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Corrigendum and addendum to "Hierarchic control for a coupled parabolic system", *Portugaliae Math.* 73 (2016), 2: 115–137

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Abstract. In [2] we used three controls for a system of two coupled parabolic equations. We defined three functionals to be minimized and a hierarchy on the controls obtaining from the optimality condition a system of six coupled equations. In order to prove the null controllability, by means of the leader control acting only on the first equation, we give a proof of a Carleman inequality (Proposition 6.4) that in fact is incorrect. In this corrigendum we slightly modify the followers functionals given by (3) page 118 [2] in such a way that for the corresponding hierarchic system a correct Carleman inequality can be proved. This modification allows to introduce a coefficient $a_{12} \neq 0$ (a_{12} was zero in [2]).

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1. Formulation of the problem

Let Ω be an open and bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^2 and ω be an open and nonempty subset of Ω. Given $T > 0$, we consider the following system of coupled parabolic PDEs with leader control localized in ω and follower controls localized in $\omega_1, \omega_2 \subset \Omega$ with $\omega_i \cap \omega = \emptyset$. More precisely

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11} y_1 + a_{12} y_2 = h\chi_\omega + v^1 \chi_{\omega_1} + v^2 \chi_{\omega_2} & \text{in } Q = \Omega \times (0,T), \\ y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = 0 \quad \text{in } Q = \Omega \times (0,T), \\ y_j(x,0) = y_j^0(x) \text{ in } \Omega, \quad y_j = 0 \text{ on } \Sigma = \partial\Omega \times (0,T),\ j = 1,2, \end{cases} \tag{1}$$

where $a_{ij} = a_{ij}(x,t) \in L^\infty(Q)$ and $y_j^0 \in L^2(\Omega)$ are prescribed.

We assume that we have a hierarchy in our wishes and we will describe the Stackelberg-Nash strategy for system (1). Let $\mathcal{O}_d \subset \Omega$ be an open subset, representing the observation domain of the followers, which are localized arbitrarily in Ω. Define the followers functionals

$$J_i(h, v^1, v^2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_d \times (0,T)} |y_1 - y^i_{1,d}|^2 + |y_2 - y^i_{2,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \iint_{\omega_i \times (0,T)} |v^i|^2 \, dx \, dt, \qquad i = 1, 2, \tag{2}$$

and the main functional

$$J(h) = \frac{1}{2} \iint_{\omega \times (0,T)} |h|^2 \, dx \, dt, \tag{3}$$

where $\alpha_i, \mu_i > 0$ are constants and $y^i_d = (y^i_{1,d}, y^i_{2,d})^*$ are given functions in $L^2(\omega_{i,d} \times (0,T))$, $i = 1, 2$.

The main goal is to choose h such that the following general objective (of null controllability) is achieved

$$y(\cdot, T; h, v^1, v^2) = 0 \qquad \text{in } \Omega. \tag{4}$$

The second priority is the following. Given the functions $y^i_{1,d}$ and $y^i_{2,d}$, we want to choose the controls v^i such that throughout the interval $t \in (0,T)$

$$y(x, t; h, v^1, v^2) \text{ “do not deviate much” from } y^i_d(x,t), \text{ in the observability domain } \omega_{i,d},\ i = 1, 2. \tag{5}$$

To achieve simultaneously (4) and (5) for a fixed leader control h, find controls $(\bar{v}^1, \bar{v}^2)$ that depend on h and the corresponding state solution $y = y(h, \bar{v}^1, \bar{v}^2)$ of equation (1) satisfying the Nash equilibrium related to (J_1, J_2), that is,

$$J_1(h, \bar{v}^1, \bar{v}^2) = \min_{v^1} J_1(h, v^1, \bar{v}^2), \tag{6}$$

$$J_2(h, \bar{v}^1, \bar{v}^2) = \min_{v^2} J_2(h, \bar{v}^1, v^2). \tag{7}$$

In [2] we prove that given $h \in L^2(\omega \times (0,T))$, the pair $(\bar{v}^1, \bar{v}^2)$ is a Nash equilibrium of problem (6)–(7) if and only if

