k be the category of non-singular (also called smooth) varieties over k.

Theorem 6.3 ([HJ] Theorem 1). *Let X be a variety. Specifying an* h*-form $\omega \in \Omega^q_{\mathrm{h}}(X)$ is equivalent to specifying a form $\omega_f \in \Omega^q(Y)$ for every $f : Y \to X$ with Y non-singular, in a compatible way for all X-morphisms of non-singular varieties.*

$$\Omega^q_{\mathrm{h}}(X) \cong \left\{ (\omega_f)_{f:Y\to X} \in \prod_{\substack{f:Y\to X \\ Y\ \mathit{smooth}}} \Omega^q_Y(Y) \;\middle|\; \begin{array}{ccc} Y' & & \\ {\scriptstyle\phi}\downarrow & \searrow^{f'} & \\ Y & \xrightarrow[f]{} & X \end{array} \Longrightarrow \phi^*\omega_f = \omega_{f'} \right\}.$$

In fancier language: Ω^q_{h} agrees with the right presheaf extension of Ω^q from Sm to Sch. The extension theorem then can be read as saying that the extension is an h-sheaf. This is an intrinsic property of the presheaf Ω^q. From this point of view, we can justify the somewhat arbitrary decision to use the h-topology in treating differential forms.

We will see below that this approach is particularly fruitful in positive characteristic.

6.3. Special cases. We now want to list some cases where h-differentials agree with ad hoc replacements introduced before.

We call a variety a *normal crossings variety* if it is isomorphic to a divisor with normal crossings in some non-singular variety.

Lemma 6.4 ([HJ] Proposition 4.9). *Let X be a normal crossings scheme. Then* h-*differentials agree with torsion free differentials* (*see Definition* 4.6)

$$\Omega^q_{\mathrm{h}}(X) = (\Omega^q/T_q)(X).$$

Theorem 6.5 ([HJ] Theorem 5.4). *Let X be a klt base space. Then* h-*differentials agree with reflexive differentials* (*see Definition* 4.10)

$$\Omega^q_{\mathrm{h}}(X) = \Omega^{[q]}(X).$$

The letters klt stand for Kawamata log terminal. By klt base space we meant that there is a divisor on X such that (X, D) is klt. We do not choose to go into the definition of klt singularities here, see [KM98] Definition 2.34. They are mild singularities characterized in terms of the canonical sheaf. They are of particular importance in the minimal model program.

As an immediate consequence, we recover a result of Kebekus (see [Keb13b]).

Corollary 6.6 ([HJ] Corollary 5.6). *Let $f : X \to Y$ be a morphism of klt base spaces. Then there is a natural pull-back on reflexive differential forms*

$$f^* : \Omega^{[q]}_Y \to \Omega^{[q]}_X.$$

Proof. The sheaf Ω^q_{h} is a presheaf. □

Behind the comparison theorem is a general structural result, which is of independent interest. Roughly, a variety over $\bar{k}$ is called *rationally chain connected*, if any two closed points can be linked by a chain of $\mathbb{P}^1$'s. We refer to [Kol96] Section IV.3 for the formal definition.

Theorem 6.7 ([HJ] Theorem 5.12). *Let $f : X \to Y$ be a proper surjective morphism such that for every closed point $y \in Y$, the fibre X_y is geometrically rationally chain connected. Then*

$$f^* : \Omega^q_{\mathrm{h}}(Y) \xrightarrow{\cong} \Omega^q_{\mathrm{h}}(X)$$

is an isomorphism.

The main input into this is the fact that there are no global differential forms on $\mathbb{P}^1$. The comparsion result for reflexive forms in Theorem 6.5 follows via deep structural results on the geometry of the desingularization of klt spaces and their consequences for differential forms (see [HM07], [GKKP11]).

Finally, basically by construction, h-differentials are very well suited to treating Hodge theory.

We denote $\mathrm{Sch}_{\mathrm{h}}$ the category of separated schemes of finite type over k equipped with the h-topology. We denote X_{Zar} the category of open subsets of X with the set of coverings given by the set ordinary coverings by Zariski open subsets. The inclusion functor induces a morphism of sites

$$\rho_X : \mathrm{Sch}_{\mathrm{h}} \to X_{\mathrm{Zar}}$$

(think: a continuous map; preimages of open sets are open, preimages of covers are covers).

