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On probability and logic

Amílcar Sernadas†, João Rasga and Cristina Sernadas

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Abstract. Within classical propositional logic, assigning probabilities to formulas is shown to be equivalent to assigning probabilities to valuations by means of stochastic valuations. A stochastic valuation is a stochastic process, that is a family of random variables one for each propositional symbol. With stochastic valuations we are able to cope with a countably infinite set of propositional symbols. A notion of probabilistic entailment enjoying desirable properties of logical consequence is defined and shown to collapse into the classical entailment when the propositional language is left unchanged. Motivated by this result, a decidable conservative enrichment of propositional logic is proposed by giving the appropriate semantics to a new language construct that allows the constraining of the probability of a formula. A sound and weakly complete axiomatization is provided using the decidability of the theory of real closed ordered fields.

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1. Introduction

Starting as far back as [36] and [8] (see also [18] for a modern view of Boole's contributions to logic and probability), adding probability features to logic has been a recurrent research topic.

The introduction of probabilities in formal logic is quite challenging since there is the need to accommodate the continuous nature of probabilities within the discrete setting of symbolic reasoning. It is also interesting from the practical point of view since probabilistic reasoning is relevant in many fields. Several ways to combine probabilities and logic have been considered. One can assign probabilities either to formulas or to models. One can either keep the original language

† Deceased February 7, 2017.

unchanged by introducing probabilities only at the meta-level or change the language in order to internalize probabilistic assertions.

For seminal examples of assigning probabilities to formulas while leaving the formal language unchanged see [2], [19], [20], [30]. Under this approach, a notion of probabilistic entailment is proposed defined by relating the probabilities of the hypotheses and the probability of the conclusion.

The approach of assigning probabilities to models was first explored in [7] and subsequently revisited by several authors, all of them choosing to change the original language in order to be able to express probabilistic assertions. Two techniques were considered when endowing models with probabilities.

The "endogenous" technique adopted by most authors consists of enriching each model of the original logic with a probability measure on some components. For instance, in modal-like logic this approach was followed for Kripke structures by assigning probabilities to worlds [7], [12], [14], [40] or to the pairs in the accessibility relation [43], [44]. This technique has been quite pervasive in probabilistic versions of logics for reasoning about computer programs involving random operations, for example in [9], [16], [22]. It was also used in [1], [11], [24], [25], [27], [31], [37] for probabilizing predicate logic by assigning probabilities to the individuals in the domain.

The "exogenous" technique consists of assigning a probability to each model of the original logic or to a class of models of the original logic [1], [3], [10], [15], [21], [23], [29], [30], [34], [39]. A similar technique was used in [28] for assigning amplitudes to models in order to set-up a logic for reasoning about quantum systems.

The existence of so diverse proposals of incorporating probability into formal logic raises the problem of expressivity namely, for instance, if nesting probability operators will be more expressive. The negative answer is given in [5]. Additionally, expressing probabilistic reasoning under contradiction has been investigated in the context of logics of formal inconsistency (see [6], [33]).

In this paper, within the setting of propositional logic, we propose in Section 3 the novel notion of stochastic valuation and its main properties. A stochastic valuation is a stochastic process, that is a family of random variables one for each propositional symbol. This notion allows us to deal with the case where the set of propositional symbols is countably infinite which was not considered before. We discuss the probabilistic halting problem and the meeting problem for illustrating the need for a countably infinite set of propositional symbols. The section ends with the proof of the equivalence of assigning probabilities to formulas and assigning probabilities to valuations.

First, leaving the language unchanged, in Section 4 we analyze a notion of probabilistic entailment enjoying the usual properties of a logical consequence. The section ends with the proof of the collapse of the probabilistic entailment into classical entailment.

Second, since almost nothing is gained by probabilizing formulas or models while keeping the language unchanged, we propose in Section 5 a small enrichment (PPL) of propositional logic by providing the appropriate stochastic-valuation semantics to a new language constructor that allows (without nesting) the constraining of the probability of a formula. At the end of Section 5, capitalizing on the decidability of the theory of real closed ordered fields, we present an axiomatization of PPL. This axiomatization is shown to be sound and weakly complete in Section 6. Moreover, we prove in Section 7 that PPL is a decidable conservative extension of classical propositional logic. The paper ends with an assessment of what was achieved and a brief discussion of possible future work in Section 8.

2. Assigning probabilities to formulas

Throughout the paper, L is the propositional language generated by the set $B = \{B_j : j \in \mathbb{N}\}$ of propositional symbols using the connectives $\neg$ and $\supset$. The other connectives, as well as tt (verum) and ff (falsum), are introduced as abbreviations as usual. Recall that a (classical) valuation is a map $v : B \to \{0, 1\}$. We use $v \Vdash_{\mathsf{c}} \alpha$ for stating that valuation v satisfies formula α and $\Delta \vDash_{\mathsf{c}} \alpha$ for stating that the set of formulas Δ entails formula α, that is $v \Vdash_{\mathsf{c}} \alpha$ whenever $v \Vdash_{\mathsf{c}} \delta$ for every $\delta \in \Delta$. Recall that according to Adams [2], a *probability assignment* is a map $P : L \to \mathbb{R}$ satisfying the following principles:

P1 $0 \leq P(\alpha) \leq 1$;

P2 if $\vDash_{\mathsf{c}} \alpha$ then $P(\alpha) = 1$;

P3 if $\alpha \vDash_{\mathsf{c}} \beta$ then $P(\alpha) \leq P(\beta)$;

P4 if $\vDash_{\mathsf{c}} \neg(\beta \wedge \alpha)$ then $P(\beta \vee \alpha) = P(\beta) + P(\alpha)$.

The value $P(\varphi) \in P(L)$ is the probability assigned by P to φ.

Proposition 2.1. *Let $\alpha \in L$. Then, $P(\neg\alpha) = 1 - P(\alpha)$.*

Proof. Observe that, by P4,

$$P\big(\alpha \vee (\neg\alpha)\big) = P(\alpha) + P(\neg\alpha)$$

since $\vDash_{\mathsf{c}} \neg\big(\alpha \wedge (\neg\alpha)\big)$. Moreover, by P2,

$$P\big(\alpha \vee (\neg\alpha)\big) = 1$$

because $\vDash_{\mathsf{c}} \alpha \vee (\neg\alpha)$. Hence, $P(\neg\alpha) = 1 - P(\alpha)$. $\square$

We now show that the set of principles can be simplified.

Proposition 2.2. *The principles* P1–P4 *are not independent. That is,* P3 *follows from* P1, P2, P4.

Proof. We start by showing that:

$$(*) \text{ If } \alpha \vDash_{\mathsf{c}} \beta \text{ then } P(\beta) = P\big(\alpha \vee ((\neg\alpha) \wedge \beta)\big).$$

Indeed, assume that $\alpha \vDash_{\mathsf{c}} \beta$. Then,

$$\vDash_{\mathsf{c}} \beta \equiv \big(\alpha \vee ((\neg\alpha) \wedge \beta)\big).$$

Hence,

$$(\dagger) \ \vDash_{\mathsf{c}} (\neg\beta) \vee \big(\alpha \vee ((\neg\alpha) \wedge \beta)\big)$$

and

$$\vDash_{\mathsf{c}} \beta \vee \big(\neg\big(\alpha \vee ((\neg\alpha) \wedge \beta)\big)\big).$$

Moreover,

$$\vDash_{\mathsf{c}} \neg\big((\neg\beta) \wedge \big(\alpha \vee ((\neg\alpha) \wedge \beta)\big)\big).$$

Thus, by P4,

$$P\big((\neg\beta) \vee \big(\alpha \vee ((\neg\alpha) \wedge \beta)\big)\big) = P(\neg\beta) + P\big(\alpha \vee ((\neg\alpha) \wedge \beta)\big).$$

Observe that, by P2 applied to $(\dagger)$,

$$P\big((\neg\beta) \vee \big(\alpha \vee ((\neg\alpha) \wedge \beta)\big)\big) = 1.$$

Hence,

$$1 = P(\neg\beta) + P\big(\alpha \vee ((\neg\alpha) \wedge \beta)\big).$$

From Proposition 2.1, we have

$$P(\neg\beta) = 1 - P(\beta).$$

Therefore, we can conclude that

$$P(\beta) = P\big(\alpha \vee ((\neg\alpha) \wedge \beta)\big).$$

Returning to the main proof, observe that

$$\vDash_{\mathsf{c}} \neg\big(\alpha \wedge ((\neg\alpha) \wedge \beta)\big).$$

Then, using P4, we get

$$(**)\ \ P\big(\alpha \vee ((\neg\alpha) \wedge \beta)\big) = P(\alpha) + P\big((\neg\alpha) \wedge \beta\big).$$

So, from $(*)$ and $(**)$, we can conclude that

$$\text{if } \alpha \vDash_{\mathsf{c}} \beta \text{ then } P(\beta) = P(\alpha) + P\big((\neg\alpha) \wedge \beta\big).$$

On the other hand, by P1,

$$P\big((\neg\alpha) \wedge \beta\big) \geq 0.$$

Hence, if $\alpha \vDash_{\mathsf{c}} \beta$ then $P(\alpha) \leq P(\beta)$. □

3. Stochastic valuations

Towards endowing propositional logic with a probabilistic semantics, we introduce here the notion of stochastic valuation and show that it induces a probability assignment to formulas that fulfils the principles postulated in [2]. We also show that each probability assignment to formulas fulfilling those principles induces a unique stochastic valuation that recovers the original assignment. These results allow us to conclude that the choice of assigning probabilities either to valuations or to formulas is immaterial. In the subsequent sections of this paper we stick to the approach of assigning probabilities to valuations using stochastic valuations.

Given $A \in \wp^+_{\mathsf{fin}} B$, we say that an A-valuation is a map $v : A \to \{0, 1\}$ (given a set S, we denote the collection of its subsets by $\wp S$, the collection of its non-empty subsets by $\wp^+ S$, the collection of its finite subsets by $\wp_{\mathsf{fin}} S$ and the collection of its non-empty finite subsets by $\wp^+_{\mathsf{fin}} S$). We use $v \Vdash_{\mathsf{c}} \alpha$ for stating that valuation v satisfies formula α and $\Delta \vDash_{\mathsf{c}} \alpha$ for stating that the set of formulas Δ entails formula α, that is $v \Vdash_{\mathsf{c}} \alpha$ whenever $v \Vdash_{\mathsf{c}} \delta$ for every $\delta \in \Delta$.

When defining a probabilistic semantics, one might be tempted to look at probabilistic valuations as random variables taking values on the set of all classical valuations. However, it turns out that it is much better to look at them as stochastic processes as follows.

A *stochastic valuation* is a family

$$V = \{V_{B_j} : B_j \in B\}$$

of discrete random variables defined over the same probability space $(\Omega, \mathscr{F}, \mu)$ and taking values in $\{0, 1\}$ (that is, each V_{B_j} is a Bernoulli random variable). The elements of Ω are called outcomes, $\mathscr{F} \subseteq \wp\Omega$ is the σ-field of events, $\mu : \mathscr{F} \to [0, 1]$ is a probability measure and each $V_{B_j} : \Omega \to \{0, 1\}$ is a measurable map (that is, such that $(V_{B_j})^{-1}(S) \in \mathscr{F}$ for every $S \subseteq \{0, 1\}$).

In other words, V is a stochastic process, that is a family of random variables over the same probability space indexed by the countably infinite set B of propositional symbols. For further details on the notion of stochastic process see, for instance, [4].

For the purposes of this paper it is convenient to identify each valuation v with the subset $\{B_j : v(B_j) = 1\}$ of B. Accordingly, restriction is achieved by intersection: given a subset A of B, $v|_A = v \cap A$.

Moreover, it becomes handy to assume that each random variable V_{B_j} takes values in $\{\emptyset, \{B_j\}\}$ with $\emptyset$ standing for 0 and $\{B_j\}$ for 1.

Then, given a non-empty finite subset $A = \{B_{j_1}, \ldots, B_{j_n}\}$ of B and $U \subseteq A$, we write

$$\mathsf{Prob}(V_A = U)$$

for the (joint) probability (given by V), where $\overline{U}$ is $\{\bar{u}_j : j \in \{j_1, \ldots, j_n\}$ and $\bar{u}_j$ is $\{B_j\}$ if $B_j \in U$ and $\emptyset$ otherwise$\}$.

$$\begin{aligned}
\mu\big(V_A^{-1}(\overline{U})\big) &= \mu\big(\{\omega \in \Omega : V_A(\omega) = \overline{U}\}\big) \\
&= \mu\Big(\bigcap_{k=1}^{n} \{\omega \in \Omega : (V_{B_{j_k}})(\omega) = U \cap \{B_{j_k}\}\}\Big) \\
&= \mu\Big(\bigcap_{k=1}^{n} (V_{B_{j_k}})^{-1}(U \cap \{B_{j_k}\})\Big).
\end{aligned}$$

Hence, $\mathsf{Prob}(V_A = U)$ is the probability of each $B_{j_k} \in U$ being true and each $B_{j_k} \in A \setminus U$ being false. In particular, $\mathsf{Prob}(V_{B_j} = \{B_j\})$ is the probability (given by V) of B_j being true while $\mathsf{Prob}(V_{B_j} = \emptyset)$ is the probability (given by V) of B_j being false.

