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Approximation rates for regularized level set power mean curvature flow

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Abstract. In [11] the evolution of hypersurfaces in $\mathbb{R}^{n+1}$ with normal speed equal to a power $k > 1$ of the mean curvature is considered and the level set solution u of the flow is obtained as the C^0-limit of a sequence u^ε of smooth functions solving the regularized level set equations. We prove a rate for this convergence.

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1. Introduction and main result

The mean curvature flow, cf. e.g. [4] and [7], evolves hypersurfaces in the direction of their normal with normal speed equal to the mean curvature. During the last thirty years many related extrinsic curvature flows have been analyzed; they differ mainly in the prescribed normal velocity and the ambient space, in which the evolution takes place; to name only a few of them cf. e.g. the inverse mean curvature flow [8], the Gaussian curvature flow [1] and the inverse mean curvature flow in a Lorentzian manifold [5]. In the first example [8] the flow is used to prove the Riemannian Penrose inequality, in the second example [1] the flow models the changing shape of a tumbling stone subjected to collisions from all directions with uniform frequency and the third example [5] implies that future ends of certain cosmological spacetimes can be foliated by the leaves of an inverse mean curvature flow and as a consequence also by hypersurfaces with constant mean curvature.

Schulze [10] considers the evolution of hypersurfaces in $\mathbb{R}^{n+1}$ in the direction of their normal, for which the normal speed is given by a power $k > 1$ of the mean curvature. In [10], [12] it is shown that this flow shrinks – similar to the mean curvature flow – a convex, embedded and closed initial hypersurface to a point and becomes spherical in the limit under a certain pinching condition for the principle curvatures of the initial hypersurface. In a further paper [11] Schulze uses a level set formulation of this flow to prove certain isoperimetric inequalities. The level set solution in [11] is obtained as the limit of a family of solutions of regularized equations.

The level set formulation is powerful since it can handle topological changes of the flow, i.e. in the case of non-convex initial hypersurfaces with positive mean curvature the parametric flow [10] might develop a singularity but the level set flow [11] continues to exist.

In [3] Deckelnick proves a rate of convergence for the approximation of the level set solution of mean curvature flow (existence of a solution is a classical result by Evans and Spruck [4]) by using a finite difference scheme; for the approximation he uses the solution of the regularized level set equation as an intermediate step and divides the error estimate correspondingly into the approximation error between the level set solution and the solution of the regularized level set equation and the error for the finite difference approximation of the regularized level set equation. Deckelnick's estimate of the error between the level set solution and the solution of the regularized level set equation is extended in [9] to certain cases of type $0 < k < 1$; concerning the existence of a solution for the level set equation in these cases we refer to the references in [9].

The goal of our paper is to prove a rate of convergence for the solutions of the regularized equations in [11].

We introduce our setting more precisely. Let M be a smooth n-dimensional compact manifold without boundary, $k > 1$ and $x_0 : M \to \mathbb{R}^{n+1}$ a smooth embedding such that $x_0(M)$ has positive mean curvature, then there exist a small $T > 0$ and a smooth mapping

$$x : [0, T) \times M \to \mathbb{R}^{n+1} \tag{1}$$

with

$$\begin{aligned} x(0, \cdot) &= x_0 \\ \dot{x}(t, \xi) &= -H^k \nu. \end{aligned} \tag{2}$$

Here, H and ν denote the mean curvature and the outer normal of $x(t, \cdot)(M)$ at $x(t, \xi)$ respectively, cf. [11], Section 1. Furthermore, 'smooth' stands here and below for C^∞.

We call this a power mean curvature flow (PMCF).

We give a level set formulation of PMCF. Let $\Omega \subset \mathbb{R}^{n+1}$ be open, connected and bounded having smooth boundary $\partial\Omega$ with positive mean curvature. We call the level sets $\Gamma_t = \{x \in \bar{\Omega} : u(x) = t\}$ of the continuous function $0 \le u \in C^0(\bar{\Omega})$ a level set PMCF, if u is a viscosity solution of

$$\begin{aligned} \operatorname{div}\left(\frac{Du}{|Du|}\right) &= -\frac{1}{|Du|^{1/k}} \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{3}$$

Note, that in equation (3) the nonlinearity coming from the exponent $k > 1$ affects only lower order (spatial) derivatives of the level set function which would be different in case of a time-dependent level set formulation as in [3].