Proposition 6.8 ([HJ] Theorem 7.12). *Let X be a variety. Then*

$$R\rho_{X*}\Omega^q_{\mathrm{h}} \in D^+\big(\mathrm{Sh}(X_{\mathrm{Zar}})\big)$$

is canonically isomorphic to the Du Bois complex $\underline{\Omega}^q_X$ (see Definition 4.15).

6.4. De Rham cohomology. We have defined algebraic de Rham cohomology for non-singular varieties above (see Definition 3.6). We are now able to extend this easily to the singular case.

Definition 6.9. Let X be a variety. We define *algebraic de Rham cohomology* of X as hypercohomology of the de Rham complex in the h-topology

$$H^i_{\mathrm{dR}}(X) = \mathbb{H}^i_{\mathrm{h}}(X, \Omega^*_{\mathrm{h}}).$$

Remark 6.10. It follows directly from Theorem 6.1 that this agrees with the standard definition in the non-singular case. The first definition in the singular case is due to Hartshorne (see [Har75]). He needs to embedd X into a non-singular variety Y and considers cohomology the completion of the de Rham complex on Y

with respect to the ideal of the embedding. Another definition via smooth proper hypercovers comes as a byproduct of Deligne's construction of the mixed Hodge structure on cohomology of a singular cohomology (see [Del74]). There are even more advanced approaches via the theory of mixed Hodge modules or the de Rham realization on triangulated motives. All definitions agree. We feel that the above is the simplest.

Properties of de Rham cohomology of non-singular varieties now translate to the singular case.

Proposition 6.11 ([HJ] Corollary 7.6). *Let* X *a variety over* $\mathbb{C}$. *Then there is a natural isomorphism*

$$H^i_{\mathrm{dR}}(X) \cong H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}).$$

As in Remark 3.11 this allows us to define a period isomorphism for varieties over $\mathbb{Q}$. This point of view is explained in full detail in [HMS]. As mentioned before, this was actually the starting point of the present project.

7. Results in positive characteristic

We report on joint work with S. Kebekus and S. Kelly. Throughout the section, k is a perfect field of positive characteristic p.

7.1. Topologies. As mentioned in Section 5, the existence of Frobenius implies that $\Omega^q_{\mathrm{h}} = 0$ in positive characteristic. This means that we have to work in a weaker topology instead.

A cover is called *completely decomposed* if every (possibly non-closed) point has a preimage with the same residue field. An étale cover is called *Nisnevich cover* if it is completely decomposed.

Definition 7.1 ([VSF] Chapter 4, Definition 3.2). The cdh-*topology* on Sch_k is the Grothendieck topology generated by Nisnevich covers and the covers defined by abstract blow-up squares (see Section 6.1).

Example 7.2. If X is non-singular and $\tilde{X} \to X$ is proper and birational, then it defines a cdh-cover (see [HKK] Proposition 2.13).

Remark 7.3. Alternatively, one can work with the eh-topology (generated by proper cdh-covers and étale covers) or the rh-topology (generated by proper cdh-covers and Zariski covers). So far all our results hold in all three topologies.

Moreover, under the assumption that resolutions of singularities exist (see [HKK] Proposition 5.13)

$$\Omega^q_{\mathrm{rh}} = \Omega^q_{\mathrm{cdh}} = \Omega^q_{\mathrm{eh}}.$$

An unconditional argument can be found in [HK]. Hence we concentrate on the best known case of the cdh-topology in this exposition.

7.2. Descent.

Theorem 7.4 ([HKK] Theorem 5.11). *Let X be non-singular. Then*

$$\Omega^q_{\mathrm{cdh}}(X) = \Omega^q(X).$$

The main point about this result is that it is unconditional, i.e., it does *not* assume that resolutions of singularities exist. We will explain in Section 7.4 some the concepts going into the proof.

Note that this is a lot weaker than Theorem 6.1 as we do not know about cohomology, but only about sections.

The blow-up sequence is still available, but much less useful because the existence of a birational desingularization is an open question. On the other hand, its existence has unexpected consequences. It is easy to write down a blow-up square

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

with $\tilde{X}$, Z and E non-singular, but $E \to Z$ an *inseparable* finite cover, see [HKK] Example 5.15. This means that the restriction map $\Omega^q(Z) \to \Omega^q(E)$ vanishes. The blow-up sequence starts

$$0 \to \Omega^q_{\mathrm{cdh}}(X) \to \Omega^q(\tilde{X}) \oplus \Omega^q(Z) \to \Omega^q(E)$$

This implies that any class in $\Omega^q(\tilde{X})$ which vanishes along E gives rise to a torsion class on X!