Observe that no independence assumption is made on $V_{B_{j_k}}$ for $k = 1, \ldots, n$. Therefore, it may be the case that

$$\mu\big(V_A^{-1}(\overline{U})\big) \neq \prod_{k=1}^{n} \mu\big((V_{B_{j_k}})^{-1}(U \cap \{B_{j_k}\})\big)$$

meaning that the random variables $B_{j_1}, \ldots, B_{j_n}$ are not independent. That is, it is not always the case that

$$\mathsf{Prob}(V_A = U) = \prod_{k=1}^{n} \mathsf{Prob}(V_{B_{j_k}} = U \cap \{B_{j_k}\}).$$

We consider a simple example of dependence between random variables in the stochastic valuation. Let $(\{\omega_1, \omega_2, \omega_3, \omega_4\}, \wp\{\omega_1, \omega_2, \omega_3, \omega_4\}, \mu)$ be a probability space where

$$\mu(\omega_1) = \frac{1}{2}, \qquad \mu(\omega_2) = 0, \qquad \mu(\omega_3) = \frac{1}{3}, \qquad \mu(\omega_4) = \frac{1}{6}$$

and $V_{B_j} : \Omega \to \wp\{B_j\}$ be such that

$$V_{B_1}(\omega_1) = V_{B_1}(\omega_2) = V_{B_1}(\omega_3) = \{B_1\} \quad \text{and} \quad V_{B_1}(\omega_4) = \emptyset$$

and

$$V_{B_2}(\omega_1) = V_{B_2}(\omega_2) = V_{B_2}(\omega_3) = \emptyset \quad \text{and} \quad V_{B_2}(\omega_4) = \{B_2\}.$$

Then,

$$\mu\big(\{\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} : V_{B_1}(\omega) = \{B_1\}\}\big) = \frac{5}{6},$$

$$\mu\big(\{\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} : V_{B_2}(\omega) = \{B_2\}\}\big) = \frac{1}{6}$$

and

$$\mu\big(\{\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} : V_{\{B_1, B_2\}}(\omega) = \{\{B_1\}, \{B_2\}\}\}\big) = 0$$

and so

$$\mathsf{Prob}(V_{\{B_1, B_2\}} = \{B_1, B_2\}) \neq \mathsf{Prob}(V_{B_1} = \{B_1\}) \times \mathsf{Prob}(V_{B_2} = \{B_2\}).$$

Therefore, random variables V_{B_1} and V_{B_2} are not independent. On the other hand, let $(\{\omega_1, \omega_2, \omega_3, \omega_4\}, \wp\{\omega_1, \omega_2, \omega_3, \omega_4\}, \mu)$ be a probability space where

$$\mu(\omega_j) = \frac{1}{4}, \qquad \text{for } j = 1. \ldots, 4$$

and $V_{B_j} : \Omega \to \wp\{B_j\}$, $j = 1, 2$, be such that

$$V_{B_1}(\omega_1) = V_{B_1}(\omega_3) = \{B_1\} \quad \text{and} \quad V_{B_1}(\omega_2) = V_{B_1}(\omega_4) = \emptyset$$

and

$$V_{B_2}(\omega_1) = V_{B_2}(\omega_2) = \{B_2\} \quad \text{and} \quad V_{B_2}(\omega_3) = V_{B_2}(\omega_4) = \emptyset.$$

Then,

$$\mu\big(\{\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} : V_{B_1}(\omega) = \{B_1\}\}\big) = \frac{1}{2},$$

$$\mu\big(\{\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} : V_{B_2}(\omega) = \{B_2\}\}\big) = \frac{1}{2}$$

and

$$\mu\big(\{\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} : V_{\{B_1, B_2\}}(\omega) = \{\{B_1\}, \{B_2\}\}\}\big) = \frac{1}{4}.$$

Moreover,

$$\mathsf{Prob}(V_{\{B_1, B_2\}} = \bar{u}_1 \cup \bar{u}_2) = \mathsf{Prob}(V_{B_1} = \bar{u}_1) \times \mathsf{Prob}(V_{B_2} = \bar{u}_2)$$

for every $\bar{u}_1 \in \wp\{B_1\}$ and $\bar{u}_2 \in \wp\{B_2\}$. Therefore, random variables V_{B_1} and V_{B_2} are independent.

Each stochastic valuation V induces the family

$$\{U \mapsto \mathsf{Prob}(V_A = U) : \wp A \to [0, 1]\}_{A \in \wp^+_{\mathsf{fin}} B}$$

of finite-dimensional (joint probability) distributions that we may call *family of finite-dimensional probabilistic valuations* which is consistent in the sense that the following *marginal condition* holds:

$$\mathsf{Prob}(V_{A'} = U') = \sum_{\substack{U \subseteq A \\ U \cap A' = U'}} \mathsf{Prob}(V_A = U) \qquad \forall A \in \wp^+_{\mathsf{fin}} B \;\; \forall A' \in \wp^+ A \;\; \forall U' \in \wp A'.$$

Assume that $A = \{B_j, B_k, B_\ell\}$, $A' = \{B_j, B_k\}$ and $U' = \{B_j\}$. Then, the marginal condition states that

$$\mathsf{Prob}(V_{A'} = \{B_j\}) = \mathsf{Prob}(V_A = \{B_j, B_\ell\}) + \mathsf{Prob}(V_A = \{B_j\}).$$

Conversely, given a consistent system of finite-dimensional distributions (in our case, finite-dimensional probabilistic valuations), the Kolmogorov existence theorem (see Section 36 of [4]) guarantees the existence of a unique stochastic process (in our case, a unique stochastic valuation) that induces those finite-dimensional distributions. Thus, in our case, the Kolmogorov existence theorem states that there is a unique countably infinite family $\{V_{B_j} : B_j \in B\}$ of discrete random variables defined on a probability space $(\Omega, \mathscr{F}, \mu)$ induced by the given consistent family

$$\{U \mapsto \mathsf{Prob}(V_A = U) : \wp A \to [0,1]\}_{A \in \wp^+_{\mathsf{fin}} B}$$

of finite-dimensional probabilistic valuations.

This theorem will be frequently used in the paper. Its availability well justifies our claim that it is much better to probabilize valuations using stochastic processes, namely when the set of propositional symbols in not finite (see the example in Subsection 5.4).

3.1. Stochastic valuations versus probability assignments. Given $\alpha \in L$, let B_α be the set of propositional symbols occurring in α and $[\![\alpha]\!]$ be the set $\{v \cap B_\alpha : v \Vdash_{\mathsf{c}} \alpha\}$ of the restrictions to B_α of the valuations that satisfy α.

With these notions and notation at hand we are ready to compute the probability that a stochastic valuation assigns to a formula.

Given $\alpha \in L$ and a stochastic valuation V, the *probability of* α *under* V is computed as follows:

$$\mathsf{Prob}_V(\alpha) = \sum_{U \in [\![\alpha]\!]} \mathsf{Prob}(V_{B_\alpha} = U).$$

That is, the probability of α under V is the sum of the probabilities of the restrictions to B_α of the classical valuations that satisfy α. These probabilities are provided by the finite-dimensional probabilistic valuation

$$U \mapsto \mathsf{Prob}(V_{B_\alpha} = U) : \wp B_\alpha \to [0,1]$$

induced by V on B_α.

The probabilities that a stochastic valuation assigns to formulas fulfil the principles postulated by Adams (in [2]) as we now proceed to show. First, we introduce some notation and prove a few auxiliary results.

Given $U \subseteq A \subseteq B$, we use the abbreviation

$$\phi_A^U \text{ for } \Big(\bigwedge_{B_j \in U} B_j\Big) \wedge \Big(\bigwedge_{B_j \in A \setminus U} \neg B_j\Big).$$

Clearly, this formula identifies the A-valuation that makes each B_j in U true and each B_j not in U false. Observe that, for each such U and A, the set

$$\{v \cap A : v \Vdash_{\mathsf{c}} \phi_A^U\}$$

is the singleton $\{U\}$. Note also that the set $B_{\phi_A^U}$ of propositional symbols occurring in ϕ_A^U coincides with A.

Proposition 3.1. *Let $U_1, U_2 \subseteq A \subseteq B$ be such that $U_1 \neq U_2$. Then*

$$\vDash_{\mathsf{c}} \neg(\phi_A^{U_1} \wedge \phi_A^{U_2})$$

Proof. Let v be a valuation. Assume, by contradiction, that $v \Vdash_{\mathsf{c}} \phi_A^{U_1}$ and $v \Vdash_{\mathsf{c}} \phi_A^{U_2}$. Without loss of generality, let B_j be a symbol in U_1 but not in U_2. Then, $v \Vdash_{\mathsf{c}} B_j$ and $v \Vdash_{\mathsf{c}} \neg B_j$ which is a contradiction. □

Proposition 3.2. *Let $A \subseteq B$. Then*

$$\vDash_{\mathsf{c}} \bigvee_{U \subseteq A} \phi_A^U.$$

Proof. Let v be a valuation. Then, it is straightforward that $v \Vdash_{\mathsf{c}} \phi_A^{v \cap A}$. □

Proposition 3.3. *Let $U' \subseteq A' \subseteq A \subseteq B$. Then*

$$\vDash_{\mathsf{c}} \Big(\bigvee_{\substack{U \subseteq A \\ U \cap A' = U'}} \phi_A^U \Big) \equiv \phi_{A'}^{U'}.$$

Proof. Let v be a valuation.

($\rightarrow$) Assume that

$$v \Vdash_{\mathsf{c}} \Big(\bigvee_{\substack{U \subseteq A \\ U \cap A' = U'}} \phi_A^U \Big).$$

Then, $v \Vdash_{\mathsf{c}} \phi_A^U$ for some $U \subseteq A$ such that $U \cap A' = U'$. Let $B_j \in U'$. Then, $B_j \in U \cap A'$ and so $v \Vdash_{\mathsf{c}} B_j$ since $v \Vdash_{\mathsf{c}} \phi_A^U$. Let $B_j \in A' \setminus U'$. Then, $B_j \notin U \cap A'$ and so $B_j \notin U$. Hence, $v \not\Vdash_{\mathsf{c}} B_j$ since $v \Vdash_{\mathsf{c}} \phi_A^U$. Thus, $v \Vdash_{\mathsf{c}} \phi_{A'}^{U'}$.

($\leftarrow$) Assume that $v \Vdash_{\mathsf{c}} \phi_{A'}^{U'}$. Observe that $v \Vdash_{\mathsf{c}} \phi_A^{v \cap A}$. Moreover, $v \cap A \subseteq A$ and $(v \cap A) \cap A' = v \cap A' = U'$ since $v \Vdash_{\mathsf{c}} \phi_{A'}^{U'}$. □

The next result shows that probabilities do not change when adding new propositional symbols.

Proposition 3.4. *Let δ, ϕ be formulas and V a stochastic valuation. Then*

$$\sum_{U \in \{v \cap (B_\delta \cup B_\phi) : v \Vdash_{\mathsf{c}} \delta\}} \mathsf{Prob}(V_{B_\delta \cup B_\phi} = U) = \sum_{U' \in \{v \cap B_\delta : v \Vdash_{\mathsf{c}} \delta\}} \mathsf{Prob}(V_{B_\delta} = U').$$

Proof. Observe that:

$$\begin{aligned}
&\sum_{U' \in \{v \cap B_\delta : v \Vdash_{\mathsf{c}} \delta\}} \mathsf{Prob}(V_{B_\delta} = U') \\
&\quad = \sum_{U' \in \{v \cap B_\delta : v \Vdash_{\mathsf{c}} \delta\}} \sum_{\substack{U \subseteq B_\delta \cup B_\phi \\ U \cap B_\delta = U'}} \mathsf{Prob}(V_{B_\delta \cup B_\phi} = U) && (*) \\
&\quad = \sum_{\substack{U \subseteq B_\delta \cup B_\phi \\ U \cap B_\delta \in \{v \cap B_\delta : v \Vdash_{\mathsf{c}} \delta\}}} \mathsf{Prob}(V_{B_\delta \cup B_\phi} = U) \\
&\quad = \sum_{U \in \{v \cap (B_\delta \cup B_\phi) : v \Vdash_{\mathsf{c}} \delta\}} \mathsf{Prob}(V_{B_\delta \cup B_\phi} = U) && (**)
\end{aligned}$$

where $(*)$ follows by the marginal condition and $(**)$ is proved now. Indeed

$$U \subseteq B_\delta \cup B_\phi \text{ and } U \cap B_\delta \in \{v \cap B_\delta : v \Vdash_{\mathsf{c}} \delta\} \quad \text{iff } U \in \{v \cap (B_\delta \cup B_\phi) : v \Vdash_{\mathsf{c}} \delta\}$$

since:

($\rightarrow$) Assume that $U \subseteq B_\delta \cup B_\phi$ and $U \cap B_\delta \in \{v \cap B_\delta : v \Vdash_{\mathsf{c}} \delta\}$. Let v be such that $U \cap B_\delta = v \cap B_\delta$ and $v \Vdash_{\mathsf{c}} \delta$. Let v' be such that $v' \cap B_\delta = v \cap B_\delta$ and $v' \cap (B_\delta \cup B_\phi) = U$. Then, $v' \Vdash_{\mathsf{c}} \delta$. Therefore, $U \in \{v \cap (B_\delta \cup B_\phi) : v \Vdash_{\mathsf{c}} \delta\}$.

($\leftarrow$) Assume that $U \in \{v \cap (B_\delta \cup B_\phi) : v \Vdash_{\mathsf{c}} \delta\}$. Let v be such that $U = v \cap (B_\delta \cup B_\phi)$ and $v \Vdash_{\mathsf{c}} \delta$. Thus, $U \subseteq B_\delta \cup B_\phi$. Moreover $U \cap B_\delta = v \cap (B_\delta \cup B_\phi) \cap B_\delta = v \cap B_\delta$. Hence $U \cap B_\delta \in \{v \cap B_\delta : v \Vdash_{\mathsf{c}} \delta\}$. □

Proposition 3.5. *Given a formula α and a stochastic valuation V,*

$$\vDash_{\mathsf{c}} \alpha \quad \textit{implies} \quad \mathsf{Prob}_V(\alpha) = 1.$$

Proof. Assume $\vDash_{\mathsf{c}} \alpha$. Then,

$$[\![\alpha]\!] = \wp B_\alpha.$$

Indeed it is immediate that $[\![\alpha]\!] \subseteq \wp B_\alpha$. For the other direction, let $U \subseteq B_\alpha$. Pick a valuation v such that $v \cap B_\alpha = U$. Then, $U \in [\![\alpha]\!]$ since $v \Vdash_{\mathsf{c}} \alpha$.

