Linearization and categorification

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Abstract. We discuss the notion of linearization through examples, which include the
Price map, PageRank, representation theory, the Euler characteristic and quantum link
invariants. We also review categorification, which adds an additional layer of structure,
in the context of the last two examples.

1. Linearization

We are going to start by discussing *linearization*. We loosely define linearization
as mapping instances of discrete, set-theoretical, geometric, topological, and even
real-life structures (the latter cannot be fully described in a purely mathematical
language) to elements of a linear structure. A linear structure has addition in the
background, or, equivalently, has the structure of an abelian group. Often, a vec-
tor space over a field appears as the background structure, allowing, among other
benefits, to use convenient and powerful tools of linear algebra.

   To convince the reader of the ubiquity of linearization let us provide some
examples. We give two examples from real life and several from mathematics.
The real-life examples are

- The Price map
- PageRank

1.1. The Price map. To naively define the Price map let us start with the semir-
ing of non-negative integers $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Consider the set Items, which is
the union of sets of Goods and Services:

$$\text{Items} = \text{Goods} \cup \text{Services}.$$

To value goods and services, in the simplest possible case, we set up a map

$$\text{Price} : \text{Items} \to \mathbb{Z}_+.$$

Setting price of a sandwich, for instance, to 6, we value the sandwich at \$6 (using dollars as the currency, for example).

A more refined version of this setup would map an item together with the data of location and time to an element of $\mathbb{Z}_+$:

$$\text{Price} : (\text{item}, \text{time}, \text{location}) \to \mathbb{Z}_+,$$

since the price of an item might depend on where and when it is offered for sale.

Thousands of books have been written on economics and finance, studying the Price map in great detail. This one-page example cannot even attempt to start on Economics 101. Instead, we would like to point out that the Price map is an example of *linearization*. Various objects varying from simple to incredibly sophisticated are mapped to elements of $\mathbb{Z}_+$. Applying this map loses almost all the information about the objects, retaining only a nonnegative integer.

Most characteristics of an item are lost, but the Price map is incredibly convenient. It also needs a developed framework to function well (Money, Government, Enforcement).

The structure of the Price map is enhanced by enlarging its target to

- $\mathbb{Z}$ (with sinister consequences),
- $\frac{1}{N}\mathbb{Z}$ (where $N = 100$ is a common choice),
- A more refined approximation to $\mathbb{Q}$.

Enlarging the target of the map to $\mathbb{Z}$, that is, allowing negative prices, is a natural step from both the real-life and mathematical perspectives. This sometimes leads, in our interactions with the Price map, to unpleasant consequences, such as being in debt.

Upon a brief reflection, it becomes clear that the structure of $\mathbb{Z}$ as a commutative ring is not fully necessary for the Price map. We rarely if ever encounter objects measured in dollars squared $\$^2$, indicating that the multiplication in the target $\mathbb{Z}$ can be avoided. Rather, it seems enough to make $\mathbb{Z}$, the target of the Price map, a free rank one module over the ring $\mathbb{Z}$. The natural order $<$ on this module is, of course, of paramount importance.

That the target is a module rather than a ring becomes further obvious when fractions are introduced, by enlarging $\mathbb{Z}$ to

$$\frac{1}{N}\mathbb{Z} = \left\{ \frac{a}{N}, a \in \mathbb{Z} \right\}.$$

$N = 100$ is a common choice (dollars and cents). Notice that $\frac{1}{N}\mathbb{Z}$ is not naturally a ring. We can view it as a subset of the ring

$$\mathbb{Z}\left[\frac{1}{N}\right] = \left\{ \frac{a}{N^n}, a \in \mathbb{Z}, n \in \mathbb{Z}_+ \right\},$$

but this subset is not closed under multiplication, indicating, again, that the target of the Price map is a module rather than a ring. Thus, the enhanced target $\frac{1}{N}\mathbb{Z}$ of the Price map is a $\mathbb{Z}$-module together with the order $<$.