$$\bar{v}^i = -\frac{1}{\mu_i} p^i_1 \chi_{\omega_i}, \qquad i = 1, 2, \tag{8}$$

where (y, p^i) is solution of the coupled system

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11} y_1 + a_{12} y_2 = h\chi_\omega - \frac{1}{\mu_1} p_1^1 \chi_{\omega_1} - \frac{1}{\mu_2} p_1^2 \chi_{\omega_2} & \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = 0 & \text{in } Q, \\ -p^i_{1,t} - \Delta p^i_1 + a_{11} p^i_1 + a_{21} p^i_2 = \alpha_i (y_1 - y^i_{1,d}) \chi_{\mathcal{O}_d} & \text{in } Q, \\ -p^i_{2,t} - \Delta p^i_2 + a_{12} p^i_1 + a_{22} p^i_2 = \alpha_i (y_2 - y^i_{2,d}) \chi_{\mathcal{O}_d} & \text{in } Q, \\ y_j(0) = y_j^0,\ p_j^i(T) = 0, \quad y_j = p_j^i = 0 \text{ on } \Sigma,\ i,j = 1,2. \end{cases} \tag{9}$$

In [2], in order to prove the null controllability of system (11) we introduced the adjoint system to (11) and presented Proposition 6.4, a Carleman inequality, which proof is incorrect.

In fact, the proof is correct until equation (36). At this point we implicitly assume that $p_1 < p_2 + 4$. However, after equation (39), in order to eliminate the local term in φ_2, we took $p_2 + 4 < p_1$. This obviously cannot be done, so the proof is not valid and with the same assumptions, we don't think the result is correct.

2. Solution

In this corrigendum we present a new functional for the followers in such a way that the proof of the Carleman inequality can be obtained.

We will modify slightly (2) by adding a weighted norm in the functional minimized by the followers controls. That is, given $\alpha(x,t)$ as in Proposition 6.4 in [2] we take a new weight $\alpha_* = \max_{x\in\Omega} \alpha(x,t)$. For $\rho_*(t) \geq e^{s\alpha^*/2}$, fixed we take the follower weighted functionals

$$J_i(h, v^1, v^2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{d\times(0,T)}} |y_1 - y^i_{1,d}|^2 + |y_2 - y^i_{2,d}|^2 \,dx\,dt$$
$$+ \frac{\mu_i}{2} \iint_{\omega_i \times (0,T)} \rho_*^2(t) |v^i|^2 \,dx\,dt, \qquad i = 1,2, \tag{10}$$

and conserve the main functional as in [2]. With this new functionals the pair $(\bar{v}_1, \bar{v}_2)$ is a Nash equilibrium if

$$\bar{v}_i = -\frac{1}{\mu_i} \rho_*^{-2} p_1^i, \qquad i = 1,2,$$

and p_j^i, y_j, $i,j = 1,2$ are solution of the system:

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11} y_1 + a_{12} y_2 = h\chi_\omega - \frac{1}{\mu_1} \rho_*^{-2} p_1^1 \chi_{\omega_1} - \frac{1}{\mu_2} \rho_*^{-2} p_1^2 \chi_{\omega_2} & \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = 0 & \text{in } Q, \\ -p^i_{1,t} - \Delta p^i_1 + a_{11} p^i_1 + a_{21} p^i_2 = \alpha_i (y_1 - y^i_{1,d}) \chi_{\mathcal{O}_d} & \text{in } Q, \\ -p^i_{2,t} - \Delta p^i_2 + a_{12} p^i_1 + a_{22} p^i_2 = \alpha_i (y_2 - y^i_{2,d}) \chi_{\mathcal{O}_d} & \text{in } Q, \\ y_j(0) = y_j^0,\ p_j^i(T) = 0, \quad y_j = p_j^i = 0 \text{ on } \Sigma,\ i,j = 1,2. \end{cases} \tag{11}$$

Theorem 2.1 in [2] now reads as follows:

Theorem 2.1. *Assume that* $\mathcal{O}_d \cap \omega \neq \emptyset$ *and that* μ_i *for* $i = 1, 2$, *are sufficiently large. If*

$$a_{21} \geq a_0 > 0 \quad \text{or} \quad -a_{21} \geq a_0 > 0 \qquad \text{in } (\mathcal{O}_d \cap \omega) \times (0, T), \tag{12}$$

there exists a positive function $\rho = \rho(t)$ *blowing up at* $t = T$ *such that if*

$$\iint_{\mathcal{O}_d \times (0,T)} \rho^2 |y^i_{j,d}|^2 \, dx \, dt < +\infty, \qquad i, j = 1, 2,$$

then for any $y^0 \in L^2(\Omega)^2$ *there exists a control* $h \in L^2\big(\omega \times (0, T)\big)$ *such that the solution of* (11) *satisfies*

$$y_1(T) = y_2(T) = 0.$$

That is, there exists a Stackelberg-Nash strategy $(h, \bar{v}^1, \bar{v}^2)$ *for the functionals given by* (3) *and* (10), *with* h *subject to* $y_1(T) = y_2(T) = 0$.