Hence

$$\mathsf{Prob}_V(\alpha) = \sum_{U \in [\![\alpha]\!]} \mathsf{Prob}(V_{B_\alpha} = U) = \sum_{U \subseteq B_\alpha} \mathsf{Prob}(V_{B_\alpha} = U) = 1$$

and so $\mathsf{Prob}_V(\alpha)$. □

With these results in hand we are ready to show that every stochastic valuation assigns probabilities to formulas fulfilling Adams' principles.

Theorem 3.6. *Let V be a stochastic valuation. Then, $\hat{V} = \mathsf{Prob}_V$ is a probability assignment.*

Proof. Indeed, all the properties of probability assignments are satisfied:

P1 Direct from the fact that V_{B_α} is a probability distribution for every α.

P2 Follows immediately from Proposition 3.5.

P4 Assume that $\vDash_{\mathsf{c}} \neg(\beta \wedge \alpha)$. Then, there is no valuation v such that $v \Vdash_{\mathsf{c}} \beta$ and $v \Vdash_{\mathsf{c}} \alpha$. Hence,

$$\begin{aligned}
\mathsf{Prob}_V(\beta \vee \alpha) &= \sum_{U \in [\![\beta \vee \alpha]\!]} \mathsf{Prob}(V_{B_\beta \cup B_\alpha} = U) \\
&= \sum_{U \in \{v \cap (B_\beta \cup B_\alpha) : v \Vdash_{\mathsf{c}} \beta \vee \alpha\}} \mathsf{Prob}(V_{B_\beta \cup B_\alpha} = U) \\
&= \sum_{U \in \{v \cap (B_\beta \cup B_\alpha) : v \Vdash_{\mathsf{c}} \beta \text{ or } v \Vdash_{\mathsf{c}} \alpha\}} \mathsf{Prob}(V_{B_\beta \cup B_\alpha} = U) \\
&= \sum_{U \in \{v \cap (B_\beta \cup B_\alpha) : v \Vdash_{\mathsf{c}} \beta\}} \mathsf{Prob}(V_{B_\beta \cup B_\alpha} = U) \\
&\quad + \sum_{U \in \{v \cap (B_\beta \cup B_\alpha) : v \Vdash_{\mathsf{c}} \alpha\}} \mathsf{Prob}(V_{B_\beta \cup B_\alpha} = U) \\
&= \sum_{U \in \{v \cap B_\beta : v \Vdash_{\mathsf{c}} \beta\}} \mathsf{Prob}(V_{B_\beta} = U) \\
&\quad + \sum_{U \in \{v \cap B_\alpha : v \Vdash_{\mathsf{c}} \alpha\}} \mathsf{Prob}(V_{B_\alpha} = U) \qquad (*) \\
&= \sum_{U \in [\![\beta]\!]} \mathsf{Prob}(V_{B_\beta} = U) + \sum_{U \in [\![\alpha]\!]} \mathsf{Prob}(V_{B_\alpha} = U) \\
&= \mathsf{Prob}_V(\beta) + \mathsf{Prob}_V(\alpha)
\end{aligned}$$

where $(*)$ follows by Proposition 3.4. Observe that Prob_V also satisfies P3 due to Proposition 2.2. □

We now show the converse result: each probability assignment induces a stochastic valuation giving back the original assignment. To this end, we first spell-out the family of finite-dimensional probabilistic valuations induced by a probability assignment and show that it fulfils the marginal condition. Afterwards, the envisaged stochastic valuation is obtained using Kolmogorov's existence theorem.

Given a probability assignment P, let

$$\eta^P = \{\eta^P_A = U \mapsto P(\phi^U_A) : \wp A \to [0,1]\}_{A \in \wp^+_{\mathrm{fin}} B}.$$

Proposition 3.7. *Let P be a probability assignment. There exists a unique stochastic valuation*

$$\check{P}$$

such that $\mathsf{Prob}(\check{P}_A = U) = P(\phi^U_A)$.

Proof. (1) Each η^P_A is a finite-dimensional probabilistic valuation:

(a) $\eta^P_A(U) \in [0,1]$. Follows immediately from P1.

(b) $\sum_{U \subseteq A} \eta^P_A(U) = 1$. Indeed:

$$\begin{aligned} \sum_{U \subseteq A} \eta^P_A(U) &= \sum_{U \subseteq A} P(\phi^U_A) \\ &= P\Big(\bigvee_{U \subseteq A} \phi^U_A\Big) \qquad (*) \\ &= 1 \qquad (**) \end{aligned}$$

where $(*)$ follows from Proposition 3.1 and P4 and $(**)$ follows from Proposition 3.2 and P2.

(c) Additivity is trivial since we are dealing with a measure over a finite set of outcomes.

(2) The family η^P fulfils the marginal condition. Assume that $A' \subseteq A$ and $U' \subseteq A'$. Then,

$$\begin{aligned} \sum_{\substack{U \subseteq A \\ U \cap A' = U'}} \eta^P_A(U) &= \sum_{\substack{U \subseteq A \\ U \cap A' = U'}} P(\phi^U_A) \\ &= P\Big(\bigvee_{\substack{U \subseteq A \\ U \cap A' = U'}} \phi^U_A\Big) \qquad (\dagger) \\ &= P(\phi^{U'}_{A'}) \qquad (\ddagger) \\ &= \eta^P_{A'}(U') \end{aligned}$$

where (†) follows from Proposition 3.1 and P4 and (‡) follows from Proposition 3.3 and Proposition 2.2.

Hence, using Kolmogorov's existence theorem, there exists a unique stochastic valuation having these finite-dimensional distributions. Let $\check{P}$ be this stochastic valuation. □

We now proceed to show that $\check{P}$ induces back the original probability assignment P and, conversely, that a stochastic valuation V induces the probability assignment $\hat{V}$ that gives back the original V. To this end, we need the following auxiliary result.

Proposition 3.8. *Let $U' \subseteq A' \subseteq A \subseteq B$ and β a formula in L. Then*

$$\vDash_{\mathsf{c}} \Big(\bigvee_{U \in \{v \cap B_\beta : v \Vdash_{\mathsf{c}} \beta\}} \phi_{B_\beta}^{U} \Big) \equiv \beta.$$

Proof. Let v be a valuation.

($\rightarrow$) Assume that

$$v \Vdash_{\mathsf{c}} \Big(\bigvee_{U \in \{v \cap B_\beta : v \Vdash_{\mathsf{c}} \beta\}} \phi_{B_\beta}^{U} \Big).$$

Let $U \in \{v \cap B_\beta : v \Vdash_{\mathsf{c}} \beta\}$ be such that

$$v \Vdash_{\mathsf{c}} \phi_{B_\beta}^{U}.$$

Then, there is v' such that $U = v' \cap B_\beta$ and $v' \Vdash_{\mathsf{c}} \beta$. Hence

$$v \Vdash_{\mathsf{c}} \phi_{B_\beta}^{v' \cap B_\beta} \qquad (\dagger).$$

We now show that $v' \cap B_\beta = v \cap B_\beta$. Let $B_j \in v' \cap B_\beta$. Then, $v \Vdash_{\mathsf{c}} B_j$ by (†) and so $B_j \in v \cap B_\beta$. For the other direction let $B_j \in v \cap B_\beta$. By (†), $B_j \in v'$. Hence $B_j \in v' \cap B_\beta$.

Therefore,

$$U = v \cap B_\beta$$

and so $v \Vdash_{\mathsf{c}} \beta$.

($\leftarrow$) Assume that $v' \Vdash_{\mathsf{c}} \beta$. Observe that $v' \Vdash_{\mathsf{c}} \phi_{B_\beta}^{v' \cap B_\beta}$. Then, $v' \cap B_\beta \in \{v \cap B_\beta : v \Vdash_{\mathsf{c}} \beta\}$ and so the thesis follows. □

Theorem 3.9. *Let V be a stochastic valuation and P a probability assignment. Then,*

$$\check{\hat{V}} = V \quad \text{and} \quad \hat{\check{P}} = P.$$

Proof. Let $A \in \wp^+_{\mathrm{fin}} B$ and $U \subseteq A$. Observe that we have:

$$\begin{aligned}
\mathsf{Prob}(\check{\hat{V}}_A = U) &= \hat{V}(\phi_A^U) \\
&= \mathsf{Prob}_V(\phi_A^U) \\
&= \sum_{U' \in [\![\phi_A^U]\!]} \mathsf{Prob}(V_{B_{\phi_A^U}} = U') \\
&= \mathsf{Prob}(V_{B_{\phi_A^U}} = U) \\
&= \mathsf{Prob}(V_A = U).
\end{aligned}$$

Therefore, the stochastic valuations $\check{\hat{V}}$ and V have the same finite-dimensional probabilistic valuations and, so, by the Kolmogorov's existence theorem, they are equivalent.

Moreover, let $\beta \in L$. Then:

$$\begin{aligned}
\hat{\check{P}}(\beta) = \mathsf{Prob}_{\check{P}}(\beta) &= \sum_{U \in [\![\beta]\!]} \mathsf{Prob}(\check{P}_{B_\beta} = U) \\
&= \sum_{U \in \{v \cap B_\beta : v \Vdash_{\mathsf{c}} \beta\}} P(\phi_{B_\beta}^U) \\
&= P\Big(\bigvee_{U \in \{v \cap B_\beta : v \Vdash_{\mathsf{c}} \beta\}} \phi_{B_\beta}^U\Big) && (*) \\
&= P(\beta) && (**)
\end{aligned}$$

where $(*)$ follows from Proposition 3.1 and P4 and $(**)$ is a consequence of Proposition 3.8 and P3. □

In short, there is a strict Galois connection between stochastic valuations and probability assignments to (classical) formulas. Therefore, we can freely choose to assign probabilities to formulas or to valuations. In the remainder of this paper we adopt the latter approach, using stochastic valuations for the purpose.

4. Probabilistic entailment

In this section, we compare the entailment in CPL (classical propositional logic with valuations as semantics) with the probabilistic entailment that we are able to define in svPL (a variant of CPL with the same language but adopting stochastic valuations as semantics). The key result of this section is the collapse of the probabilistic entailment into the classical entailment.

It is possible to define in svPL a family $\models_p^q$ of probabilistic entailments depending on the minimal probability p required from the hypotheses in order to obtain the conclusion with at least probability q. To this end, we first define satisfaction by a stochastic valuation of a formula with a minimal probability p.

Let V be a stochastic valuation, $\alpha \in L$ and $p \in [0,1]$. We say that α is *p-satisfied* by V, written

$$V \Vdash_p \alpha,$$

whenever $\mathsf{Prob}_V(\alpha) \geq p$. That is, a formula is p-satisfied by V whenever its probability under V is at least p.

Given $\Delta \cup \{\alpha\} \subseteq L$ and $p, q \in [0,1]$, one would say that Δ pq-entails α, written here

$$\Delta \mathrel{\ddot{\models}_p^q} \alpha,$$

whenever, for every stochastic valuation V,

$$\text{if } V \Vdash_p \delta \text{ for every } \delta \in \Delta \text{ then } V \Vdash_q \alpha.$$

That is, if the probability under V of each hypothesis is at least p then the probability under V of the conclusion is at least q. This notion is a particular case of the one proposed in [19] (pages 196 and 197) and also in [20][1] by considering that $a_1 = \cdots = a_n = [p,1]$ and $a = [q,1]$.

However, we find this definition wanting since $\ddot{\models}_p^q$ does not enjoy the following desirable property:

$$\delta_1, \delta_2 \mathrel{\ddot{\models}_p^q} \alpha \qquad \text{iff } \delta_1 \wedge \delta_2 \mathrel{\ddot{\models}_p^q} \alpha.$$

Indeed, for instance,

$$B_1 \wedge (\neg B_1) \mathrel{\ddot{\models}_{1/2}^{1/4}} \mathsf{ff}$$

while

$$B_1, \neg B_1 \mathrel{\ddot{\not\models}_{1/2}^{1/4}} \mathsf{ff}.$$

[1]The set $\{\delta_1, \ldots, \delta_n\}$ entails α with respect to $a_1, \ldots, a_n$, $a \subseteq [0,1]$ (all of them non-empty sets) if for all probability models M

$$\text{if } P_M(\delta_1) \in a_1, \ldots, P_M(\delta_n) \in a_n \text{ then } P_M(\alpha) \in a,$$

where $P_M(\varphi)$ is the probability given by M to φ.

The former holds vacuously because $[\![B_1 \wedge (\neg B_1)]\!] = \emptyset$. Concerning the latter, observe that it is easy to find a stochastic valuation V such that $\mathsf{Prob}_V(B_1) = \mathsf{Prob}(\neg B_1) = \frac{1}{2}$ and note that every stochastic valuation assigns probability zero to ff since $[\![\mathsf{ff}]\!] = \emptyset$.

In order to overcome this difficulty, we propose to use the following notion of probabilistic entailment where, as usual, for any finite set Φ of formulas, we write $\bigwedge \Phi$ for the conjunction of the formulas in Φ, with $\bigwedge \emptyset$ standing for tt.