Further enhancement would take us from $\frac{1}{N}\mathbb{Z}$ to $\mathbb{Q}$ (viewed as a module over itself), or even to $\mathbb{R}$, with the natural order $<$ extended to $\mathbb{Q}$ and $\mathbb{R}$. Various approximations to $\mathbb{Q}$ and $\mathbb{R}$, such as the floating point type, are used as practical implementations of this enhancement.

In all cases, the target carries a natural linear structure (abelian semigroup $\mathbb{Z}_+$ in the initial example, and abelian groups $\mathbb{Z}$, $\frac{1}{N}\mathbb{Z}$, and $\mathbb{Q}$ in the generalizations).

Problem 1.1. Find and implement modifications of the Price map to other, more refined, targets.

For instance, is there a useful modification such that the analogue of the addition operation on the target is noncommutative? Is there a modification where the target is (a free module over) a noncommutative ring?

Bitcoin is a recent perplexing example of the Price and Money framework where the target structure is a subset of $\mathbb{Z}_+$ that monotonically grows with time and was designed to be bounded from above at all times by approximately 20.3 million [7]. Bitcoins are awarded for solving hard meaningless instances of a computational problem, increasing in complexity as more bitcoins are minted, which results in the above upper bound. This artificial upper bound on the total number of bitcoins is only partially resolved by introducing fractional bitcoins.

1.2. PageRank. Already back in the late 90's the structure of the Internet was incredibly complex. The PageRank idea [6] was to take the Internet—the set of all webpages together with their content and data—and forget almost everything about it, reducing it to a directed graph, the web graph. Vertices of the web graph Γ are webpages and there is an oriented edge from i to j if $i \neq j$ and there is a link from i to j.

This first step converts an enormously complicated structure to a simple one, at least on the theoretical level. The second step is linearization. We form the web matrix A. Its rows and columns are labelled by vertices, and the entry

$$a_{ji} = \frac{1}{N_i}$$

if there is an edge from i to j, where N_i is the number of oriented edges out of i. Otherwise $a_{ji} = 0$.

Assuming irreducibility (there is a path from any i to any j), we can consider the maximum eigenvalue and the unique (up to scaling) eigenvector with this eigenvalue—the PageRank vector v. In greater generality, this is known as the Perron–Frobenius eigenvalue and eigenvector.

In this linearization we form a real vector space V with the basis labelled by webpages and convert the web graph Γ to a linear transformation on V given by the matrix A. Coefficients of the maximum eigenvector v are positive and rank the webpages by a measure of their popularity.

Just like the Price map, this conversion of the entire Internet to a linear transformation and its maximum eigenvector loses almost the entire informational content of the Internet, yet it proved to carry phenomenal value.

We conclude our discussion of real-life linearizations with an exercise and a project for the reader.

Exercise 1.2. Find more examples of real-life linearizations.

Problem 1.3. Discover a new real-life linearization and develop it.

1.3. Representation theory as linearization. The notion of a group G acting on a set X is one of the earliest fundamental concepts we encounter in modern algebra. A very natural generalization of a (left) group action on a set is that of a monoid (a semigroup with the unit element) G acting on a set X, via a map

$$G \times X \to X$$

subject to the associativity and unitality constraints.

The notion of an action of a monoid on a set admits a linearization. Linearization of a monoid is a ring R, linearization of a set on which a monoid acts is that of a (left) module M over R, with the action being a bilinear map

$$R \times M \to M$$

subject to the usual axioms. Study of modules over rings, also known as representation theory, is of utmost importance in modern mathematics.

Transformation from (G, X) to (R, M) can be achieved in two steps. First, linearize monoid G to the semigroup algebra FG, where F is a field and elements of FG are finite linear combinations of elements of G with coefficients in F. Linearize X to the F-vector space FX with basis X. Algebra FG acts on its module FX. This action is often much more interesting than the corresponding action of G on X, as we can see, for example, from the case when G is the symmetric group

S_n. Orbits of group-theoretical actions of S_n on sets are classified by conjugacy classes of subgroups of S_n, while actions of FS_n on F-vector spaces (representations of the symmetric group) constitute a beautiful theory with many applications to geometry, topology and algebra.

Furthermore, some parts of representation theory, especially representations of simple Lie algebras, quantum groups, and Hecke algebras, can be categorified. This note does not discuss such categorifications, instead restricting to more topological examples that include the Euler characteristic of a topological space and the Jones polynomial.