Proposition 7.5 ([HKK] Corollary 5.16, 5.17). *The torsion subgroup $T^q_{\mathrm{cdh}}(X) \subset \Omega^q_{\mathrm{cdh}}(X)$ is non-trivial in general. Moreover, T^q_{cdh} is not functorial in X.*

Remark 7.6. This means that passing to Ω^q_{cdh} is an improvement on the original notion of a differential form on a singular variety, but it does not address the very first objection that we had—that there are torsion forms.

7.3. What about alterations?. It is tempting to try to build a stronger topology which also uses alterations as covers. The advantage would be that every variety would be locally non-singular. The short answer is that this approach does not work.

In [HKK] Section 6 we introduce the sdh-topology. It has the covers generated by étale covers and proper h-covers such that every point has a preimage such that the residue field extension is *separable*. By [HKK] Example 6.5 we find a finite sdh-cover of a non-singular variety by a non-singular variety, such that the ramification locus and its preimage are also non-singular. However, in this example the sheaf condition for the sdh-topology is not satisfied.

Proposition 7.7 ([HKK] Proposition 6.6). *The presheaf* Ω^1 *does not satisfy* sdh-*descent on non-singular varieties.*

7.4. dvr-differentials and Riemann–Zariski spaces. We return to the alternative point of view developed in Section 6.2 now in positive characteristic.

Definition 7.8. Let X be a k-variety. We define $\Omega^q_{\mathrm{dvr}}(X)$ as the presheaf extension of Ω^q from the category of non-singular k-varieties Sm to Sch.

More explicitly, $\Omega^q_{\mathrm{dvr}}(X)$ is given by the formula of Theorem 6.3.
We now want to explain the notation Ω^q_{dvr}.

Definition 7.9. (1) Let Dvr be the subcategory of the category of k-schemes with objects of the form $\operatorname{Spec} R$ with R essentially of finite type over k and R either a field or a discrete valuation ring.

(2) For $X \in \mathrm{Sch}_k$ let Val_X be the category of X-schemes $\operatorname{Spec} R \to X$ with R either a finitely generated field extension of k or a valuation ring of a (possibly non-discrete) k-valuation of such a field.

Example 7.10. (1) The spectra of fields in Dvr and Val_k are the spectra of function fields of non-singular k-varieties.

(2) Let $x \in X$ be a point of codimension 1. The normalization of $\mathcal{O}_{X,x}$ is a semilocal ring whose localisations are discrete valuation rings with closed point over x.

(3) Let $k = \bar{k}$. We get a non-discrete valuation ring on the rational function field in two variables by the following procedure: choose a sequence of smooth projective surfaces with function field K and a sequence of points $x_n \in X_n$ by putting $X_0 = \mathbb{P}^2$, x_0 an arbitrary closed point, $\pi_n : X_n \to X_{n-1}$ the blow-up with center in x_{n-1}, x_n a closed point of the exceptional fibre of π_n. Then $R = \bigcup_{n=0}^{\infty} \mathcal{O}_{X_n, x_n}$ is a valuation ring. (See [Ha77] Excercises II.4.12 and V.5.6).

Doing algebraic geometry in terms of valuations is a very old idea going back to Zariski. Indeed it is well-known that the points of a non-singular projective curve can be identified with the set of k-valuations on the function field. They are all discrete in this case. More generally:

Definition 7.11 ([ZS75], §17, p. 110). Let A be an integral finitely generated k-algebra with quotient field K. The *Riemann–Zariski space* $\mathsf{RZ}(A)$, as a set, is the set of (not necessarily discrete) valuation rings of K. To a finitely generated sub-A-algebra A' of K is associated the set $E(A') = \{R \in \mathsf{RZ}(A) \mid A' \subset R\}$ and one defines a topology on $\mathsf{RZ}(A)$ taking the $E(A')$ as a basis.

This topological space is quasi-compact, in the sense that every open cover admits a finite subcover [ZS75], Theorem 40. The idea of using Riemann–Zariski spaces in treating cdh-differentials is due to Kelly.