If u is smooth in a neighborhood of $x \in \Omega$ with non vanishing gradient and satisfies there (3), then the level set $\{y \in \bar{\Omega} : u(y) = u(x)\}$ moves locally at x according to (2).

Using elliptic regularization of level set PMCF we obtain the equation

$$\begin{aligned} \operatorname{div}\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\right) &= -(\varepsilon^2 + |Du^\varepsilon|^2)^{-1/2k} \quad \text{in } \Omega \\ u^\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

which has unique smooth solutions u^ε for sufficiently small $\varepsilon > 0$; moreover, there is $c_0 > 0$ such that

$$\|u^\varepsilon\|_{C^1(\bar{\Omega})} \le c_0 \tag{5}$$

and (for a subsequence)

$$u^\varepsilon \to u \in C^{0,1}(\bar{\Omega}) \tag{6}$$

in $C^0(\bar{\Omega})$. Note, that in view of (5) and the uniform convergence in (6) u is lipschitz continuous with lipschitz constant c_0. We call u a weak solution of (3), which is unique for $n \le 6$.

All the above facts are proved in [11], Section 4.

The limit function u satisfies (3) in the viscosity sense, cf. Lemma 2.4. We formulate our main result.

Theorem 1.1. *For every $\lambda > 2k$ there is a positive constant $c = c(\lambda, k, \Omega)$ so that*

$$\|u - u^\varepsilon\|_{C^0(\bar{\Omega})} \le c\varepsilon^{1/\lambda}. \tag{7}$$

From interpolation we immediately obtain the following corollary.

Corollary 1.2. *For every* $0 < \Theta < 1$ *and* $\lambda > 2k$ *there is a positive constant* $c = c(\lambda, k, \Theta, \Omega)$ *so that*

$$\|u - u^\varepsilon\|_{C^{0,\Theta}(\bar{\Omega})} \le c\varepsilon^{(1/\lambda)(1-\Theta)}. \tag{8}$$

In the case $k = 1$ which means mean curvature flow we can realize in Theorem 1.1 every power of ε which lies in $\left(0, \frac{1}{2}\right)$. This is in accordance with the corresponding rate in Deckelnick's paper [3], Theorem 1.2 and Mitake's paper [9], Theorem 1 for the time dependent level set regularization.

In the remaining part of the paper we prove Theorem 1.1. We remark that we use the summation convention to sum over repeated indices from 1 to $n+1$ without indicating this explicitly. Partial derivatives of a function $u = u(x)$, $x \in \mathbb{R}^{n+1}$, are denoted by $D_i u$, $D_i D_j u$, Du, etc., and for a function $\varphi = \varphi(x, y)$, $x, y \in \mathbb{R}^{n+1}$ by $D_{x^i}\varphi$, $D_{x^i}D_{x^j}\varphi$, $D_x\varphi$, etc. with obvious meanings.

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2. Proof of Theorem 1.1

We state the definition of a viscosity solution of (3) by adapting the corresponding definitions in [4], Sections 2.2 and 2.3 and [2], Section 2. From (3) we obtain that

$$F(u) := -|Du|^{1/k-1}\left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2}\right) D_i D_j u = 1. \tag{9}$$

and from (4) that

$$F_\varepsilon(u^\varepsilon) := -(|Du^\varepsilon|^2 + \varepsilon^2)^{1/2k-1/2}\left(\delta_{ij} - \frac{D_i u^\varepsilon D_j u^\varepsilon}{|Du^\varepsilon|^2 + \varepsilon^2}\right) D_i D_j u^\varepsilon = 1. \tag{10}$$

The second order 'semijets' are defined as follows, cf. [2], Section 2.