Remark 2.2. Observe that we eliminate the assumption of Theorem 2.1 in [2] where we assume $a_{12} = 0$.

For the proof of Theorem 2.1, we need to prove an appropriate observability estimate, that can be obtained following exactly the proof in [2] but introducing the weight ρ_*^{-2}. That is, we prove a Carleman inequality for the "reduced" adjoint system to (11) (see [2] (23), p. 125). That is we will consider system:

$$\begin{cases} -\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2 = \psi_1 \chi_{\mathcal{O}_d} & \text{in } Q, \\ -\varphi_{2,t} - \Delta\varphi_2 + a_{12}\varphi_1 + a_{22}\varphi_2 = \psi_2 \chi_{\mathcal{O}_d} & \text{in } Q, \\ \psi_{1,t} - \Delta\psi_1 + a_{11}\psi_1 + a_{12}\psi_2 = -\rho_*^{-2}\left(\frac{\alpha_1}{\mu_1}\chi_{\omega_1} + \frac{\alpha_2}{\mu_2}\chi_{\omega_2}\right)\varphi_1 & \text{in } Q, \\ \psi_{2,t} - \Delta\psi_2 + a_{21}\psi_1 + a_{22}\psi_2 = 0 & \text{in } Q, \\ \varphi_j(T) = f_j, \ \psi_j(0) = 0 \text{ in } \Omega, \quad \varphi_j = \psi_j = 0 \text{ on } \Sigma, \ j = 1, 2, \end{cases} \tag{13}$$

We recall the definitions in [2]:

$$I(m, z) := \iint_Q e^{-2s\alpha}(s\gamma)^{m-2}|\nabla z|^2 \, dx \, dt + \iint_Q e^{-2s\alpha}(s\gamma)^m |z|^2 \, dx \, dt \tag{14}$$

and

$$\mathcal{L}_{\mathcal{B}}(m, z) := \iint_{\mathcal{B} \times (0,T)} e^{-2s\alpha}(s\gamma)^m |z|^2 \, dx \, dt, \qquad \gamma(t) := \frac{1}{t(T-t)}.$$

where

$$\alpha(x,t) = \frac{\alpha_0(x)}{t(T-t)}, \qquad \text{for } (x,t) \in Q,$$

is the classical function used in Carleman estimates (see e.g. [3] and [2]).

Proposition 6.4 in [2] is now:

Proposition 2.3. *Suppose that* (12) *holds and that* $\mathcal{O}_d \cap \omega \neq \emptyset$. *Then, for an adequate selection of parameters* d_i *and* $p_i \in \mathbb{R}$, *for* $i = 1, 2$, *there exist a function* $\alpha_0 \in C^2(\overline{\Omega})$ *and positive constants* C *and* σ_2 *such that, for every* $(f_1, f_2) \in [L^2(\Omega)]^2$, *the solution to system* (13) *satisfies*

$$\begin{aligned} I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(p_1,\psi_1) + I(p_2,\psi_2) \\ \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2-d_2+12}|\varphi_1|^2\,dx\,dt, \end{aligned} \tag{15}$$

$$\begin{aligned} \forall s \geq s_2 = \sigma_2\big(T + T^2 + T^2 \max\{ \max_{j=1,2} \|a_{jj}\|_\infty^{2/3}, \\ \max_{\substack{1\leq i,j\leq 2\\ i\neq j}} [\|a_{ij}\|_\infty^{2/(d_i-(d_j-3))}, \|a_{ij}\|_\infty^{2/(p_j-(p_i-3))}]\}\big). \end{aligned}$$