2. Euler characteristic and homology of topological spaces

The Euler characteristic can be thought of a map from the set of sufficiently nice topological spaces to the ring of integers

$$\chi : \text{Nice topological spaces} \rightarrow \mathbb{Z}$$

At first, it is defined naively for only, say, finite simplicial complexes. The latter are spaces given by starting with finitely many disjoint simplices of various dimensions and identifying their subsimplices via linear maps. If the resulting simplicial complex M has $|M|_n$ simplices of dimension n, the Euler characteristic of M is

$$\chi(M) = |M|_0 - |M|_1 + |M|_2 - \cdots = \sum_n (-1)^n |M|_n,$$

the alternating sum of the number of simplices in M of various dimensions.

Theorem 2.1. *$\chi(M)$ is an invariant of M.*

There are various ways in which χ is an invariant. The simplest way to phrase the invariance is as the independence of $\chi(M)$ on the choice of the simplicial decomposition of M, but, in fact, it has a much stronger invariance property, depending only on the homotopy type of M.

We can think of the Euler characteristic map χ as a linearization. A set of sufficiently well-behaved topological spaces, those that have a realization as finite simplicial complexes, is mapped to integers. Almost all topological information about a space is lost under this map, but it does provide an invariant of the space. Simple operations of the disjoint union and direct product of spaces correspond to addition and multiplication on their Euler characteristics:

$$\chi(M \sqcup N) = \chi(M) + \chi(N), \quad \chi(M \times N) = \chi(M)\chi(N).$$

The Euler characteristic is a useful but rather basic invariant of topological spaces. It can be enhanced by keeping track of the boundaries of simplices. The boundary of an n-simplex is a union of its $n+1$ facets, each an $(n-1)$-simplex. The boundary operation can be though of as assigning to an n-simplex the set of its facets, subsimplices of codimension 1. We linearize the boundary operation by taking a simplex v to the linear combination of its facets, with ± 1 coefficients, that is, signs

$$\partial(v) = \sum \pm\text{face}(v).$$

Signs come from keeping track of orientations of v and its facets.

The intuitive observation

The boundary of the boundary is empty

linearizes to

$$\partial\partial = 0$$

or $\partial^2 = 0$. This happens since each $(n-2)$-simplex on the boundary of v appears twice in the expansion of $\partial^2(v)$, with opposite signs, leading to $\partial^2(v) = 0$ for all v, thus $\partial^2 = 0$.

To make this more formal, let S_n be the set of all n-simplices of the decomposition, and $V_n = \mathbb{Q}\langle S_n \rangle$, a $\mathbb{Q}$-vector space with basis S_n. (Instead of $\mathbb{Q}$ one can use $\mathbb{Z}$, $\mathbb{R}$, or any commutative ring). Extend ∂ in a linear fashion to a map $V_n \to V_{n-1}$. The result is a complex

$$\cdots \xrightarrow{\partial} V_{n+1} \xrightarrow{\partial} V_n \xrightarrow{\partial} V_{n-1} \xrightarrow{\partial} \cdots.$$

Since $\partial^2 = 0$, we can define the n-th homology groups of the complex V:

$$H_n(V) = (\ker \partial : V_n \to V_{n-1})/(\operatorname{im} \partial : V_{n+1} \to V_n),$$

by taking the quotient of the subspace of V_n which is the kernel of the boundary map by the image of V_{n+1} under ∂.

This construction allows one to define homology groups $H_n(M)$ of a simplicial complex M as $H_n(V)$, for the above complex V of vector spaces. An important result establishes that $H_n(M)$ are isomorphic to homology groups (singular homology) defined in a more invariant way, via the complex generated in degree n by all continuous maps from a fixed n-complex to M, and the boundary operator given by essentially the same formula as above.

Singular homology, which we also denote $H_n(M)$, has the benefit of being defined for any topological space M. To M we associate the total homology groups

$$H(M) = \bigoplus_{n \geq 0} H_n(M).$$

$H(M)$ is a graded vector space (or an abelian group, if the ground ring in $\mathbb{Z}$).