Definition 2.1. Let $u \in C^0(\Omega)$ and $\hat{x} \in \Omega$, then we define

$$J_\Omega^{2,+}u(\hat{x}) = \Big\{(p, X) \in \mathbb{R}^{n+1} \times S(n+1) : u(x) \le u(\hat{x}) + \langle p, x - \hat{x}\rangle + \frac{1}{2}\langle X(x - \hat{x}), x - \hat{x}\rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x}\Big\} \tag{11}$$

and for $x \in \Omega$

$$\bar{J}_\Omega^{2,+}u(x) = \{(p, X) \in \mathbb{R}^{n+1} \times S(n+1) : \text{there are } x_k \in \Omega, \text{ and } (p_k, X_k) \in J_\Omega^{2,+}u(x_k), \text{ so that } (x_k, p_k, X_k) \to (x, p, X)\}, \tag{12}$$

where $S(l)$, $l \in \mathbb{N}$, denotes the set of symmetric $l \times l$ matrices. Furthermore, we set $J_\Omega^{2,-}u(\hat{x}) = -J_\Omega^{2,+}(-u)(\hat{x})$ and $\bar{J}_\Omega^{2,-}u(x) = -\bar{J}_\Omega^{2,+}(-u)(x)$ where we use the notation

$$-(A \times B) = \{(-p, -X) : (p, X) \in A \times B\} \tag{13}$$

for subsets $A \subset \mathbb{R}^{n+1}$, $B \subset S(n+1)$.

Now, we can state the definition of a viscosity solution.

Definition 2.2. (i) A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of (3), if for all $(\eta, X) \in J_\Omega^{2,+}(u)(x)$, $x \in \Omega$, there holds

$$-|\eta|^{1/k-1}\left(\delta_{ij} - \frac{\eta_i\eta_j}{|\eta|^2}\right)X_{ij} \leq 1, \tag{14}$$

if $\eta \neq 0$ and

$$-(\delta_{ij} - \tilde{\eta}_i\tilde{\eta}_j)X_{ij} \leq 0 \tag{15}$$

for some $\tilde{\eta}$ with $|\tilde{\eta}| \leq 1$, if $\eta = 0$.

(ii) A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of (3), if for all $(\eta, X) \in J_\Omega^{2,-}(u)(x)$, $x \in \Omega$, there holds

$$-|\eta|^{1/k-1}\left(\delta_{ij} - \frac{\eta_i\eta_j}{|D\eta|^2}\right)X_{ij} \geq 1, \tag{16}$$

if $\eta \neq 0$ and

$$-(\delta_{ij} - \tilde{\eta}_i\tilde{\eta}_j)X_{ij} \geq 0 \tag{17}$$

for some $\tilde{\eta}$ with $|\tilde{\eta}| \leq 1$, if $\eta = 0$.

(iii) A function u, which is supersolution and subsolution of (3) is a viscosity solution of (3).

Remark 2.3. A simple inspection shows that we could have replaced $J_\Omega^{2,+}(u)(x)$ in the preceding definition by $\bar{J}_\Omega^{2,+}(u)(x)$ and $J_\Omega^{2,-}(u)(x)$ by $\bar{J}_\Omega^{2,-}(u)(x)$.

Lemma 2.4. *The function u in* (6) *is a viscosity solution of* (3).

Proof. The claim follows analogously to the argumentation in [4], Section 4.3. □

The strategy for the proof of Theorem 1.1 is to apply the maximum principle for semicontinuous functions, cf. [2], Theorem 3.2, to a suitable auxiliary function which leads to an estimate for $u^\varepsilon - u$, cf. Theorem 2.5. In a further step the obtained rate will be calculated more explicitly. In order to define the auxiliary function we need some constants, which will be specified in the following. Let

$$\gamma > 1 + k \tag{18}$$

and $\alpha, s > 0$ be small so that

$$\beta_1(\alpha, s) > \beta_2(\alpha, s), \tag{19}$$

where

$$\beta_1(\alpha, s) := \frac{2 - s + \alpha\left(2 - \frac{1}{k}\right)}{\gamma\left(2 - \frac{1}{k}\right) + \frac{1}{k} - 1}, \qquad \beta_2(\alpha, s) := \frac{\alpha + ks}{\gamma - k - 1} \tag{20}$$

and choose

$$0 < r < \frac{\alpha}{\gamma}. \tag{21}$$

Theorem 2.5. *There is $c = c(k, \Omega) > 0$ such that*

$$\|u^\varepsilon - u\|_{C^0(\bar{\Omega})} \leq c\varepsilon^{\min(r,s)} \tag{22}$$

for all $\varepsilon > 0$.