Proof. Define

$$\mathcal{O} := \mathcal{O}_d \cap \omega,$$

and since $\mathcal{O} \neq \emptyset$, there exists a non-empty open set $\mathcal{O}_0 \subset\subset \mathcal{O}$. Let α_0 and α be the functions associated to $\mathcal{B} = \mathcal{O}_0$ provided by Lemma 6.2 in [2]. The proof of Proposition 6.4 in [2] until equation (33) is correct, that is, we have that,

$$\begin{aligned} I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(p_1,\psi_1) + I(p_2,\psi_2) \\ \leq C_2\Big(\sum_{j=1}^{2} \mathcal{L}_{\mathcal{O}_0}(d_j,\varphi_j) + \sum_{j=1}^{2} \mathcal{L}_{\mathcal{O}_0}(p_j,\psi_j)\Big), \qquad \forall s \geq s_2. \end{aligned}$$

with C_2 and σ_2 two new positive constants only depending on Ω, $\mathcal{O}_0$, d_j, p_j, α_j and $\|a_{ij}\|_\infty$. To this point we have choosen

$$d_1 - 3 < p_1, \qquad d_1 - 3 < d_2 < d_1 + 3, \qquad d_2 - 3 < p_2, \qquad p_1 - 3 < p_2 < p_1 + 3.$$

Proceeding exactly as in [2], that is, doing local energy estimates, we can eliminate from the right hand side the local terms in ψ_1 and ψ_2, obtaining,

$$\begin{aligned} I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(p_1,\psi_1) + I(p_2,\psi_2) \\ \leq C \sum_{j=1}^{2} \mathcal{L}_{\tilde{\mathcal{O}}}(J_j,\varphi_j), \qquad \forall s \geq s_2. \end{aligned} \tag{16}$$

with $J_1 = \max\{p_1 + 4, 2p_1 - p_2\}$, $J_2 = \max\{p_2 + 4, p_1, 2p_2 - p_1\}$ and $\mathcal{O}_0 \subset\subset \tilde{\mathcal{O}} \subset\subset \mathcal{O}$. We can take $J_1 = p_1 + 4$ and $J_2 = p_2 + 4$.

Now, we want to eliminate the local terms corresponding to φ_2 in the right-hand side of (16). Given a set $\tilde{\mathcal{O}} \subset\subset \mathcal{O} \subset\subset \omega$, we consider a function $\eta \in C^\infty(\mathbb{R}^N)$ verifying: $0 \leq \eta \leq 1$ in $\mathbb{R}^N$, $\eta \equiv 1$ in $\tilde{\mathcal{O}}$, $\operatorname{supp} \eta \subset \omega$ and

$$\frac{\Delta \eta}{\eta^{1/2}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\nabla \eta}{\eta^{1/2}} \in L^\infty(\Omega)^N.$$

We set $u = e^{-2s\alpha}(s\gamma)^{p_2+4}$. Recall that the coefficient a_{21} satisfies (12) and, for simplicity, assume that $a_{21} \geq a_0$ in $\mathcal{O} \times (0,T)$. We multiply the equation satisfied by φ_1 in system (13) by $u\eta\varphi_2$ and integrate in Q. We obtain

$$\begin{aligned} a_0 \mathcal{L}_{\tilde{\mathcal{O}}}(p_2 + 4, \varphi_2) \leq \iint_Q u\eta a_{21}|\varphi_2|^2 &= \iint_Q (\varphi_{1,t} + \Delta\varphi_1 - a_{11}\varphi_1)u\eta\varphi_2 \\ &\quad + \iint_Q \psi_1 \chi_{\mathcal{O}_d} u\eta\varphi_2 = \sum_{n=1}^{4} K_n. \end{aligned} \tag{17}$$

We proceed to estimate each of the terms K_i. This is,

$$\begin{aligned} |K_1| &= \left| \iint_Q e^{-2s\alpha}(s\gamma)^{p_2+4}\eta\varphi_2\varphi_{1,t}\,dx\,dt \right|, \\ &= \left| \iint_Q \left(e^{-2s\alpha}(s\gamma)^{p_2+4}\eta\varphi_2\right)_t \varphi_1\,dx\,dt \right|, \\ &\leq \varepsilon_1 \iint_Q e^{-2s\alpha}(s\gamma)^{d_2-4}|\varphi_{2,t}|^2 + \varepsilon_2 \iint_Q e^{-2s\alpha}(s\gamma)^{d_2}|\varphi_2|^2 \\ &\quad + C_{\varepsilon_{1,2}} \iint_{\omega\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2-d_2+12}|\varphi_1|^2 \end{aligned}$$