Theorem 2.2. *The Euler characteristic*

$$\chi(M) = \sum_{n \geq 0} (-1)^n \dim H_n(M).$$

Thus, we can recover the Euler characteristic χ from a more refined invariant—simplicial or singular homology groups. The benefits of homology groups over the Euler characteristic are plentiful; what follows is a rather incomplete list.

(1) Singular homology groups are defined for all topological spaces, not only nice ones (which in our case meant finite simplicial complexes).

(2) They carry much higher informational content than the Euler characteristic.

(3) Homology groups are *functorial*. To a continuous map of topological spaces $f : X \to Y$ there is associated a homomorphism of group (or vector spaces)

$$f_* : H_n(X) \to H_n(Y)$$

These homomorphisms together form a functor

$$\mathrm{Top} \xrightarrow{H} \mathrm{GrAb}$$

from the category of topological spaces and continuous maps to the category of graded abelian groups.

(4) Any commutative ring can be used for coefficients (the most common choices are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$).

(5) Beyond functoriality, homology carries additional structures. One of them (comultiplication) is easier to understand on the dual object to $H_*(M)$, the cohomology groups $H^*(M)$, which are naturally a graded super-commutative ring for any M.

(6) Homology and cohomology generalize to extraordinary (co)homology theories, carrying even more information.

   Passing from the Euler characteristic to homology of topological spaces is an example of *categorification*. Homology functor

$$\text{Top} \xrightarrow{H} \text{GrAb}$$

lifts the Euler characteristic (linearization) map

$$(\text{Nice}) \text{ Topological Spaces} \xrightarrow{\chi} \text{Integers}$$

We now provide a very basic categorification dictionary which helps to explain the lifting of integer-valued invariants that categorification provides.

Structure	Elements and operations	Categorification
$\mathbb{Z}_+$	n, m $n + m$ $n \cdot m$	vector spaces V, W direct sum $V \oplus W$ tensor product $V \otimes W$
$\mathbb{Z}$	n, m $n - m$	complexes V, W of vector spaces cone of a map $f : W \to V$
$\mathbb{Q}$	n/m	It is an open problem to categorify division

   Let us start with the first of the three rows in this dictionary. Category Vect of finite-dimensional vector spaces over a field categorifies the semiring $\mathbb{Z}_+$ of non-negative integers. To an object V of Vect we assign its dimension $\dim(V) \in \mathbb{Z}_+$. Direct sum and tensor product of vector spaces decategorify to the addition and multiplication of nonnegative integers

$$\dim(V \oplus W) = \dim(V) + \dim(W), \quad \dim(V \otimes W) = \dim(V) \otimes \dim(W).$$

Upon decategorification, all information about the morphisms (and most information about the objects) is lost.

   In this model example, decategorification maps objects of the additive monoidal category Vect to elements of the semiring $\mathbb{Z}_+$. Instead of Vect the category of finitely-generated free abelian groups can be used as well, with the rank of the free group taking place of dimension.

   The semiring $\mathbb{Z}_+$ naturally sits inside the ring $\mathbb{Z}$. To lift subtraction of integers to the categorical level, we enlarge the category from vector spaces to that of complexes of vector spaces. A possible natural restriction is to require that complexes are bounded and finite-dimensional in each degree. This can be relaxed to requiring that the total homology groups of a complex are finite-dimensional, so that the Euler characteristic of an object in this category is well-defined.

At first, the morphisms in this category are just homomorphisms of complexes (linear maps of vector spaces in each degree that intertwine differentials in the two complexes). To get a useful category, morphisms need to be modified—we mod out by the ideal of null-homotopic morphisms. The resulting category $\mathscr{C}$ of complexes modulo chain homotopies is *triangulated*, and its Grothendieck group is the corresponding group of a triangulated category. Tensor product of vector spaces naturally extends to complexes, respecting the ideal of null-homotopic morphisms, and leads to a tensor structure on $\mathscr{C}$. The Grothendieck group $K_0(\mathscr{C})$ of $\mathscr{C}$ acquires the structure of a ring, and there is a natural ring isomorphism

$$K(\mathscr{C}) \cong \mathbb{Z}.$$

This isomorphism is induced by the map that takes an object V of $\mathscr{C}$ to its Euler characteristic $\chi(V)$.