Let $\Delta \cup \{\alpha\} \subseteq L$ and $p, q \in (0, 1]$ such that $p \geq q$. We say that Δ *pq-entails* α, written

$$\Delta \vDash_p^q \alpha,$$

whenever there is a finite subset Φ of Δ such that, for every stochastic valuation V,

$$\text{if } V \Vdash_p \bigwedge \Phi \text{ then } V \Vdash_q \alpha.$$

Clearly,

$$\Delta \vDash_p^q \alpha \quad \text{iff } \exists \Phi \in \wp_{\mathsf{fin}}\Delta : \bigwedge \Phi \ddot{\vDash}_p^q \alpha.$$

Thus, when Δ is a singleton or the empty set the two definitions coincide.

Note that the requirement $q > 0$ is well justified because $\vDash_p^0$ is trivial. Indeed, every formula is $p0$-entailed by any set of hypotheses since $\mathsf{Prob}_V(\alpha) \geq 0$ for every stochastic valuation V and formula α.

Observe also that the requirement $p \geq q$ is essential since otherwise the induced *pq-entailment operator*

$$\Delta \mapsto \Delta^{\vDash_p^q} = \{\alpha \in L : \Delta \vDash_p^q \alpha\} : \wp L \to \wp L$$

would not be extensive. Indeed, for instance,

$$B_1 \nvDash_{1/4}^{3/4} B_1.$$

It is straightforward to verify that the pq-entailment operator is extensive if $p \geq q$, as well as monotonic for arbitrary p and q. On the other hand, it is not clear from the definition if it is idempotent. In fact, each pq-entailment operator is indeed idempotent but the proof is not trivial. Idempotence is not used on the way to the collapsing theorem at the end of this section. Moreover, it follows immediately from that theorem. Therefore, we refrain from attempting at this point to prove the idempotence of each pq-entailment operator. Observe also that it follows directly from its definition that each operator is compact.

The aim now is to compare the probabilistic entailments of svPL with the entailment of CPL.

To this end, we need to explain how a classical valuation canonically induces a stochastic valuation. Observe that it was proved in [19] that a classical valuation can be seen as a probability assignment with target set $\{0,1\}$ and vice-versa. Herein, we make explicit the map $v \mapsto V^v$. Given a valuation v, consider the family of maps

$$\eta^v = \{\eta^v_A : \wp A \to [0,1]\}_{A \in \wp^+_{\mathsf{fin}} B}$$

where each map is as follows:

$$\eta^v_A(U) = \begin{cases} 1 & U = v \cap A \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.1. *Given a valuation v, η^v is a consistent family of finite dimensional probability valuations.*

Proof. Since it is straightforward to check that each η^v_A is a probability measure over A, we focus on showing that η^v fulfils the marginal condition. Let $A \in \wp^+_{\mathsf{fin}} B$, $A' \in \wp^+ A$ and $U' \in \wp A'$. Then, consider two cases:

(i) $v \cap A' = U'$. Observe that $v \cap A \subseteq A$ and $v \cap A \cap A' = v \cap A' = U'$. Note that

$$\eta_{A'}(U') = \eta_{A'}(v \cap A') = 1.$$

On the other hand, $(v \cap A) \subseteq A$ is such that $v \cap A \cap A' = v \cap A'$. Thus,

$$\eta^v_A(v \cap A) = 1.$$

Take $U \subseteq A$ such that $U \neq (v \cap A)$ and $U \cap A' = v \cap A'$. Then,

$$\eta^v_A(U) = 0.$$

Therefore,

$$\sum_{\substack{U \subseteq A \\ U \cap A' = U'}} \eta^v_A(U) = 1 = \eta_{A'}(U').$$

(ii) $v \cap A' \neq U'$. Then, $v \cap A \cap A' = v \cap A' \neq U'$ and so

$$\sum_{\substack{U \subseteq A \\ U \cap A' = U'}} \eta^v_A(U) = 0 = \eta_{A'}(U')$$

since $\eta^v_A(U) = 0$ for every U. $\square$

Therefore, using Kolmogorov's existence theorem, there exists a unique stochastic valuation V^v inducing the finite-dimensional probability valuations in η^v. We say that V^v is the *stochastic valuation induced by* v. Observe that

$$\mathsf{Prob}(V^v_A = U) = \eta^v_A(U)$$

for each $A \in \wp^+_{\mathsf{fin}} B$ and $U \subseteq A$. The next result establishes the envisaged relationship between satisfaction by a valuation and satisfaction by its induced stochastic valuation.

Proposition 4.2. *Given a formula* $\alpha \in L$, *a valuation* v *and* $p \in (0, 1]$

$$v \Vdash_{\mathsf{c}} \alpha \quad \text{iff} \;\; V^v \Vdash_p \alpha.$$

Proof. $(\rightarrow)$ Assume that $v \Vdash_{\mathsf{c}} \alpha$. Then

$$v \cap B_\alpha \in [\![\alpha]\!]$$

by definition of $[\![\alpha]\!]$. Hence,

$$\mathsf{Prob}_{V^v}(\alpha) = \sum_{U \in [\![\alpha]\!]} \mathsf{Prob}(V^v_{B_\alpha} = U) = \mathsf{Prob}(V^v_{B_\alpha} = v \cap B_\alpha) = 1 \geq p,$$

by definition of V^v. So, $V^v \Vdash_p \alpha$.

$(\leftarrow)$ Assume that $V^v \Vdash_p \alpha$. Then, $\mathsf{Prob}_{V^v}(\alpha) \geq p$. Hence,

$$\sum_{U \in [\![\alpha]\!]} \mathsf{Prob}(V^v_{B_\alpha} = U) \geq p > 0.$$

Note that $\mathsf{Prob}(V^v_{B_\alpha} = v \cap B_\alpha) = 1$ and $\mathsf{Prob}(V^v_{B_\alpha} = U) = 0$, for every $U \neq v \cap B_\alpha$. Therefore, $v \cap B_\alpha \in [\![\alpha]\!]$. Thus, $v \Vdash_{\mathsf{c}} \alpha$. $\square$

We now proceed with the investigation of the relationship between the probabilistic entailment and the classical entailment. To this end, we need the following auxiliary result.

Proposition 4.3. *Given formulas* δ *and* α *with* $\delta \vDash_{\mathsf{c}} \alpha$ *and* $p, q \in (0, 1]$ *with* $p \geq q$,

$$\text{if } V \Vdash_p \delta \text{ then } V \Vdash_q \alpha$$

for every stochastic valuation V.

Proof. Let V be a stochastic valuation such that $V \Vdash_p \delta$. Hence,

$$\mathsf{Prob}_V(\delta) \geq p.$$

So, since Prob_V is a probability assignment, then by Proposition 2.2,

$$\mathsf{Prob}_V(\alpha) \geq \mathsf{Prob}_V(\delta) \geq p \geq q.$$

Thus, $V \Vdash_q \alpha$. □

The next result shows that each probabilistic entailment (for $p, q \in (0, 1]$) collapses into the classical entailment. So every probabilistic assertion written with the language L can be proved in propositional logic.

Theorem 4.4. *Given a set of formulas Δ, a formula α and $p, q \in (0, 1]$ with $p \geq q$,*

$$\Delta \vDash_{\mathsf{c}} \alpha \quad \textit{iff } \Delta \vDash_p^q \alpha.$$

Proof. ($\rightarrow$) Assume that $\Delta \vDash_{\mathsf{c}} \alpha$, and let Φ be a finite subset of Δ such that $\Phi \vDash_{\mathsf{c}} \alpha$. Hence, $\bigwedge \Phi \vDash_{\mathsf{c}} \alpha$. Thus, by Proposition 4.3, if $V \Vdash_p \bigwedge \Phi$ then $V \Vdash_q \alpha$, for every stochastic valuation V. Therefore, by definition, $\Delta \vDash_p^q \alpha$.

($\leftarrow$) Assume that $\Delta \vDash_p^q \alpha$, and let Φ be a finite subset of Δ such that, for every stochastic valuation V,

$$\text{if } V \Vdash_p \bigwedge \Phi \text{ then } V \Vdash_q \alpha.$$

Let v be a valuation such that $v \Vdash_{\mathsf{c}} \delta$ for each $\delta \in \Delta$. Then, $v \Vdash_{\mathsf{c}} \bigwedge \Phi$. Observe that $p > 0$. Then, $V^v \Vdash_p \bigwedge \Phi$, by Proposition 4.2. So, $V^v \Vdash_q \alpha$ because $\Delta \vDash_p^q \alpha$. Thus, again by Proposition 4.2, $v \Vdash_{\mathsf{c}} \alpha$ since $q > 0$. □

Moreover, we say that Δ *g-entails* α, written

$$\Delta \vDash^{\mathrm{g}} \alpha$$

if $\Delta \vDash_p^q \alpha$ for every $p \in (0, 1]$. Observe that $\vDash^{\mathrm{g}}$ is extensive, monotonic and idempotent. It is immediate that:

$$\Delta \vDash_{\mathsf{c}} \alpha \quad \text{iff } \Delta \vDash^{\mathrm{g}} \alpha.$$

In conclusion, it is not possible to get a reasonable definition of probabilistic entailment (not collapsing into classical entailment) by enriching only the semantics of the classical propositional logic and keeping the same set of formulas. It is necessary to extend the language with probabilistic constructs.

5. Probabilistic propositional logic

The objective of this section is to define an enrichment of CPL that captures the probabilistic nature of the semantics provided by stochastic valuations. The idea is to add as little as possible to the propositional language L. It turns out that adding a symbolic construct allowing the constraining of the probability of a formula is enough.

Before proceeding with the presentation of the envisaged probabilistic propositional logic (PPL), we need to adopt some notation concerning the first-order theory of real closed ordered fields (RCOF), having in mind the use of its terms for denoting probabilities and other quantities.

Recall that the first-order signature of RCOF contains the constants 0 and 1, the unary function symbol $-$, the binary function symbols $+$ and $\times$, and the binary predicate symbols $=$ and $<$. We take the set $X = X_{\mathbb{N}} \cup X_L$, where $X_{\mathbb{N}} = \{x_k : k \in \mathbb{N}\}$ and $X_L = \{x_\alpha : \alpha \in L\}$, as the set of variables. In the sequel, by T_{RCOF} we mean the set of terms in RCOF that do not use variables in X_L. As we shall see, the variables in X_L become handy in the proposed axiomatization of PPL, for representing within the language of RCOF the probability of α.

We write $t_1 \leq t_2$ for $(t_1 < t_2) \vee (t_1 = t_2)$, $t_1 t_2$ for $t_1 \times t_2$ and t^n for

$$\underbrace{t \times \cdots \times t}_{n \text{ times}}.$$

Furthermore, we also use the abbreviations for any given $m \in \mathbb{N}^+$ and $n \in \mathbb{N}$:

- m for $\underbrace{1 + \cdots + 1}_{\text{addition of } m \text{ units}}$;
- m^{-1} for the unique z such that $m \times z = 1$;
- $\dfrac{n}{m}$ for $m^{-1} \times n$.

The last two abbreviations might be extended to other terms, but we need them only for numerals. For the sake of simplicity, we do not notationally distinguish between a natural number and the corresponding numeral.

In order to avoid confusion with the other notions of satisfaction used herein, we adopt $\Vdash_{\mathsf{fo}}$ for denoting satisfaction in first-order logic (over the language of RCOF).

The fact that the theory RCOF is decidable (see [41]) is put to good use in the axiomatization of PPL (in this section) and, further on (in Section 7), for proving the decidability of PPL. Furthermore, every model of RCOF satisfies the theorems and only the theorems of RCOF (Corollary 3.3.16 in [26]). We take advantage of this result in the semantics of PPL by adopting the field $\mathbb{R}$ of the real numbers as the model of RCOF.

5.1. Language. The language L_{PPL} of the propositional probability logic PPL is inductively defined as follows:

- $\int \alpha \mathbin{@} p \in L_{\mathsf{PPL}}$ where $\alpha \in L$, $p \in T_{\mathsf{RCOF}}$ and $@ \in \{=, <\}$;
- $\varphi_1 \supset \varphi_2 \in L_{\mathsf{PPL}}$ whenever $\varphi_1, \varphi_2 \in L_{\mathsf{PPL}}$.

Propositional abbreviations can be introduced as usual. For instance,

$$\neg\varphi \text{ for } \varphi \supset (\textstyle\int \mathsf{tt} < 1)$$

and similarly for $\wedge$, $\vee$ and $\equiv$. Comparison abbreviations also become handy. For instance,

$$\textstyle\int \alpha \leq p \quad \text{for } (\int \alpha = p) \vee (\int \alpha < p) \qquad \text{and} \qquad \int \alpha \geq p \quad \text{for } \neg(\int \alpha < p).$$

5.2. Semantics. Given a term t and an assignment $\rho : X \to \mathbb{R}$, we write $t^{\mathbb{R}\rho}$ for the denotation of term t in $\mathbb{R}$ for ρ. When t does not contain variables we may use $t^{\mathbb{R}}$ for the denotation of t in $\mathbb{R}$.

Let V be a stochastic valuation and ρ an assignment. *Satisfaction of formulas* by V and ρ is inductively defined as follows:

- $V\rho \Vdash \int \alpha \mathbin{@} p$ whenever $\mathsf{Prob}_V(\alpha) \mathbin{@} p^{\mathbb{R}\rho}$;
- $V\rho \Vdash \varphi_1 \supset \varphi_2$ whenever $V\rho \nVdash \varphi_1$ or $V\rho \Vdash \varphi_2$.

We may omit the reference to the assignment ρ whenever the formula does not include variables.

Let $\Gamma \subseteq L_{\mathsf{PPL}}$ and $\varphi \in L_{\mathsf{PPL}}$. We say that Γ *entails* φ, written $\Gamma \vDash \varphi$, whenever, for every stochastic valuation V and assignment ρ, if $V\rho \Vdash \gamma$ for each $\gamma \in \Gamma$ then $V\rho \Vdash \varphi$. As expected, φ is said to be *valid* when $\vDash \varphi$.