$$\begin{aligned} |K_2| &= \left| \iint_Q e^{-2s\alpha}(s\gamma)^{p_2+4}\eta\varphi_2\Delta\varphi_1\,dx\,dt \right|, \\ &= \left| \iint_Q \Delta\left(e^{-2s\alpha}(s\gamma)^{p_2+4}\eta\varphi_2\right)\varphi_1\,dx\,dt \right|, \\ &\leq \varepsilon_3 \iint_Q e^{-2s\alpha}(s\gamma)^{d_2-4}|\Delta\varphi_2|^2 + C_{\varepsilon_3} \iint_{\omega\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2-d_2+12}|\varphi_1|^2 \\ &\quad + \varepsilon_4 \iint_Q e^{-2s\alpha}(s\gamma)^{d_2-2}|\nabla\varphi_2|^2 + C_{\varepsilon_4} \iint_{\omega\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2-d_2+12}|\varphi_1|^2 \\ &\quad + \varepsilon_5 \iint_Q e^{-2s\alpha}(s\gamma)^{p_2+4}\eta|\varphi_2|^2 + C_{\varepsilon_5} \iint_{\omega\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2+8}|\varphi_1|^2. \end{aligned}$$

The estimate of K_3 is straightforward. For K_4 we get

$$|K_4| = \left|\iint_Q e^{-2s\alpha}(s\gamma)^{p_2+4}\eta\varphi_2\psi_1\,dx\,dt\right|,$$
$$\leq \frac{1}{2}\iint_Q e^{-4s\alpha}(s\gamma)^{2p_2+8}\eta|\varphi_2|^2\,dx\,dt + \frac{1}{2}\iint_Q |\psi_1|^2\,dx\,dt.$$

Observe that given $\varepsilon_6 > 0$ for s large enough $e^{-2s\alpha}(s\gamma)^{2p_2-d_2+8} < \varepsilon_6$, therefore we obtain

$$|K_4| \leq \frac{\varepsilon_6}{2}\iint_Q e^{-2s\alpha}(s\gamma)^{d_2}\eta|\varphi_2|^2\,dx\,dt + \frac{1}{2}\iint_Q |\psi_1|^2\,dx\,dt.$$

Putting all together, and choosing appropriate constants ε_i, $i = 1, \dots, 6$, we obtain from (16) and (17)

$$I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(p_1,\psi_1) + I(p_2,\psi_2)$$
$$\leq C\iint_{\omega\times(0,T)} e^{-2s\alpha}(s\gamma)^{2p_2-d_2+12}|\varphi_1|^2\,dxd + C\iint_Q |\psi_1|^2\,dx\,dt. \qquad (18)$$

To eliminate the last term in the right hand side of the previous equation, we obtain energy estimates for the third and fourth equation in system (13), more precisely

$$\iint_Q (|\psi_1|^2 + |\psi_2|^2)\,dx\,dt \leq C\left(\frac{\alpha_1^2}{\mu_1^2} + \frac{\alpha_2^2}{\mu_2^2}\right)\iint_Q |\varphi_1\rho_*^{-2}|^2\,dx\,dt,$$
$$\leq C\left(\frac{\alpha_1^2}{\mu_1^2} + \frac{\alpha_2^2}{\mu_2^2}\right)\iint_Q e^{-2s\alpha^*}|\varphi_1|^2\,dx\,dt.$$

Since $e^{-2s\alpha^*} \leq e^{-2s\alpha}$ and provided that μ_i are large enough, we can put the above estimate in (18) and absorb the remaining term into the left hand side. Therefore the proof is complete. □

With the new Carleman estimate (15), we can obtain an observability inequality that implies Theorem 2.1 following the same procedure as in [2].

Remark 2.4. It is possible to modify the proof of the observability inequality (not the Carleman one) in such a way that instead of ρ_*^2 in (10), a new weight going to zero as $t \to T$ but not as $t \to 0$ modifies the functional. Since our mistake was on the Carleman inequality, we don't propose this alternative. This possibility uses a Carleman inequality with local terms in φ_1 and ψ_1. Then, a modification on the weight, as in [2], page 132, leads to the observability inequality.

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