The analogue of the subtraction operation on integers is the cone of a map of complexes. Given a map $f : V \to W$ of complexes, shift the source complex V one step to the left, form direct sums of vector spaces $V^{n-1} \oplus W^n$, over all n, and define the differential in the new complex as $-d_V + d_W + f$. The cone complex $\mathrm{Cone}(f)$ has Euler characteristic the difference of those for W and V:

$$\chi\big(\mathrm{Cone}(f)\big) = \chi(W) - \chi(V).$$

Category $\mathscr{C}$ is one of the most fundamental and useful monoidal triangulated categories with the Grothendieck group $\mathbb{Z}$. In the hierarchy of structures we built out of integers the next object, in complexity, is the ring of rational numbers $\mathbb{Q}$, which, unlike integers, allows division by a nonzero number. At this point we are already at a limit of current mathematical knowledge—it is not known how to categorify division and the ring of rational numbers $\mathbb{Q}$. We record this as an open problem.

Problem 2.3. Describe a triangulated monoidal category $\mathscr{C}$ with the Grothendieck ring isomorphic to $\mathbb{Q}$.

Even the following apparently simpler problem appears to be open (an approach to the $n = 2$ case is being considered in [5]).

Problem 2.4. Construct a triangulated monoidal category $\mathscr{C}$ with the Grothendieck ring isomorphic to $\mathbb{Z}\left[\frac{1}{n}\right]$.

3. Jones polynomial and its categorification

3.1. Jones polynomial and the Kauffman bracket.
Knots and links are smooth or piecewise-linear embeddings of a single circle S^1 (knots) or a disjoint union

of finitely-many circles (links) into $\mathbb{R}^3$, with the embeddings considered up to isotopies. They appear toyish at first glance, but in the past few decades have been related to an amazing plethora of deep structures in mathematics and mathematical physics.

On such structure is the Jones polynomial [2]. It is an invariant of links that assigns a Laurent polynomial $J(L)$ in a single variable q to an oriented link in $\mathbb{R}^3$, and can be thought as a map

$$J : \text{Links} \to \mathbb{Z}[q, q^{-1}]$$

from the set of oriented links in $\mathbb{R}^3$ to a linear structure—the ring of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$.

The Jones polynomial is uniquely determined by the conditions:

- The Jones polynomial of the trivial knot is $q + q^{-1}$,

- For any three links that differ only in the neighbourhood of a small ball as depicted below, there is a linear relation on their Jones polynomials

$$q^2 J\left(\times\right) - q^{-2} J\left(\times\right) = (q - q^{-1}) J\left(\big)\big(\right). \tag{1}$$

Shortly after the discovery of the Jones polynomial, Louis Kauffman [3] found a recursive construction of the polynomial that allows an elementary proof that the polynomial is well-defined and provides a wealth of other structural information. His construction is known as the Kauffman bracket.

To define the Kauffman bracket, take a generic projection D of an oriented link L onto the plane $\mathbb{R}^2$ (generic in the sense of not having triple intersection points and tangency points) and temporarily forget about the orientation of L.

If projection D has no crossings, we define its Kauffman bracket

$$\langle D \rangle = (q + q^{-1})^c,$$

where c is the number of components (circles) in the projection.

$$\left(\bigcirc\,\bigcirc\right) \mapsto (q + q^{-1})^{\#\ of\ circles}$$

D having crossings is an interesting case. We pick a crossing of D and define the bracket of D recursively, as a linear combination of brackets of projections with one crossing less:

$$\times = \smile - q^{-1} \big)\big($$

If projection D has n crossings, the full expansion of $\langle D \rangle$ will have 2^n terms, each of the form $\pm q^a(q + q^{-1})^b$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_+$. This construction uniquely determines $\langle D \rangle$ for any generic diagram D. To get a link invariant, recall that L is oriented and define

$$J(D) = (-1)^{x(D)}q^{2x(D)-y(D)}\langle D \rangle, \tag{2}$$

where $x(D)$ and $y(D)$ is the number of negative $\times$ and positive $\times$ crossings of D.