Observe that entailment in PPL is not compact. Let $\alpha \in L$. We start by showing that in the real closed ordered field $\mathbb{R}$

$$(\dagger) \quad \left\{ \textstyle\int \alpha \leq \dfrac{1}{n} : n \in \mathbb{N}^+ \right\} \vDash \textstyle\int \alpha = 0.$$

Let V be a stochastic valuation such that

$$\mathsf{Prob}_V(\alpha) \leq \frac{1}{n}, \qquad \text{for every } n \in \mathbb{N}^+.$$

Assume, by contradiction, that $\mathsf{Prob}_V(\alpha) \neq 0$. Then, because

$$\left\{ \mathsf{Prob}_V(\alpha) \leq \frac{1}{n} : n \in \mathbb{N}^+ \right\} = \left\{ n \leq \frac{1}{\mathsf{Prob}_V(\alpha)} : n \in \mathbb{N}^+ \right\},$$

we would have that $\frac{1}{\mathsf{Prob}_V(\alpha)}$ is an upper bound of $\mathbb{N}$. But $\mathbb{N}$ is not bounded from above taking into account that $\mathbb{R}$ has the least upper bound property. Therefore, (†) is true. However, for every $k \geq 1$, we have

$$\left\{ \int \alpha \leq \frac{1}{n_j} : n_j \in \mathbb{N}^+, 1 \leq j \leq k \right\} \not\vDash \int \alpha = 0.$$

Hence, PPL cannot be complete in the presence of hypotheses.

5.3. Calculus. The PPL calculus combines propositional reasoning with RCOF reasoning. We intend to use the RCOF reasoning to a minimum, namely to prove assertions like

$$(\int \alpha_1 \, @_1 \, p_1 \wedge \cdots \wedge \int \alpha_k \, @_k \, p_k) \supset \int \alpha_{k+1} \, @_{k+1} \, p_{k+1}.$$

To this end, we represent in RCOF the probability $\int \alpha$ of each propositional formula α by variable x_α and impose conditions on that variable that effect the properties of the probability.

Recall that the probability of a formula α is the sum of the probabilities of the B_α-valuations that satisfy the formula and that there is a disjunctive normal form of α where each disjunct can be seen as identifying a B_α-valuation that satisfies the formula. Hence, for calculating the probability of α it is enough to sum the probabilities of each such disjunct.

As we proceed to explain, we collect these conditions in a formula of RCOF. We say that $\Lambda = \{\alpha_{11}, \ldots, \alpha_{1m_1}, \ldots, \alpha_{k1}, \ldots, \alpha_{km_k}\} \subset L$ is an *adequate set of DNF-conjuncts* for $\{\alpha_1, \ldots, \alpha_k\} \subset L$ whenever

(1) $B_{\alpha_{11}} = \cdots = B_{\alpha_{km_k}} = B_{\alpha_1} \cup \cdots \cup B_{\alpha_k} = B_\Lambda$;

(2) each $\alpha_{j\ell}$ is a conjunction of literals;

(3) $\vDash_{\mathsf{c}} \neg(\alpha_{j\ell} \wedge \alpha_{j\ell'})$ for $1 \leq \ell \neq \ell' \leq m_j$;

(4) $\vDash_{\mathsf{c}} \alpha_j \equiv \bigvee_{\ell=1}^{m_j} \alpha_{j\ell}$ for each $j = 1, \ldots, k$.

Observe that clauses 2 and 4 ensure that $\bigvee_{\ell=1}^{m_j} \alpha_{j\ell}$ is a disjunctive normal form of α_j. Moreover, clause 3 guarantees that there are no redundant disjuncts in the disjunctive normal form of each α_j. Given such an adequate set Λ of DNF-conjuncts for $\alpha_1, \ldots, \alpha_k$, we use the abbreviation

$$Q^{\alpha_1, \ldots, \alpha_k}_{\alpha_{11}, \ldots, \alpha_{km_k}}$$

for the RCOF formula

$$\Big(\bigwedge_{U \subseteq B_\Lambda} 0 \leq x_{\phi^U_{B_\Lambda}} \leq 1 \Big) \wedge \Big(\sum_{U \subseteq B_\Lambda} x_{\phi^U_{B_\Lambda}} = 1 \Big) \wedge \Big(\bigwedge_{j=1}^{k} \Big(x_{\alpha_j} = \sum_{\ell=1}^{m_j} x_{\alpha_{j\ell}} \Big) \Big).$$

Recall that each propositional formula $\phi^U_{B_\Lambda}$ identifies the B_Λ-valuation that assigns true to the propositional symbols in U and assigns false to the propositional symbols in $B_\Lambda \backslash U$. Hence,

$$0 \leq x_{\phi^U_{B_\Lambda}} \leq 1$$

imposes that the probability of each B_Λ-valuation should be in the interval $[0, 1]$ and

$$\sum_{U \subseteq B_\Lambda} x_{\phi^U_{B_\Lambda}} = 1$$

imposes in RCOF that the sum of the probabilities of all B_Λ-valuations is 1. The conjunct

$$x_{\alpha_j} = \sum_{\ell=1}^{m_j} x_{\alpha_{j\ell}}$$

imposes that the probability of α_j is the sum of the probabilities of the valuations that satisfy the formula.

The calculus for PPL is an extension of the classical propositional calculus containing the following axioms and rules:

TT $\dfrac{}{\varphi}$

provided that φ is a tautological formula;

RR $\dfrac{}{(\int \alpha_1 \,@_1\, p_1 \wedge \cdots \wedge \int \alpha_k \,@_k\, p_k) \supset \int \alpha_{k+1} \,@_{k+1}\, p_{k+1}}$

provided that there is an adequate set

$$\{\alpha_{11}, \ldots, \alpha_{1m_1}, \ldots, \alpha_{(k+1)1}, \ldots, \alpha_{(k+1)m_{k+1}}\}$$

of DNF-conjuncts for $\{\alpha_1, \ldots, \alpha_{k+1}\}$ such that

$$\forall \Big(\big(Q^{\alpha_1, \ldots, \alpha_{k+1}}_{\alpha_{11}, \ldots, \alpha_{k+1 m_{k+1}}} \wedge \bigwedge_{j=1}^{k} (x_{\alpha_j} \,@_j\, p_j) \big) \supset (x_{\alpha_{k+1}} \,@_{k+1}\, p_{k+1}) \Big)$$

is a theorem of RCOF;

MP $\dfrac{\varphi_1 \quad \varphi_1 \supset \varphi_2}{\varphi_2}$.

Axioms TT and rule MP extend the propositional reasoning to formulas in L_{PPL}. Axioms RR import to PPL all we need from RCOF. It is worthwhile to refer that the metatheorem of deduction holds for PPL.

1	$\int(\alpha \supset \mathsf{tt}) = 1$	RR
2	$\int \mathsf{tt} = 1$	RR
3	$\big(\int(\alpha \supset \mathsf{tt}) = 1\big) \supset \big((\int \mathsf{tt} = 1) \supset \big((\int(\alpha \supset \mathsf{tt}) = 1\big) \wedge (\int \mathsf{tt} = 1)\big)\big)$	TT
4	$(\int \mathsf{tt} = 1) \supset \big((\int(\alpha \supset \mathsf{tt}) = 1\big) \wedge (\int \mathsf{tt} = 1)\big)$	MP 1, 3
5	$\big(\int(\alpha \supset \mathsf{tt}) = 1\big) \wedge (\int \mathsf{tt} = 1)$	MP 2, 4
6	$\big(\big(\int(\alpha \supset \mathsf{tt}) = 1\big) \wedge (\int \mathsf{tt} = 1)\big) \supset (\int \alpha \leq 1)$	RR
7	$\int \alpha \leq 1$	MP 5, 6

Figure 1. $\vdash \int \alpha \leq 1$.

5.4. Examples. Consider the derivation in Figure 1 for establishing that

$$\vdash \int \alpha \leq 1$$

holds for arbitrary $\alpha \in L$. The use of RR axioms can be involved. We explain their use only for obtaining $\int \mathsf{tt} = 1$ (step 2 of the derivation). Assume that tt is an abbreviation of $B_1 \vee (\neg B_1)$. Then the following formula is in RCOF:

$$\forall\big(Q^{\mathsf{tt}}_{B_1,\neg B_1} \supset (x_{\mathsf{tt}} = 1)\big)$$

where $Q^{\mathsf{tt}}_{B_1,\neg B_1}$ is

$$\begin{gathered}(0 \leq x_{B_1} \leq 1) \wedge (0 \leq x_{\neg B_1} \leq 1)\\ \wedge\\ x_{B_1} + x_{\neg B_1} = 1\\ \wedge\\ x_{\mathsf{tt}} = x_{B_1} + x_{\neg B_1},\end{gathered}$$

and, so, $\int \mathsf{tt} = 1$ is obtained by RR.

Next, let us see how we can express and derive (binary) additivity, i.e., how we can establish that

$$\int \alpha = x_1, \qquad \int \beta = x_2, \qquad \int(\alpha \wedge \beta) = x_3 \vdash \int(\alpha \vee \beta) = x_1 + x_2 - x_3 \qquad \text{(BA)}$$

holds for arbitrary $\alpha, \beta \in L$. In fact, let

- α_1 be α;
- α_2 be β;
- α_3 be $\alpha \wedge \beta$ and each $\alpha_{3\ell}$ a common disjunct in the disjunctive normal form of α_1 and α_2;
- α_4 be $\alpha \vee \beta$ and each $\alpha_{4\ell}$ a disjunct in the disjunctive normal form of α_1 and α_2 with no repetitions.

1	$\int(B_1 \wedge (\neg B_2)) = x_1$	HYP
2	$\int(B_1 \wedge B_2) = x_2$	HYP
3	$(\int(B_1 \wedge (\neg B_2)) = x_1) \supset ((\int(B_1 \wedge B_2) = x_2) \supset$ $((\int(B_1 \wedge (\neg B_2)) = x_1) \wedge (\int(B_1 \wedge B_2) = x_2)))$	TT
4	$(\int(B_1 \wedge B_2) = x_2) \supset$ $((\int(B_1 \wedge (\neg B_2)) = x_1) \wedge (\int(B_1 \wedge B_2) = x_2))$	MP 1, 3
5	$(\int(B_1 \wedge (\neg B_2)) = x_1) \wedge (\int(B_1 \wedge B_2) = x_2)$	MP 2, 4
6	$((\int(B_1 \wedge (\neg B_2)) = x_1) \wedge (\int(B_1 \wedge B_2) = x_2)) \supset$ $(\int B_1 = x_1 + x_2)$	RR
7	$\int B_1 = x_1 + x_2$	MP 5, 6

Figure 2. $\int(B_1 \wedge (\neg B_2)) = x_1, \int(B_1 \wedge B_2) = x_2 \vdash \int B_1 = x_1 + x_2$.

Then, the formula

$$\forall\Big((Q^{\alpha_1,\ldots,\alpha_4}_{\alpha_{11},\ldots,\alpha_{4m_4}} \wedge \bigwedge_{j=1}^{3}(x_{\alpha_j} = x_j)) \supset (x_{\alpha_4} = x_1 + x_2 - x_3)\Big)$$

is a theorem of RCOF. Then, by RR we have

$$\vdash (\int \alpha = x_1 \wedge \int \beta = x_2 \wedge \int(\alpha \wedge \beta) = x_3) \supset \int(\alpha \vee \beta) = x_1 + x_2 - x_3.$$

Hence, (BA) follows by applying MP.

The marginal condition is also expressible and derivable. For instance, let $A = \{B_1, B_2\}$, $A' = \{B_1\}$ and $U' = \{B_1\}$. We present in Figure 2 a derivation for showing that

$$\int(B_1 \wedge (\neg B_2)) = x_1, \qquad \int(B_1 \wedge B_2) = x_2 \vdash \int B_1 = x_1 + x_2$$

holds.

The next example shows how *modus ponens* for classical formulas is lifted to PPL formulas. Observe that

$$\int \alpha_1 = 1, \qquad \int \alpha_1 \supset \alpha_2 = 1 \vdash \int \alpha_2 = 1$$

holds, as can be seen in Figure 3. Therefore, the rule

$$\mathrm{MP}^* \quad \frac{\int \alpha_1 = 1 \quad \int \alpha_1 \supset \alpha_2 = 1}{\int \alpha_2 = 1}$$

is admissible in the PPL calculus.

1	$\int(\alpha_1) = 1$	HYP
2	$\int(\alpha_1 \supset \alpha_2) = 1$	HYP
3	$\big(\int(\alpha_1) = 1\big) \supset \big(\big(\int(\alpha_1 \supset \alpha_2) = 1\big) \supset$ $\big(\big(\int(\alpha_1) = 1\big) \wedge \big(\int(\alpha_1 \supset \alpha_2) = 1\big)\big)\big)$	TT
4	$\big(\int(\alpha_1 \supset \alpha_2) = 1\big) \supset \big(\big(\int(\alpha_1) = 1\big) \wedge \big(\int(\alpha_1 \supset \alpha_2) = 1\big)\big)$	MP 1, 3
5	$\big(\int(\alpha_1) = 1\big) \wedge \big(\int(\alpha_1 \supset \alpha_2) = 1\big)$	MP 2, 4
6	$\big(\big(\int(\alpha_1) = 1\big) \wedge \big(\int(\alpha_1 \supset \alpha_2) = 1\big)\big) \supset \big(\int \alpha_2 = 1\big)$	RR
7	$\int \alpha_2 = 1$	MP 5, 6

Figure 3. $\int \alpha_1 = 1, \int \alpha_1 \supset \alpha_2 = 1 \vdash \int \alpha_2 = 1$.