Definition 7.12. Let Ω^q_{val} be the right Kan extension of Ω^q from Val_k to Sch_k, i.e., for $X \in \mathrm{Sch}_k$ we put

$$\begin{aligned}\Omega^q_{\mathrm{val}}(X) &= \varprojlim_{\mathrm{Val}_X} \Omega^q \\ &= \{(\omega_Y)_{Y \in \mathrm{Val}_X} | \omega_Y|_{Y'} = \omega_{Y'} \text{ for all } Y' \to Y \text{ over } X\}\end{aligned}$$

Remark 7.13. Actually, by Theorem 7.16 below, an element of $\Omega^q_{\mathrm{val}}(X)$ is already uniquely determined by its values on all $Y = \mathrm{Spec}(\kappa(x))$ where $x \in X$ is a (not necessarily closed) point of X.

Proposition 7.14 ([HKK] Proposition 4.14, [HK]). (1) *The presheaf* Ω^q_{dvr} *on* Sch *is equal to the right Kan extension of* Ω^q *from* Dvr *to* Sch_k.

(2) *The presheaf* Ω^q_{cdh} *on* Sch *is equal to* Ω^q_{val}.

Theorem 7.15 ([HKK], Observation 5.3, Proposition 5.12, Theorem 5.11, Proposition 5.13). *The presheaf* Ω^q_{dvr} *is a* cdh-*sheaf.*

Let X be a variety. Then the natural map

$$\Omega^q_{\mathrm{cdh}}(X) \to \Omega^q_{\mathrm{dvr}}(X)$$

is injective and an isomorphism if X is non-singular. It is always an isomorphism if we assume weak resolution of singularities.

Actually, we deduce the Descent Theorem 7.4 from this. The proof of Theorem 7.15 uses the geometry of Riemann–Zariski spaces and a key fact about differential forms on valuation rings.

Theorem 7.16 ([HKK] Remark A.2). *Let R be a (possibly non-descrete) valuation ring with quotient field K. Then Ω^q is torsion free on* Spec R, *i.e., the map*

$$\Omega^q(R) \to \Omega^q(K)$$

is injective.

Proof. The essential case is $q = 1$. It is due to Gabber and Romero, [GR03] Corollary 6.5.21. The general case follows with a bit of commuative algebra, see [HKK] Lemma A.4. □

Another consequence is that we understand the local behaviour of Ω^q_{cdh}.

Corollary 7.17 ([HK]). *Let $X \in \mathrm{Sch}_k$. The Zariski-sheaf $\Omega^q_{\mathrm{cdh}}|_{X_{\mathrm{Zar}}}$ on X is a coherent $\mathcal{O}_X$-module.*

The case of 0-differentials is simpler than the general case.

Lemma 7.18. *Let X be normal. Then*

$$\mathcal{O}_{\mathrm{cdh}}(X) = \mathcal{O}_X(X).$$

Proof. Elements in $k(X)$ which are integral on all discrete valuation rings of $k(X)$ always extend to an open subset such that the codimension is at least 2. In the normal case functions on such an open extend to all of X. □

This is particularly useful because all varieties are cdh-locally normal.

Definition 7.19 ([HK]). Let ρ_X be the restriction form $(\mathrm{Sch}_k)_{\mathrm{cdh}}$ to X_{Zar}. We call X *Du Bois* if

$$R\rho_{X*}\mathcal{O}_{\mathrm{cdh}} = \mathcal{O}_X[0].$$

If cohomological descent holds and X admits a smooth proper hypercover (e.g. in characteristic 0) this agrees with Steenbrink's original definition given earlier, see Definition 4.15. The advantage is that we can make an unconditional definition.

7.5. Open questions. The main open question concerns the analogue of Theorem 6.1.

Conjecture 7.20. *Let X be non-singular over a perfect field k. Then*

$$H^i_{\mathrm{cdh}}(X, \Omega^q_{\mathrm{cdh}}) = H^i(X, \Omega^q_X).$$

Under strong resolution of singularities this was shown by Geisser in [Gei06] Theorem 4.7. The case $i = 0$ is our Theorem 7.4.