Theorem 3.1 (L. Kauffman). *The resulting polynomial $J(D)$ does not depend on a choice of projection D of a link L and equals the Jones polynomial $J(L)$.*

The theorem claims the invariance of $J(D)$ under the Reidemeister moves of link diagrams and has a direct computational proof. Kauffman's bracket gives the easiest way to see that the Jones polynomial is well-defined.

3.2. Graded complexes. We now move on to categorification of the Jones polynomial. It takes values in $\mathbb{Z}[q, q^{-1}]$, and we start by realizing this ring as the Grothendieck ring of a suitable category. Recall that the ring $\mathbb{Z}$ was lifted, at first, to the category of finite-dimensional vector spaces, and later, to the category of complexes of vector spaces. Analogous lifting of $\mathbb{Z}[q, q^{-1}]$ is realized via graded vector spaces. Consider the category GVect with objects—graded finite-dimensional vector spaces (say, over $\mathbb{Q}$)

$$V = \bigoplus_{n \in \mathbb{Z}} V_n = \cdots \oplus V_{-2} \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus V_2 \oplus \cdots$$

In particular, only finitely many of V_n's are nonzero. Morphisms in GVect are linear maps $V \to W$ that preserve the grading, that is, take V_n to W_n for all n. To V assign its graded dimension

$$\mathrm{gdim}(V) = \sum_{n \in \mathbb{Z}} \dim(V_n) \cdot q^n \in \mathbb{Z}[q, q^{-1}].$$

One can "add" and "multiply" graded vector spaces, by forming their direct sum and tensor product. These operations turn GVect into a linear monoidal category. Its Grothendieck ring is naturally $\mathbb{Z}[q, q^{-1}]$, and variable q becomes a grading shift upon this lifting.

$\mathrm{gdim}(V)$, for a graded vector space V, has non-negative coefficients. To allow arbitrary integer coefficients we need to pass to complexes, while maintaining the

grading. A complex of graded vector spaces V is a bigraded vector space

$$V = \bigoplus_{n,m \in \mathbb{Z}} V_n^m$$

with a differential $\partial : V \to V$, $\partial^2 = 0$, that respects additional grading, that is, restricts to

$$\partial : V_n^m \to V_{n-1}^m$$

for all n, m. We can think of V as the direct sum of complexes of vector spaces

$$V^m = \cdots \xrightarrow{\partial} V_{n+1}^m \xrightarrow{\partial} V_n^m \xrightarrow{\partial} V_{n-1}^m \xrightarrow{\partial} \cdots$$

in each q-degree m.

To a complex of graded vector spaces V we associate homology groups

$$H_n^m(V) = (\ker \partial : V_n^m \to V_{n-1}^m)/(\operatorname{im} \partial : V_{n+1}^m \to V_n^m)$$

Homology of V is a bigraded vector space

$$H(V) = \bigoplus_{n,m \in \mathbb{Z}} H_n^m(V).$$

As for complexes, we pick a suitable category $\mathscr{C}$ to work with, by requiring that homomorphisms and homotopies of graded complexes respect the extra grading m, and imposing the finite-dimensionality condition $\dim\big(H(V)\big) < \infty$.

For an object V of $\mathscr{C}$ the Euler characteristic of each complex $V^m = \bigoplus_{n \in \mathbb{Z}} V_n^m$ is an integer

$$\chi(V^m) = \sum_n (-1)^n \dim\big(H_n^m(V)\big).$$

If V^m, and not just its homology, is finite-dimensional, the Euler characteristic can be computed from the spaces themselves,

$$\chi(V^m) = \sum_n (-1)^n \dim(V_n^m).$$

We make these integers into coefficients of a Laurent polynomial

$$\chi(V) = \sum_m \chi(V^m) \cdot q^m = \sum_{m,n} (-1)^n \dim\big(H_n^m(V)\big) \cdot q^m.$$

When V is finite-dimensional, we also have

$$\chi(V) = \sum_{n,m}(-1)^n \dim(V_n^m) \cdot q^m$$

3.3. Categorification.

The idea behind categorification of the Jones polynomial $J(L)$ is to look at the formula for the Kauffman bracket $\langle D \rangle$ of a diagram D and consistenly lift all the terms there into a complex $C(D)$ of graded vector spaces with the Euler characteristic $\langle D \rangle$.