Our first goal is to prove Theorem 2.5 by adapting the proof of [3], Theorem 1.2. For $\varepsilon > 0$ we define $w_\varepsilon : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ by

$$w_\varepsilon(x, y) := \mu u(x) - u^\varepsilon(y) - \frac{\varepsilon^{-\alpha}}{\gamma}|x - y|^\gamma, \qquad x, y \in \bar{\Omega}, \tag{23}$$

where

$$\mu = \mu(\varepsilon) = (1 - \varepsilon^s)^k. \tag{24}$$

We use the abbreviation

$$\varphi(x, y) := \frac{\varepsilon^{-\alpha}}{\gamma}|x - y|^\gamma. \tag{25}$$

Let $\hat{x}, \hat{y} \in \bar{\Omega}$ such that

$$w_\varepsilon(\hat{x}, \hat{y}) = \sup_{\bar{\Omega}\times\bar{\Omega}} w_\epsilon. \tag{26}$$

Lemma 2.6. *There holds $\hat{x} \in \partial\Omega$ or $\hat{y} \in \partial\Omega$.*

Proof. We assume $\hat{x}, \hat{y} \in \Omega$. From the maximum principle for semicontinuous function which is introduced in [2], Section 3, cf. especially [2], Theorem 3.2, we deduce that for every $\rho > 0$ there are $X, Y \in S(n+1)$ such that

$$\big(D_x\varphi(\hat{x}, \hat{y}), X\big) \in \bar{J}^{2,+}_{\Omega}(\mu u)(\hat{x}) \ \wedge \ \big(D_y\varphi(\hat{x}, \hat{y}), Y\big) \in \bar{J}^{2,+}_{\Omega}(-u^\varepsilon)(\hat{y}) \tag{27}$$

and

$$-\left(\frac{1}{\rho} + \|A\|\right)I \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le A + \rho A^2, \tag{28}$$

where $A := D^2\varphi(\hat{x}, \hat{y})$. We calculate

$$D_x\varphi(\hat{x}, \hat{y}) = \varepsilon^{-\alpha}|\xi|^{\gamma-2}\xi = -D_y\varphi(\hat{x}, \hat{y}), \qquad \xi = \hat{x} - \hat{y}, \tag{29}$$

and

$$A = \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}, \qquad B = \varepsilon^{-\alpha}|\xi|^{\gamma-4}\big((\gamma-2)\xi\otimes\xi + |\xi|^2 I\big). \tag{30}$$

Using

$$F(\mu u) = \mu^{1/k}, \qquad F_\varepsilon(-u^\varepsilon) = -1, \tag{31}$$

we conclude from (27) that

$$-\left(\delta_{ij} - \frac{D_{x^i}\varphi D_{x^j}\varphi}{|D_x\varphi|^2}\right)X_{ij} \le \mu^{1/k}|D_x\varphi|^{1-1/k} \qquad \text{at } (\hat{x}, \hat{y}) \tag{32}$$

if $D_x\varphi(\hat{x}, \hat{y}) \neq 0$ and

$$-(\delta_{ij} - \eta_i\eta_j)X_{ij} \le 0 \tag{33}$$

for some $\eta \in \mathbb{R}^n$ with $|\eta| \le 1$ if $D_x\varphi(\hat{x}, \hat{y}) = 0$; furthermore, there holds

$$-\left(\delta_{ij} - \frac{D_{y^i}\varphi D_{y^j}\varphi}{|D_y\varphi|^2 + \varepsilon^2}\right)Y_{ij} \le -\big(|D_y\varphi|^2 + \varepsilon^2\big)^{1/2-1/2k} \qquad \text{at } (\hat{x}, \hat{y}). \tag{34}$$

From (28) we get for all $\zeta \in \mathbb{R}^n$

$$\begin{aligned} \zeta^t(X+Y)\zeta &= (\zeta^t, \zeta^t)\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \\ &\le (\zeta^t, \zeta^t)\left\{\begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + 2\rho\begin{pmatrix} B^2 & -B^2 \\ -B^2 & B^2 \end{pmatrix}\right\}\begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \\ &= 0, \end{aligned} \tag{35}$$

i.e.

$$X + Y \le 0, \tag{36}$$

and

$$\begin{aligned} \xi^t Y \xi &= (0, \xi^t)\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} 0 \\ \xi \end{pmatrix} \\ &\le \xi^t B \xi + 2\rho \xi^t B^2 \xi \\ &\le (\gamma - 1)\varepsilon^{-\alpha}|\xi|^\gamma + 2\rho\xi^t B^2 \xi. \end{aligned} \tag{37}$$