By specialising to X affine the conjecture immediately implies:

Conjecture 7.21. *Let X be non-singular and affine. Let $\pi : \tilde{X} \to X$ be proper and birational and an isomorphism outside $Z \subset X$ with preimage E. Then the map*

$$\Omega^q_{\mathrm{cdh}}(\tilde{X}) \oplus \Omega^q_{\mathrm{cdh}}(Z) \to \Omega^q_{\mathrm{cdh}}(E)$$

is surjective.

Lemma 7.22 ([HK]). *Conjecture* 7.21 *follows from Conjecture* 7.20. *It holds unconditionally for $q = \dim X$.*

Proof. By the blow-up sequence, we have the exact sequence

$$0 \to \Omega^q_{\mathrm{cdh}}(X) \to \Omega^q_{\mathrm{cdh}}(\tilde{X}) \oplus \Omega^q_{\mathrm{cdh}}(Z) \to \Omega^q_{\mathrm{cdh}}(E) \to H^1_{\mathrm{cdh}}(X, \Omega^q_{\mathrm{cdh}})$$

By Conjecture 7.20, the last term is equal to $H^1(X, \Omega^q_X)$. As X is affine, this vanishes.

The case of $q = \dim X$ is obvious because $\Omega^{\dim X}(E) = 0$. □

Actually, we expect a stronger result comparison:

Conjecture 7.23. *Let X, $\tilde{X}$, Z, E be as in Conjecture* 7.21. *Then*

$$\Omega^q_{\mathrm{cdh}}(Z) \to \Omega^q_{\mathrm{cdh}}(E)$$

is an isomorphism.

This stronger statement is true in characteristic 0 by Theorem 6.7 because $E \to Z$ has geometrically rationally chain connected fibres. It is not obvious what the correct condition in positive characteristic is.

8. Earlier work

(1) *Deligne* in 1974 ([Del74]) was using proper h-covers in his definition of a mixed Hodge structure on singular cohomology of singular algebraic varieties. Subsequently, Hodge theory was and is a most important tool in the study of algebraic varieties. However, the step to turning it into an actual Grothendieck topology was not made for a long time.

(2) In 1996 *Voevodsky* together with Suslin and Friedlander defined and studied the h-topology and the related weaker topologies cdh (completely decomposed

h) and qfh (quasi-finite h) (see [Voe96], [VSF]). The same papers also establish many of the basic properties. His aim was the definition of an algebraic version of singular homology. This eventually led to his definition of geometric motives over a field. In this, they switched to an alternative point of view with finite correspondences and only the Nisnevich topology. This seems to be the reason that the h-topology itself was a bit forgotten.

(3) *Geißer* in 2006 (see [Gei06]) introduced the eh-topology and studied its properties. He was considering special values of ζ-functions of varieties over finite fields. As part of this project, he also considered Ω^q_{eh} and was able to show under resolution of singularities that their cohomology agrees with ordinary sheaf cohomology in the non-singular case.

(4) *Lee* in 2009 (see [Lee09]) studied the h-sheafification of Ω^q on the category of non-singular varieties of characteristic 0 and established cohomological descent. This is very similar to Theorem 6.1. His approach is technially made more complicated by the fact that the category of non-singular varieties does not have fibre products. He also obtained a description of Du Bois differentials analoguous to Proposition 6.8.

(5) In a series of papers *Cortiñas, Haesemeyer, Schlichting, Walker and Weibel* 2008–2013 ([CHSWei08], [CHWei08], [CHWaWei09], [CHWaWei10], [CHWaWei11], [CHWaWWe13]) studied homotopy invariance properties of algebraic K-theory. On the way they also studied Ω^q_{cdh} in characteristic 0. In particular they already obtained the cdh-versions of Theorem 6.1 and the comparison statement with reflexive differentials (Theorem 6.5) for toric varieties.

(6) Beilinson in 2012 (see [B12]) gave a spectacular application by giving a new proof of the p-adic comparison theorem between étale and algebraic de Rham cohomology. His approach is a lot more subtle as he is working in mixed characteristic. His starting point is the contangent complex instead of the sheaf of differential forms, which is then p-adically completed in a derived way.

What all these papers have in common is that they are interested in algebraic de Rham cohomology and other *homotopy invariant* cohomology theories. All are explicitly or in spirit related to motives. What we want to advertise is applying these same techniques also in a non-homotopy invariant setting: the individual Ω^q.

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