In the same vein it is easy to show that the rule

$$\mathrm{TT}^* \quad \frac{}{\int \alpha = 1} \qquad \text{provided that } \vDash_{\mathsf{C}} \alpha$$

is also admissible in the PPL calculus. Just observe that

$$\{\phi_{B_\alpha}^U : U \subseteq B_\alpha\}$$

is an adequate set of DNF-conjuncts for α, since α is a tautology. Then, it is immediate to conclude that $(\int \mathsf{tt} = 1) \supset (\int \alpha = 1)$ is an RR axiom and, so, $\int \alpha = 1$ is derived from MP.

Oblivious transfer protocol. As a further illustration of the expressive power of PPL, we want to specify what is envisaged with an Oblivious Transfer Protocol (OTP) by expressing the assumed state before and the required state after a run of the protocol. To this end, it becomes handy to use the abbreviation

$$\alpha \text{ for } \int \alpha = 1$$

to which we will return in Section 7 for establishing a conservative embedding of CPL in PPL.

Simplifying from [35], an OTP is a protocol to be followed by two agents (say John and Mary) so that John sends a bit to Mary, but remains oblivious as to if the bit reached Mary or not, while these two alternatives are equiprobable. Rabin proposed a protocol for solving a more general problem (a message with several bits is to be obliviously sent by John to Mary). The existence of such oblivious transfer protocols is quite significant because by building upon them one can solve other types of cryptographic problems.

The state that is assumed before the run of the protocol and the state required after the run can be specified using the following propositional symbols (each with the indicated intended meaning):

$$\begin{array}{rl} \mathsf{JB0} & \text{(John holds bit 0)} \\ \mathsf{JB1} & \text{(John holds bit 1)} \\ \mathsf{MB0} & \text{(Mary holds bit 0)} \\ \mathsf{MB1} & \text{(Mary holds bit 1)} \\ \mathsf{JKMB0} & \text{(John knows that Mary holds bit 0)} \\ \mathsf{JKMB1} & \text{(John knows that Mary holds bit 1).} \end{array}$$

Indeed, the assumed initial state can be specified by the conjunction of the following PPL formulas:

$$\begin{array}{rl} \mathsf{JB0} \vee \mathsf{JB1} & \text{(John holds bit 1 or holds bit 0)} \\ (\neg\mathsf{MB0}) \wedge (\neg\mathsf{MB1}) & \text{(Mary does not hold bit 1 or bit 0),} \end{array}$$

and the envisaged final state by the conjunction of the following PPL formulas:

$$\begin{array}{rl} \mathsf{JB0} \vee \mathsf{JB1} & \text{(John holds bit 1 or holds bit 0)} \\ (\neg\mathsf{JKMB0}) \wedge (\neg\mathsf{JKMB1}) & \text{(John does not know} \\ & \quad \text{what, if anything, Mary holds)} \\ \mathsf{MB0} \supset \mathsf{JB0} & \text{(If Mary holds bit 0 then so does John)} \\ \mathsf{MB1} \supset \mathsf{JB1} & \text{(If Mary holds bit 1 then so does John)} \\ \int(\mathsf{MB0} \vee \mathsf{MB1}) = \frac{1}{2} & \text{(Mary holds a bit with probability } \frac{1}{2}\text{).} \end{array}$$

It is also necessary to impose the relevant epistemic requirements:

$$\mathsf{JKMB0} \supset \mathsf{MB0} \quad \text{and} \quad \mathsf{JKMB1} \supset \mathsf{MB1}.$$

This example shows the practical interest of developing a probabilistic epistemic dynamic logic as an enrichment of PPL, endeavour that we leave for future work. Observe that in the previous example we only needed a finite number of propositional symbols. But a key novelty of PPL is the possibility of working with a countably infinite set of propositional symbols. This capability of PPL adds a lot to its expressive power.

Halting problem. As an illustration, consider the encoding in PPL of the halting problem (as originally introduced in [42]):

Does Turing machine i halt on input j?

In this case we need the following propositional symbols (with the indicated intended meaning):

$$\begin{array}{lll} \mathsf{H}_{ijk} & \text{(Machine } i \text{ halts on input } j \text{ in } k \text{ steps)} & \text{for each } i,j,k \in \mathbb{N} \\ \mathsf{H}_{ij} & \text{(Machine } i \text{ halts on input } j\text{)} & \text{for each } i,j \in \mathbb{N}. \end{array}$$

Using this countably infinite set of propositional symbols, consider the PPL theory with the following set of proper axioms:

$$\begin{aligned} \mathrm{Ax}_{\mathsf{H}} = \{\mathsf{H}_{ijk} \supset \mathsf{H}_{ij} : i,j,k \in \mathbb{N}\} \\ \cup \\ \{\mathsf{H}_{ijk} : \text{machine } i \text{ halts on input } j \text{ in } k \text{ steps}\}. \end{aligned}$$

It is straightforward to establish the following fact about this theory for each $i, j \in \mathbb{N}$:

$$\text{(i)}\ \ \mathrm{Ax}_{\mathsf{H}} \vdash \mathsf{H}_{ij} \quad \text{iff} \quad \text{(ii)}\ \ \text{machine } i \text{ halts on input } j.$$

Indeed, it is easy to present a derivation for obtaining (i) from (ii). On the other hand, obtaining (ii) from (i) requires the (strong) soundness of the calculus of PPL, the first result in Section 6. Observe that Ax_{H} is decidable as required of a set of axioms. However, $(\mathrm{Ax}_{\mathsf{H}})^{\vdash}$ is undecidable (since otherwise, thanks to the fact above, the halting problem would also be decidable). So, this example shows that, in PPL, $\Gamma^{\vdash}$ may be undecidable even when Γ is assumed to be decidable. But $\emptyset^{\vdash}$ is decidable, the last result of Section 7.

The probabilistic capabilities of PPL would be needed for developing a similar theory for probabilistic Turing machines. To this end, we may take

$$\begin{gathered} \{\textstyle\int \mathsf{H}_{ijk} = x_1 \supset \int \mathsf{H}_{ij} \geq x_1 : i,j,k \in \mathbb{N}\} \\ \cup \\ \{\textstyle\int \mathsf{H}_{ijk} = p : \text{machine } i \text{ halts on input } j \text{ in } k \text{ steps with probability } p^{\mathbb{R}}\} \end{gathered}$$

as the set of proper axioms.

Meeting problem. Consider the meeting problem M as presented in [13]. The problem involves two persons travelling independently on $\mathbb{Z}$ beginning at the same time at point 0. Each traveller at a position s can either move to position $s-1$ or to position $s+1$ with probability $\frac{1}{2}$. Observe that the probability of a traveller to return to zero in an odd number of steps is zero. On the other hand, the probability of returning in 0 steps is 1, in two steps is $\frac{1}{2}$ since it can either go $0 \to 1 \to 0$ or $0 \to -1 \to 0$ and in 4 steps is $\frac{3}{8}$ since it can follow the steps $0 \to 1 \to 2 \to 1 \to 0$, $0 \to -1 \to -2 \to -1 \to 0$, $0 \to 1 \to 0 \to 1 \to 0$, $0 \to 1 \to$

$0 \to -1 \to 0$, $0 \to -1 \to 0 \to 1 \to 0$ and $0 \to -1 \to 0 \to -1 \to 0$. That is, each traveller behaves according to a random walk on the integer number line $\mathbb{Z}$. It is possible to prove that the probability of returning to point 0 in $2k$ steps is greater than or equal to $\frac{1}{\sqrt{4k}}$ (see [13]) and the probability of returning to point 0 in $2k+1$ steps is 0.

The objective is to prove that the travellers will meet again at point 0 in the future with probability one. In this case, we need the following propositional symbols (with the indicated intended meaning):

T^j_{s+1}	variable that takes value 1 when traveller j returns to point 0 in the $(s+1)$-th step
A_s	variable that takes value 1 if both travellers reach point 0 in the s-th step
B_{s+1}	variable that takes value 1 if both travellers reach point 0 in the $(s+1)$-th step, without a previous meeting at point 0
C_s	variable that takes value 1 if up to and including the s-th step, the travellers have not met at point 0

for each $s \in \mathbb{N}$ and $j = 1, 2$. Using this countably infinite set of propositional symbols and an auxiliary propositional symbol B_1, consider the PPL theory with the following set Ax_{M} of proper axioms:

$$
\begin{array}{lll}
\text{(Ax0)} & \int \mathsf{T}^j_{2k+1} = 0 & k \in \mathbb{N},\ j = 1,2 \\
& \left((\int B_1 = x \times x) \wedge \left(\int B_1 = \frac{1}{4k}\right) \right) \supset \int \mathsf{T}^j_{2k} \geq x & k \in \mathbb{N}^+,\ j = 1,2 \\
\text{(Ax1)} & (\int \mathsf{T}^1_s \wedge \mathsf{T}^2_s = x) \equiv (\int \mathsf{T}^1_s \times \int \mathsf{T}^2_s = x) & s \in \mathbb{N}^+ \\
\text{(Ax2)} & \int \mathsf{A}_s \wedge \mathsf{C}_s = 0 & s \in \mathbb{N} \\
\text{(Ax3)} & \int \mathsf{A}_0 = 1 & \\
\text{(Ax4)} & \left((\int \mathsf{A}_{s-i} = x_1) \wedge (\int \mathsf{B}_i = x_2) \right) & \\
& \qquad \supset (\int \mathsf{A}_s \wedge \mathsf{B}_i = x_1 \times x_2) & s \in \mathbb{N}^+,\ 0 < i \leq s
\end{array}
$$

as well as

$$
\begin{array}{lll}
\text{(Ax5)} & \neg(\mathsf{B}_i \wedge \mathsf{B}_j) & i < j \\
& \neg(\mathsf{B}_i \wedge \mathsf{C}_j) & i \leq j \\
\text{(Ax6)} & \mathsf{C}_s \vee \bigvee_{i=1}^{s} \mathsf{B}_i & s \in \mathbb{N}^+ \\
\text{(Ax7)} & \mathsf{A}_s \equiv (\mathsf{T}^1_s \wedge \mathsf{T}^2_s) & s \in \mathbb{N}^+.
\end{array}
$$

The objective is to show that

$$\sum_{s \in \mathbb{N}^+} \int \mathsf{B}_s = 1.$$

In order to simplify the presentation, we assume without loss of generality in the rest of this example, that the probability constructor $\int \cdot$ can occur in RCOF terms and moreover that relational assertions can contain RCOF terms at both sides. So, assume by contradiction, that there is a rational number $p \in (0,1)$ such that for every $s \in \mathbb{N}^+$

$$(\dagger) \ \vdash \sum_{j=1}^{s} \int \mathsf{B}_j \leq 1 - p.$$

The first step is to show that

$$\text{(i)} \ \vdash \Big(\sum_{i=1}^{s} \int \mathsf{A}_i\Big) = \Big(\sum_{i=0}^{s-1} \Big(\int \mathsf{A}_i \times \Big(\sum_{j=1}^{s-i} \int \mathsf{B}_j\Big)\Big)\Big).$$

In fact, by axiom (Ax5), (Ax6) and the PPL consequence (BA), we have

$$(\dagger) \ \vdash \int \mathsf{A}_i = \Big(\sum_{j=1}^{i} \int \mathsf{A}_i \wedge \mathsf{B}_j\Big) + (\int \mathsf{A}_i \wedge \mathsf{C}_i), \qquad \text{for each } i = 1, \ldots, s.$$

By RCOF substitution of equals, from (Ax4) and (†) we can conclude that

$$(\ddagger) \ \vdash \int \mathsf{A}_i = \Big(\sum_{j=1}^{i} \int \mathsf{A}_{i-j} \times \int \mathsf{B}_j\Big) + (\int \mathsf{A}_s \wedge \mathsf{C}_s), \qquad \text{for each } i = 1, \ldots, s.$$

Similarly, by (Ax2) and (‡) we have

$$(\dagger\dagger) \ \vdash \int \mathsf{A}_i = \Big(\sum_{j=1}^{i} \int \mathsf{A}_{i-j} \times \int \mathsf{B}_j\Big), \qquad \text{for each } i = 1, \ldots, s.$$

Furthermore RCOF reasoning over (††) leads to

$$\vdash \sum_{i=1}^{s} \int \mathsf{A}_i = \sum_{i=0}^{s-1} \Big(\int \mathsf{A}_i \times \Big(\sum_{j=1}^{s-i} \int \mathsf{B}_j\Big)\Big),$$

which concludes the first step.

We now show that, for any $s \in \mathbb{N}^+$,

$$\text{(ii)}\ \vdash \sum_{i=1}^{s} \int \mathsf{A}_i < \frac{1}{p}.$$

In fact, from (i), (†) and (RR),

$$\vdash \sum_{i=1}^{s} \int \mathsf{A}_i \leq (1-p)\left(1 + \sum_{i=1}^{s-1} \int \mathsf{A}_i\right).$$

On the other hand,

$$\vdash \sum_{i=1}^{s} \int \mathsf{A}_i \leq (1-p)\left(1 + \sum_{i=1}^{s} \int \mathsf{A}_i - \int \mathsf{A}_s\right).$$

Thus,

$$\vdash p \sum_{i=1}^{s} \int \mathsf{A}_i \leq (1-p) - (1-p) \int \mathsf{A}_s$$

and so

$$\vdash p \sum_{i=1}^{s} \int \mathsf{A}_i \leq (1-p).$$

Hence,

$$\vdash p \sum_{i=1}^{s} \int \mathsf{A}_i < 1,$$

concluding (ii).