Case $\hat{x} \neq \hat{y}$: We add the inequalities (32) and (34) and get

$$\begin{aligned} LHS &:= -\left(\delta_{ij} - \frac{D_{x^i}\varphi D_{x^j}\varphi}{|D_x\varphi|^2}\right)X_{ij} - \left(\delta_{ij} - \frac{D_{y^i}\varphi D_{y^j}\varphi}{|D_y\varphi|^2 + \varepsilon^2}\right)Y_{ij} \\ &\le (\mu^{1/k} - 1)|D_x\varphi|^{1-1/k}. \end{aligned} \tag{38}$$

We estimate LHS from below

$$\begin{aligned} LHS &= -\left(\delta_{ij} - \frac{D_{x^i}\varphi D_{x^j}\varphi}{|D_x\varphi|^2}\right)(X_{ij} + Y_{ij}) \\ &\quad - \varepsilon^2 \frac{D_{x^i}\varphi D_{x^j}\varphi}{|D_x\varphi|^2(|D_y\varphi|^2 + \varepsilon^2)} Y_{ij} \\ &\ge -\frac{\varepsilon^2 \xi^t Y \xi}{|\xi|^2(|\xi|^{2\gamma-2}\varepsilon^{-2\alpha} + \varepsilon^2)} \\ &\ge \frac{-(\gamma-1)\varepsilon^{2-\alpha}|\xi|^\gamma - 2\varepsilon^2\rho\xi^t B^2 \xi}{|\xi|^2(|\xi|^{2\gamma-2}\varepsilon^{-2\alpha} + \varepsilon^2)} \end{aligned} \tag{39}$$

where we used (36) and (37). Combining (38) with (39), letting $\rho \to 0$ and applying the relations (24) and (29) yield

$$-\frac{(\gamma-1)\varepsilon^{2-\alpha}|\xi|^{\gamma-2}}{|\xi|^{2\gamma-2}\varepsilon^{-2\alpha}+\varepsilon^{2}} \leq -\varepsilon^{s-\alpha(1-1/k)}|\xi|^{(\gamma-1)(1-1/k)}. \tag{40}$$

We multiply this inequality by the denominator of the left-hand side and deduce two inequalities

$$\begin{aligned} -(\gamma-1)\varepsilon^{2-\alpha}|\xi|^{\gamma-2} &\leq -\varepsilon^{s-\alpha(3-1/k)}|\xi|^{(\gamma-1)(1-1/k)+2\gamma-2} \\ -(\gamma-1)\varepsilon^{2-\alpha}|\xi|^{\gamma-2} &\leq -\varepsilon^{s+2-\alpha(1-1/k)}|\xi|^{(\gamma-1)(1-1/k)}, \end{aligned} \tag{41}$$

which lead to

$$\begin{aligned} (\gamma-1)\varepsilon^{2-s+\alpha(2-1/k)} &\geq |\xi|^{\gamma(2-1/k)-1+1/k} \\ (\gamma-1)^{k}\varepsilon^{-\alpha-ks} &\geq |\xi|^{-\gamma+k+1}. \end{aligned} \tag{42}$$

Accounting for (18) we have

$$\begin{aligned} |\xi| &\leq (\gamma-1)^{1/(\gamma(2-1/k)-1+1/k)}\varepsilon^{(2-s+\alpha(2-1/k))/(\gamma(2-1/k)+1/k-1)} =: c_1\varepsilon^{\beta_1(\alpha,s)} \\ |\xi| &\geq (\gamma-1)^{k/(-\gamma+k+1)}\varepsilon^{(\alpha+ks)/(\gamma-k-1)} =: c_2\varepsilon^{\beta_2(\alpha,s)}. \end{aligned} \tag{43}$$

In view of (19) we get a contradiction for small $\varepsilon > 0$.