Key term in the construction is $q + q^{-1}$, the Kauffman bracket of a simple circle in the plane, also equal to the Jones polynomial of the unknot. We lift this polynomial to the graded vector space

$$A = \mathbb{Q} \cdot \mathbf{1} \oplus \mathbb{Q} \cdot X$$

with basis elements denoted $\mathbf{1}$ and X, with degrees

$$\deg \mathbf{1} = -1, \quad \deg X = 1.$$

Graded dimension of A is $q + q^{-1}$. The complex associated to the simple circle diagram will be just A, placed in homological degree 0:

$$0 \to A \to 0.$$

To a crossingless diagram with two circles we assign $A \otimes A$, which has a basis $\{\mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes X, X \otimes \mathbf{1}, X \otimes X\}$. In general, to a diagram which consists of k disjoint, perhaps nested, circles we associate $A^{\otimes k}$, which has the graded dimension $(q + q^{-1})^k$.

We next move on to diagrams with a single crossing. For such a diagram D its Kauffman bracket $\langle D \rangle$ is the difference of two terms, one coming from a two-circle diagram, the other from a one-circle diagram, corresponding to two different ways to simplify (resolve) D into crossingless diagrams. One such example is depicted below.
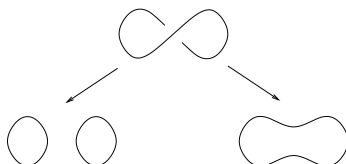
In the Kauffman formula for $\langle D \rangle$, the single circle diagram enters with the coefficient $-q^{-1}$, and the two-circle one with coefficient 1. To interpret $\langle D \rangle$ as

the Euler characteristic, we place $A^{\otimes 2}$ in homological degree 0, A in homological degree 1, and look for the differential to make this into a complex.

$$0 \to A \otimes A \xrightarrow{M} A \to 0$$

We label the differential M, for *multiplication*, since that's what it looks like. To interpret q^{-1} in the formula, we need to shift the internal grading of A down by 1. Since we want homology to be an invariant of a knot, and not just its diagram, the homology of the above complex should be isomorphic to A, perhaps up to an overall grading shift, that we can take care of later. Therefore, M is surjective and, moreover, it must preserve the internal degree. The table below lists the degrees of basis elements of the two spaces, with $\{-1\}$ denoting the degree shift down by 1.

Degree	Basis of $A \otimes A$	Basis of $A\{-1\}$
2	$X \otimes X$	
1		
0	$X \otimes \mathbf{1}, \mathbf{1} \otimes X$	X
-1		
-2	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1}$

M is now constrained to essentially a unique map, given below.

$$\begin{aligned} X \otimes X &\mapsto 0 \\ X \otimes \mathbf{1}, \mathbf{1} \otimes X &\mapsto X \\ \mathbf{1} \otimes \mathbf{1} &\mapsto \mathbf{1} \end{aligned}$$

This map makes A into a commutative associative algebra with the unit element $\mathbf{1}$.

The other case for a single crossing complex is when the $-q^{-1}$ coefficient in the Kauffman formula appears with the $(q + q^{-1})^2$ term. A possible diagram when this happens is depicted below.
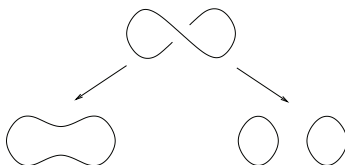
By analogy with the previous case, we want to lift the formula for $\langle D \rangle$ to a complex

$$0 \to A \xrightarrow{\Delta} A \otimes A\{-1\} \to 0$$

with the differential denoted by Δ, since it resembles comultiplication. The degrees for basis elements of these vector spaces are listed in the table below.