The third step is to show that

$$\text{(iii)}\ \vdash \sum_{i=1}^{2k} \int \mathsf{A}_i \geq \frac{1}{4} \sum_{i=1}^{k} \frac{1}{i} \quad \text{and} \quad \vdash \sum_{i=1}^{2k+1} \int \mathsf{A}_i \geq \frac{1}{4} \sum_{i=1}^{k} \frac{1}{i}.$$

In fact, observe that, by (Ax0), (Ax1) and (Ax7) we have

$$\vdash \int \mathsf{A}_{2j} \geq \frac{1}{4j} \quad \text{and} \quad \vdash \int \mathsf{A}_{2j+1} = 0$$

allowing us to conclude the third step.

Let ℓ be greater than $2^{8/p}$. Then,

$$\vdash \frac{1}{4}\sum_{i=1}^{\ell}\frac{1}{i} \geq \frac{1}{p}.$$

Then, (ii) and (iii) allows us to conclude that

$$\vdash \sum_{i=1}^{2\ell} \int \mathsf{A}_i \geq \frac{1}{p} \quad \text{and} \quad \vdash \sum_{i=1}^{2\ell} \int \mathsf{A}_i < \frac{1}{p}$$

which is a contradiction.

Notwithstanding their simplicity, the examples above should be enough to assess the power of PPL for describing probabilistic systems and reasoning about them.

6. Soundness and weak completeness

In this section we show that the calculus for PPL is (strongly) sound and weakly complete. Observe that strong completeness is obviously out of question since the PPL entailment is not compact (as mentioned in Section 5).

Theorem 6.1. *The logic* PPL *is sound.*

Proof. The rules are sound. We only check that axiom RR is sound since the proof of the others is straightforward.

(RR) is sound. Let V be a stochastic valuation and ρ an assignment over $\mathbb{R}$. Assume that

$$V\rho \Vdash \int \alpha_j \; @_j \; p_j \qquad \text{for each } j = 1, \ldots, k$$

and that the formula

$$\forall\Big(\big(Q^{\alpha_1,\ldots,\alpha_{k+1}}_{\alpha_{11},\ldots,\alpha_{k+1m_{k+1}}} \wedge \bigwedge_{j=1}^{k}(x_{\alpha_j} \; @_j \; p_j)\big) \supset (x_{\alpha_{k+1}} \; @_{k+1} \; p_{k+1})\Big)$$

is in RCOF. Let ρ' be an assignment over $\mathbb{R}$ such that,

$$\rho'(x_\alpha) = \mathsf{Prob}_V(\alpha)$$

and $\rho'(x) = \rho(x)$ for every $x \in X_{\mathbb{N}}$. Then,

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} Q^{\alpha_1,\ldots,\alpha_{k+1}}_{\alpha_{11},\ldots,\alpha_{k+1m_{k+1}}} \wedge \bigwedge_{j=1}^{k}(x_{\alpha_j} \; @_j \; p_j).$$

Therefore,

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} x_{\alpha_{k+1}} \;@_{k+1}\; p_{k+1}$$

Hence, $\rho'(x_{\alpha_{k+1}}) \;@_{k+1}\; p_{k+1}^{\mathbb{R}\rho'}$ and so $\mathsf{Prob}_V(\alpha_{k+1}) \;@_{k+1}\; p_{k+1}^{\mathbb{R}\rho}$. Therefore $V\rho \Vdash \int \alpha_{k+1} \;@_{k+1}\; p_{k+1}$. □

We now proceed towards the weak completeness of the calculus. We start by proving an important lemma showing that we can move back and forth between satisfaction of RCOF formulas expressing probabilistic reasoning and satisfaction of PPL formulas.

Proposition 6.2. *Let φ be a formula of* PPL *and $\alpha_1, \ldots, \alpha_k$ be the propositional formulas such that $\int \alpha_j \;@_j\; p_j$ occurs in φ for each $j = 1, \ldots, k$. Moreover, let $\Lambda = \{\alpha_{11}, \ldots, \alpha_{km_k}\}$ be an adequate set of DNF-conjuncts for $\{\alpha_1, \ldots, \alpha_k\}$. Let ρ be an assignment over $\mathbb{R}$. Assume that*

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}}.$$

Then, there is a stochastic valuation V such that

$$V\rho \Vdash \varphi \quad \text{ iff } \mathbb{R}\rho \Vdash_{\mathsf{fo}} \psi$$

where ψ is the RCOF *formula obtained from φ by substituting $x_\alpha \;@\; p$ for each* PPL *formula $\int \alpha \;@\; p$.*

Proof. Let $A \in \wp_{\mathsf{fin}} B$ and $\eta_A : \wp A \to [0,1]$ be such that

$$\eta_A(U) = \frac{1}{2^{|A \setminus B_\Lambda|}} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = U \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}}).$$

Note that $0 \leq \rho(x_{\phi_{B_\Lambda}^{U'}}) \leq 1$ since $\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}}$. We start by showing that η_A is a finite-dimensional probability distribution. Observe that

$$\begin{aligned}
\sum_{U \subseteq A} \eta_A(U) &= \sum_{U \subseteq A} \frac{1}{2^{|A \setminus B_\Lambda|}} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = U \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}}) \\
&= \frac{1}{2^{|A \setminus B_\Lambda|}} \sum_{U \subseteq A} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = U \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}}) \\
&= 1
\end{aligned}$$

since

$$\sum_{U \subseteq A} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = U \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}}) = \sum_{U_1 \subseteq A \setminus B_\Lambda} \sum_{U_2 \subseteq A \cap B_\Lambda} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = U_2}} \rho(x_{\phi_{B_\Lambda}^{U'}})$$

$$= \sum_{U_1 \subseteq A \setminus B_\Lambda} \sum_{U' \subseteq B_\Lambda} \rho(x_{\phi_{B_\Lambda}^{U'}}) \qquad (*)$$

$$= \sum_{U_1 \subseteq A \setminus B_\Lambda} 1 \qquad (**)$$

$$= 2^{|A \setminus B_\Lambda|}$$

where $(*)$ follows from the fact that there is a bijection from

$$\{(U', U_2) : U' \subseteq B_\Lambda, U' \cap A = U_2, U_2 \subseteq A \cap B_\Lambda\} \text{ to } \{U' : U' \subseteq B_\Lambda\}$$

and $(**)$ holds because $\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}}$.

Now we prove that $\{\eta_A\}_{A \in \wp_{\mathsf{fin}} B}$ satisfies the marginal condition. Let $A \subseteq A'$, $A' \subseteq B$ and $U \subseteq A$. Then,

$$\eta_A(U) = \frac{1}{2^{|A \setminus B_\Lambda|}} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = U \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}})$$

$$= \frac{1}{2^{|A' \setminus B_\Lambda|}} \frac{1}{2^{|A \setminus B_\Lambda|}} 2^{|A' \setminus B_\Lambda|} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = U \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}})$$

$$= \frac{1}{2^{|A' \setminus B_\Lambda|}} \frac{1}{2^{|A \setminus B_\Lambda|}} 2^{|A \setminus B_\Lambda|} \sum_{\substack{U'' \subseteq A' \\ U'' \cap A = U}} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A' = U'' \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}}) \qquad (*)$$

$$= \sum_{\substack{U'' \subseteq A' \\ U'' \cap A = U}} \frac{1}{2^{|A' \setminus B_\Lambda|}} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A' = U'' \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}})$$

$$= \sum_{\substack{U'' \subseteq A' \\ U'' \cap A = U}} \eta_{A'}(U'')$$

where $(*)$ holds since:

$$2^{|A' \setminus B_\Lambda|} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = U \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}})$$

$$= \sum_{U'' \subseteq A' \setminus B_\Lambda} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = U \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}})$$

$$= \sum_{U''' \subseteq A \setminus B_\Lambda} \sum_{\substack{U'' \subseteq A' \\ U'' \cap A = U}} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A' = U'' \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}}) \qquad (**)$$

$$= 2^{|A \setminus B_\Lambda|} \sum_{\substack{U'' \subseteq A' \\ U'' \cap A = U}} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A' = U'' \cap B_\Lambda}} \rho(x_{\phi_{B_\Lambda}^{U'}})$$

and where $(**)$ holds since there is a bijection f from

$$\{(U'', U') : U'' \subseteq A' \setminus B_\Lambda, U' \subseteq B_\Lambda, U' \cap A = U \cap B_\Lambda\}$$

to

$$\{(W''', W'', W') : W''' \subseteq A \setminus B_\Lambda, W'' \subseteq A', \\ W'' \cap A = U, W' \cap A' = W'' \cap B_\Lambda, W' \subseteq B_\Lambda\}$$

such that

$$f(U'', U') = \big(U'' \cap A, U \cup \big(U'' \cap (A' \setminus A)\big) \cup \big(U' \cap (A' \setminus A)\big), U'\big).$$

Indeed,

(a) $f(U'', U')$ is in the range of f:

  (i) $W''' \subseteq A \setminus B_\Lambda$. Note that $W''' = U'' \cap A$. Hence, $W''' \subseteq A$. Moreover, since $U'' \subseteq B \setminus B_\Lambda$ then $W''' \subseteq B \setminus B_\Lambda$.

  (ii) $W'' \subseteq A'$. Note that $W'' = U \cup \big(U'' \cap (A' \setminus A)\big) \cup \big(U' \cap (A' \setminus A)\big)$ and that $U \subseteq A \subseteq A'$, $U'' \cap (A' \setminus A) \subseteq A'$ and $U' \cap (A' \setminus A) \subseteq A'$.

  (iii) $W'' \cap A = U$. It is sufficient to note that $W'' = U \cup \big(U'' \cap (A' \setminus A)\big) \cup \big(U' \cap (A' \setminus A)\big)$ and that $U \cap A = U$, $U'' \cap (A' \setminus A) \cap A = \emptyset$ and $U' \cap (A' \setminus A) \cap A = \emptyset$.

  (iv) $W' \cap A' = W'' \cap B_\Lambda$. It is sufficient to note that $W'' = U \cup \big(U'' \cap (A' \setminus A)\big) \cup \big(U' \cap (A' \setminus A)\big)$ and that $U \cap B_\Lambda = U' \cap A$, $U'' \cap (A' \setminus A) \cap B_\Lambda = \emptyset$ and $U' \cap (A' \setminus A) \cap B_\Lambda = U' \cap (A' \setminus A)$. So, $W'' \cap B_\Lambda = U' \cap A' = W' \cap A'$.

  (v) $W' \subseteq B_\Lambda$. Immediate since $U' \subseteq B_\Lambda$.

(b) f is injective. Assume that $f(U_1'', U_1') = f(U_2'', U_2')$. Then $U_1' = U_2'$. Moreover, $U_1'' \cap A = U_2'' \cap A$ and

$$U \cup \big(U_1'' \cap (A' \setminus A)\big) \cup \big(U_1' \cap (A' \setminus A)\big) = U \cup \big(U_2'' \cap (A' \setminus A)\big) \cup \big(U_2' \cap (A' \setminus A)\big).$$

Observe that $U_i'' \cap (A' \setminus A) \cap U = \emptyset$ and $U_i'' \cap (A' \setminus A) \cap U_i' \cap (A' \setminus A) = \emptyset$ for $i = 1, 2$. Hence, $U_1'' \cap (A' \setminus A) = U_2'' \cap (A' \setminus A)$. So

$$\begin{aligned} U_1'' &= U_1'' \cap A' \\ &= U_1'' \cap \big(A \cup (A' \backslash A)\big) \\ &= (U_1'' \cap A) \cup \big(U_1'' \cap (A' \backslash A)\big) \\ &= (U_2'' \cap A) \cup \big(U_2'' \cap (A' \backslash A)\big) \\ &= U_2''. \end{aligned}$$

(c) f is surjective. Let (W''', W'', W') be in the range of f. Take

$$U'' = W''' \cup \big((W'' \backslash A) \backslash B_\Lambda\big), \qquad U' = W'.$$

(i) (U'', U') is in the domain of f:
- $U'' \subseteq A' \backslash B_\Lambda$. Note that $U'' = W''' \cup \big((W'' \backslash A) \backslash B_\Lambda\big)$, $W''' \subseteq A \subseteq A'$ and $W''' \subseteq B \backslash B_\Lambda$. So, $W''' \subseteq A' \backslash B_\Lambda$. On the other hand, $(W'' \backslash A) \backslash B_\Lambda \subseteq A'$ since $W'' \subseteq A'$ and $(W'' \backslash A) \backslash B_\Lambda \subseteq B \backslash B_\Lambda$. So, $U'' \subseteq A' \backslash B_\Lambda$.
- $U' \subseteq B_\Lambda$. Immediate since $W' \subseteq B_\Lambda$.
- $U' \cap A = U \cap B_\Lambda$. Observe that

$$\begin{aligned} U \cap B_\Lambda &= W'' \cap B_\Lambda \cap A) \\ &= W' \cap A' \cap A) \\ &= W' \cap A) \\ &= U' \cap A. \end{aligned}$$

(ii) $f(U'', U') = (W''', W'', W')$. Indeed:
- $U'' \cap A = W'''$. In fact

$$\begin{aligned} U'' \cap A &= \big(W''' \cup \big((W'' \backslash A) \backslash B_\Lambda\big)\big) \cap A\big) \\ &= (W''' \cap A) \cup \big(\big((W'' \backslash A) \backslash B_\Lambda\big) \cap A\big)\big) \\ &= W''' \cap A) \\ &= W'''. \end{aligned}$$

- $U \cup \big(U'' \cap (A' \backslash A)\big) \cup \big(U' \cap (A' \backslash A)\big) = W''$. In fact

$$\begin{aligned} & U \cup \big(U'' \cap (A' \backslash A)\big) \cup \big(U' \cap (A' \backslash A)\big) \\ &\quad = U \cup \big(\big(W''' \cup \big((W'' \backslash A) \backslash B_\Lambda\big)\big) \cap (A' \backslash A)\big) \cup \big(W' \cap (A' \backslash A)\big) \\ &\quad = U \cup \big(\big((W'' \backslash A) \backslash B_\Lambda\big) \cap (A' \backslash A)\big) \cup \big(W' \cap (A' \backslash A)\big) \\ &\quad = U \cup \big((W'' \backslash A) \backslash B_\Lambda\big) \cup \big((W'' \cap B_\Lambda) \backslash A\big) \\ &\quad = U \cup (W'' \backslash A) \end{aligned}$$

$$= (W'' \cap A) \cup (W'' \backslash A)$$
$$= W''.$$

Hence, using Kolmogorov's existence theorem, there exists a unique stochastic valuation V having these finite-dimensional distributions.