Case $\hat{x} = \hat{y}$: Due to $\gamma > 2$ and (30) we have $B = 0$, so that a calculation as in (37) (now with η instead of ξ) shows

$$\eta^t Y\eta \leq 0. \tag{44}$$

Hence, adding (33) to (34) and having (36) in mind we get

$$\begin{aligned} \varepsilon^{1-1/k} &\leq (\delta_{ij}-\eta_i\eta_j)X_{ij} + \delta_{ij}Y_{ij} \\ &\leq (\delta_{ij}-\eta_i\eta_j)(X_{ij}+Y_{ij}) + \eta^t Y\eta \\ &\leq 0, \end{aligned} \tag{45}$$

which is a contradiction. □

Lemma 2.7. *There is $c_4 > 0$ such that*

$$w_\varepsilon(\hat{x},\hat{y}) \leq c_4\varepsilon^{r}. \tag{46}$$

Proof. In view of Lemma 2.6 we have w.l.o.g. that $\hat{y} \in \partial\Omega$. (Otherwise, the succeeding argument will work analogously where now (47) holds with μ omitted.) Hence we can write

$$w_\varepsilon(\hat{x}, \hat{y}) = \mu u(\hat{x}) - \mu u(\hat{y}) - \frac{\varepsilon^{-\alpha}}{\gamma}|\hat{x} - \hat{y}|^\gamma. \tag{47}$$

In case $|\hat{x} - \hat{y}| \leq \varepsilon^r$ we get using the lipschitz continuity of u, cf. (6) and the succeeding sentence,

$$w_\varepsilon(\hat{x}, \hat{y}) \leq \mu c_0 |\hat{x} - \hat{y}| \leq \mu c_0 \varepsilon^r, \tag{48}$$

which proves the lemma.

The remaining case $|\hat{x} - \hat{y}| > \varepsilon^r$ is not available for sufficiently small $\varepsilon > 0$, for we estimate

$$w_\varepsilon(\hat{x}, \hat{y}) \leq 2\mu c_0 - \frac{\varepsilon^{r\gamma-\alpha}}{\gamma} \to -\infty, \qquad \varepsilon \to 0. \tag{49}$$

□

Now, collecting facts we finish the proof of Theorem 2.5. Let $x \in \Omega$ arbitrary. Then

$$\begin{aligned} u(x) - u^\varepsilon(x) &= \mu u(x) - u^\varepsilon(x) + (1-\mu)u(x) \\ &= w_\varepsilon(x, x) + (1-\mu)u(x) \\ &\leq c_4\varepsilon^r + c_0\varepsilon^s \\ &\leq c_5\varepsilon^{\min(r,s)}, \end{aligned} \tag{50}$$

with a positive constant c_5. Interchanging the roles of u and u^ε we see, that there is a positive constant c_6 with

$$|u(x) - u^\varepsilon(x)| \leq c_6\varepsilon^{\min(r,s)}. \tag{51}$$

This proves Theorem 2.5. It remains to prove Theorem 1.1 which will be done by rewriting the right-hand side of the estimate in Theorem 2.5. Since (19) 'improves' for decreasing s we choose $s = \frac{\alpha}{\gamma}$ and assume equality in (19). We multiply the resulting equation by γ and get

$$\frac{2 - s + \alpha\left(2 - \frac{1}{k}\right)}{2 - \frac{1}{k} + \frac{\frac{1}{k} - 1}{\gamma}} = \frac{\alpha + sk}{1 - \frac{k+1}{\gamma}}. \tag{52}$$

Now, we multiply with the denominator of the left-hand side and sort by α and s on each side which leads to

$$2 - s + \alpha\left(2 - \frac{1}{k}\right) = \alpha \frac{2 - \frac{1}{k}}{1 - \frac{k+1}{\gamma}} + s \frac{\frac{1}{k} - 1 + k\left(2 - \frac{1}{k} + \frac{\frac{1}{k} - 1}{\gamma}\right)}{1 - \frac{k+1}{\gamma}} \tag{53}$$

and after rearranging terms to

$$2 = s \frac{\frac{1}{k} - \frac{k+1}{\gamma} + k\left(2 - \frac{1}{k} + \frac{\frac{1}{k} - 1}{\gamma}\right) + \left(2 - \frac{1}{k}\right)(k+1)}{1 - \frac{k+1}{\gamma}}. \tag{54}$$

We may let γ tend to infinity without changing the value of s (by adapting α correspondingly). Hence the right-hand side of (54) converges to $4ks$ as $\gamma \to \infty$ and Theorem 1.1 follows.

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