Degree	Basis of A	Basis of $A \otimes A\{-1\}$
1	X	$X \otimes X$
0		
-1	$\mathbf{1}$	$X \otimes \mathbf{1}, \mathbf{1} \otimes X$
-2		
-3		$\mathbf{1} \otimes \mathbf{1}$

The differential should preserve the internal grading. This and other natural conditions, including that the homology of this complex should be isomorphic to A, modulo a bigrading shift, leads to a formula for Δ:

$$X \to X \otimes X$$
$$\mathbf{1} \to \mathbf{1} \otimes X + X \otimes \mathbf{1}$$

Maps M and Δ turn A into a commutative Frobenius algebra—a commutative unital algebra equipped with a nondegenerate symmetric trace form. Such algebras are in a bijection with 2-dimensional topological quantum field theories, that is, monoidal functors from the category of 2-dimensional cobordisms between 1-manifolds to the category of vector spaces. A is, in addition, graded, and the degree of the map associated to a cobordism S equals minus the Euler characteristic of S.

An arbitrary diagram D with n crossing has 2^n resolutions into crossingless diagrams. With each resolution we associate $A^{\otimes k}$, where k is the number of circles in it. These powers of A can be naturally placed into the vertices of an n-dimensional cube. Every edge of a cube corresponds to an elementary modification of a resolution, converting a k-circle planar diagram to a $(k \pm 1)$-circle diagram. This modification (which can be realized by a cobordism between the circle diagrams) induces a map between corresponding tensor powers of A (the map is either M or Δ times the identity on the remaining circles).

Furthermore, for every square face of the cube, two compositions of maps assigned to its edges commute, due to M and Δ being structure maps of a two-dimensional TQFT.

Taking direct sums of tensor powers of A (with the suitably shifted internal grading) along the hyperplanes orthogonal to the main diagonal of the cube and defining the differential to be a signed sum of the edge maps of the cube nets us a complex of graded vector spaces $\overline{C}(D)$. The Euler characteristic of $\overline{C}(D)$ is the Kauffman bracket $\langle D \rangle$.

One then shifts the bigrading of $\overline{C}(D)$ to match the coefficient in the formula (2) and gets the complex $C(D)$ associated to a planar diagram D. Its homology is denoted $H(D)$ and carries a bigrading.

Theorem 3.2. $H(D)$ *depends only on the underlying link L and not on its diagram D. The Euler characteristic of $H(D)$ is the Jones polynomial $J(L)$.*

Denoting $H(D)$ by $H(L)$, we obtain a homology theory of links in $\mathbb{R}^3$. It can be naturally thought of as a categorification of the Jones polynomial (the term *categorification* was originally introduced by Louis Crane and Igor Frenkel in a related context [1]).

Since its discovery, this homology theory of links has been greatly developed and thoroughly understood by many people. Below is a very brief list of some of the benefits and structures stemming from the homology theory H.

- H contains large amount of information about knots and links and gives rise to new structural relations between low-dimensional topology and algebra.

- H is functorial and extends to an invariant of link cobordisms (Jacobsson, Bar–Natan, Khovanov, Clark–Morrison–Walker).

- H carries homological operations (Lipshitz–Sarkar, Kriz–Kriz–Po).

- Other link polynomials have also been categorified, including the Alexander polynomial (Ozsváth–Rasmussen–Szabó), the HOMFLYPT polynomial and more general Reshetikhin–Turaev invariants.

- H relates to several areas of mathematics (geometric representation theory, symplectic topology, algebraic geometry, the Langlands program).

- H appears in mathematical physics (Gukov–Schwarz–Vafa, Witten).

We would like to emphasize that the overall structure is built in two steps. First step is constructing the Jones polynomial, which we can think of as an example of linearization, in this case going from links, topological objects (which admit a discrete combinatorial interpretation) to elements of a linear structure, the ring of Laurent polynomials. The second step is categorification, lifting linear invariants of links to vector spaces and homology groups. It is possible that categorification can be thought of as a kind of second linearization, but we will not try to carefully phrase here what this might mean.

We conclude with an exercise and a problem for the reader.

Exercise 3.3. Can the following structures and operations be interpreted as linearizations?

- Passing from a topological space to the ring of continuous functions on it.

- Quantization.

- Quantum computation.

Problem 3.4. Discover new linearizations and categorifications and develop them.

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