Finally, we show, by induction on the structure of φ, that

$$V\rho \Vdash \varphi \quad \text{ iff } \mathbb{R}\rho \Vdash_{\mathsf{fo}} \psi.$$

Base: φ is $\int \alpha_j \mathbin{@_j} p_j$. Observe first that:

$$\begin{aligned}
\mathsf{Prob}_V(\alpha_j) &= \sum_{U \in [\![\alpha]\!]} \mathsf{Prob}(V_{B_{\alpha_j}} = U) \\
&= \sum_{U \in \{v \cap B_\Lambda,\, v \Vdash_{\mathsf{c}} \alpha_j\}} \mathsf{Prob}(V_{B_\Lambda} = U) && (*) \\
&= \sum_{v \cap B_\Lambda,\, v \Vdash_{\mathsf{c}} \alpha_j} \eta_{B_\Lambda}(v \cap B_\Lambda) \\
&= \sum_{v \cap B_\Lambda,\, v \Vdash_{\mathsf{c}} \alpha_j} \rho(x_{\phi_{B_\Lambda}^{v \cap B_\Lambda}}) \\
&= \sum_{\ell = 1, \ldots, m_j} \rho(x_{\alpha_{j\ell}}) && (**) \\
&= \rho(x_{\alpha_j}) && (***)
\end{aligned}$$

where $(*)$ holds by Proposition 3.4, $(**)$ holds since $\{\alpha_{11}, \ldots, \alpha_{km_k}\}$ is an adequate set of DNF-conjuncts for $\{\alpha_1, \ldots, \alpha_k\}$, and $(***)$ holds since $\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\alpha_1, \ldots, \alpha_k}_{\alpha_{11}, \ldots, \alpha_{km_k}}$. Then,

$(\leftarrow)$ Assume that $\mathbb{R}\rho \Vdash_{\mathsf{fo}} x_{\alpha_j} \mathbin{@_j} p_j$. Then $\rho(x_{\alpha_j}) \mathbin{@_j} p_j^{\mathbb{R}\rho}$. So, $\mathsf{Prob}_V(\alpha_j) \mathbin{@_j} p_j^{\mathbb{R}\rho}$. Hence, $V\rho \Vdash \varphi$.

$(\rightarrow)$ Assume that $V\rho \Vdash \int \alpha_j \mathbin{@_j} p_j$. Then $\mathsf{Prob}_V(\alpha_j) \mathbin{@_j} p_j^{\mathbb{R}\rho}$. Thus, $\rho(x_{\alpha_j}) \mathbin{@_j} p_j^{\mathbb{R}\rho}$ and so $\mathbb{R}\rho \Vdash_{\mathsf{fo}} \psi$.

Step: φ is $\varphi_1 \supset \varphi_2$. Then

$$\begin{gathered}
\mathbb{R}\rho \Vdash_{\mathsf{fo}} \psi_1 \supset \psi_2 \\
\text{iff} \\
\mathbb{R}\rho \not\Vdash_{\mathsf{fo}} \psi_1 \text{ or } \mathbb{R}\rho \Vdash_{\mathsf{fo}} \psi_2 \\
\text{iff} \qquad\qquad \text{IH} \\
V\rho \not\Vdash \varphi_1 \text{ or } V\rho \Vdash \varphi_2 \\
\text{iff} \\
V\rho \Vdash \varphi
\end{gathered}$$

where ψ_1 and ψ_2 are formulas obtained from φ_1 and φ_2, respectively, by replacing each formula $\int \alpha @ p$ by $x_\alpha @ p$. $\square$

Observe that η_A, as defined in the proof of the result above, gives values to valuations over the propositional symbols in A. We give two illustrations of $\eta_A(U)$.

(i) Assume that $A = \{B_1, B_2, B_3\}$, $B_\Lambda = \{B_1, B_3, B_4\}$ and $U = \{B_2\}$. Then,

$$\eta_A(U) = \frac{1}{2} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = \emptyset}} \rho(x_{\phi_{B_\Lambda}^{U'}}) = \frac{1}{2}\left(\rho(x_{\phi_{B_\Lambda}^{\emptyset}}) + \rho(x_{\phi_{B_\Lambda}^{\{B_4\}}})\right)$$

$$= \frac{1}{2}\left(\rho(x_{(\neg B_1)\wedge(\neg B_3)\wedge(\neg B_4)}) + \rho(x_{(\neg B_1)\wedge(\neg B_3)\wedge B_4})\right).$$

Observe that U is the valuation that gives 1 to B_2 and 0 to B_1, B_3 and B_4. The probability of U is one half the value of $\rho(x_{(\neg B_1)\wedge(\neg B_3)})$.

(ii) Assume that $A = \{B_1, B_2, B_3, B_4\}$, $B_\Lambda = \{B_1, B_3\}$ and $U = \{B_1, B_2\}$. Then,

$$\eta_A(U) = \frac{1}{4} \sum_{\substack{U' \subseteq B_\Lambda \\ U' \cap A = \{B_1\}}} \rho(x_{\phi_{B_\Lambda}^{U'}}) = \frac{1}{4}\rho(x_{\phi_{B_\Lambda}^{\{B_1\}}}) = \frac{1}{4}\rho(x_{B_1 \wedge (\neg B_3)}).$$

Note that U is the valuation that gives 1 to B_1, B_2 and 0 to B_3, B_4. The probability of U is a quarter the value of $\rho(x_{B_1 \wedge (\neg B_3)})$.

Proposition 6.3. *Let $\varphi \in L_{\mathsf{PPL}}$. Then, there is $\psi \in L_{\mathsf{PPL}}$ such that*

$$\vdash \varphi \equiv \psi$$

and ψ is in disjunctive normal form. Moreover, if φ is consistent then there is a conjunction of literals in ψ that is also consistent.

Theorem 6.4. *The logic* PPL *is weakly complete.*

Proof. Let $\varphi \in L_{\mathsf{PPL}}$. Assume that $\nvdash \varphi$. We proceed to show that $\nvDash \varphi$. First observe that $\neg\varphi$ must be consistent in the sense that $\neg\varphi \nvdash \mathrm{ff}$ because otherwise from $\neg\varphi$ one would be able to derive every formula, including in particular, $\neg\varphi \vdash \varphi$. Hence, by the metatheorem of deduction $\vdash (\neg\varphi) \supset \varphi$. Observing that $\vdash \left((\neg\varphi) \supset \varphi\right) \supset \varphi$, because $\left((\neg\varphi) \supset \varphi\right)$ is a tautological formula, then by MP we would have $\vdash \varphi$ contradicting the hypothesis. By Proposition 6.3,

$$\vdash (\neg\varphi) \equiv \bigvee_{m \in M} \eta_m$$

where each disjunct is a conjunction of literals. Since $\neg\varphi$ is consistent, at least one of the disjuncts must also be consistent. Let η_m be one such consistent disjunct. In order to show that $\not\vDash \varphi$ it is enough to show that $\neg\varphi$ is satisfiable. Hence, it is enough to show that there is one satisfiable disjunct. Indeed, η_m is satisfiable. Towards a contradiction, assume that there are no V and ρ such that $V\rho \Vdash \eta_m$ holds. Let η_m be of the form

$$(\textstyle\int \alpha_1 \mathbin{@_1} p_1) \wedge \cdots \wedge (\int \alpha_k \mathbin{@_k} p_k).$$

Let

$$\{\alpha_{11}, \ldots, \alpha_{1m_1}, \ldots, \alpha_{k1}, \ldots, \alpha_{km_k}\} \subset L$$

be an adequate set of DNF-conjuncts for $\{\alpha_1, \ldots, \alpha_k\} \subset L$. Then, by Proposition 6.2, there would not exist ρ and such that

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}} \wedge \Big(\bigwedge_{j=1}^{k}(x_{\alpha_j} \mathbin{@_j} p_j)\Big).$$

Hence, we would have

$$\forall\Big(\Big(Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}} \wedge \bigwedge_{j=1}^{k}(x_{\alpha_j} \mathbin{@_j} p_j)\Big) \supset \mathsf{ff}\Big) \in \mathsf{RCOF}.$$

Then, by RR, we would establish

$$\eta_m \vdash \mathsf{ff}$$

in contradiction with the consistency of η_m. □

7. Conservativeness and decidability

In this section we start by working towards showing that PPL is a conservative extension of classical propositional logic.

Given $\alpha \in L$, we denote by α^* the PPL formula $\int \alpha = 1$. Moreover, given $\Delta \subseteq L$, we denote by Δ^* the set $\{\delta^* : \delta \in \Delta\}$.

Proposition 7.1. *Letting $\Delta \cup \{\alpha\} \subseteq L$, if $\Delta \vdash_c \alpha$ then $\Delta^* \vdash \alpha^*$.*

Proof. Just observe that if $\alpha_1, \ldots, \alpha_n$ is a derivation sequence of $\alpha = \alpha_n$ from Δ in CPL, then, making good use of TT* and MP* (the admissible rules established in Subsection 5.4), $\alpha_1^*, \ldots, \alpha_n^*$ is a derivation sequence of α^* from Δ^* in PPL. □

Theorem 7.2. *Let $\Delta \cup \{\alpha\} \subseteq L$. Then*

$$\Delta^* \vDash \alpha^* \quad \textit{iff } \Delta \vDash_{\mathsf{c}} \alpha.$$

Proof. $(\rightarrow)$ Assume that $\Delta^* \vDash \alpha^*$. Let v be a (classical) valuation such that $v \Vdash_{\mathsf{c}} \delta$ for every $\delta \in \Delta$. Then, by Proposition 4.2, $\mathsf{Prob}_{V^v}(\delta) \geq 1$ for every $\delta \in \Delta$. Hence, $V^v \Vdash \int \delta = 1$ for every $\delta \in \Delta$. Thus, $V^v \Vdash \int \alpha = 1$ and, so, $\mathsf{Prob}_{V^v}(\alpha) = 1$. Therefore, using the same proposition, $v \Vdash_{\mathsf{c}} \alpha$.

$(\leftarrow)$ Assume that $\Delta \vDash_{\mathsf{c}} \alpha$. Then, thanks to the previous proposition, $\Delta^* \vdash \alpha^*$ and, so, by Theorem 6.1, $\Delta^* \vDash \alpha^*$. □

We now concentrate on the decidability of the PPL validity problem. For this purpose we assume given the following two algorithms. Let $\mathscr{A}_{\mathrm{DNF}}$ be an algorithm that receives a propositional formula α and a set of propositional symbols $A \supseteq B_\alpha$, and returns a set $\{\beta_1, \ldots, \beta_m\}$ of conjunctions of literals such that each $B_{\beta_i} = A$, $\beta_1 \vee \cdots \vee \beta_m$ is a disjunctive normal form of α and $\nvDash_{\mathsf{c}} \beta_i \equiv \beta_j$ for $1 \leq i \neq j \leq m$. Furthermore, let $\mathscr{A}_{\mathsf{RCOF}}$ be an algorithm for deciding the validity of sentences in RCOF.

The procedure in Figure 4 receives a PPL formula and returns true whenever the formula is valid and false otherwise. Indeed, the following theorem establishes that the execution of $\mathscr{A}_{\mathsf{PPL}}$ always terminates and does so with the correct output.

Theorem 7.3. *The procedure $\mathscr{A}_{\mathsf{PPL}}$ is an algorithm. Moreover, $\mathscr{A}_{\mathsf{PPL}}$ is correct.*

Proof. It is straightforward to verify that the execution of $\mathscr{A}_{\mathsf{PPL}}$ always terminate, returning either true or false, so we focus on correctness:

(i) We start by showing that if $\mathscr{A}_{\mathsf{PPL}}(\varphi)$ is true then φ is a valid formula of PPL. Let φ be a formula of PPL. Assume that $\mathscr{A}_{\mathsf{PPL}}(\varphi)$ is true. Then,

$$\mathscr{A}_{\mathsf{RCOF}}\big(\forall (Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}} \supset \psi)\big)$$

Input: formula φ of PPL.

(1) Let $B_\varphi := \{B_j : B_j \in B \text{ and } B_j \text{ occurs in } \varphi\}$;

(2) Let $\{\alpha_1, \ldots, \alpha_k\} := \{\alpha : \int \alpha \mathbin{@} p \text{ occurs in } \varphi\}$;

(3) Let ψ be the formula obtained from φ by replacing each formula $\int \alpha \mathbin{@} p$ by $x_\alpha \mathbin{@} p$;

(4) For each $j = 1, \ldots, k$:

  (a) Let $\{\alpha_{j1}, \ldots, \alpha_{jm_j}\} := \mathscr{A}_{\mathrm{DNF}}(\alpha_j, B_\varphi)$;

(5) Return $\mathscr{A}_{\mathsf{RCOF}}\big(\forall (Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}} \supset \psi)\big)$.

Figure 4. Algorithm $\mathscr{A}_{\mathsf{PPL}}$.

is true. Let V be a stochastic valuation and ρ an assignment. Let ρ' be an assignment over $\mathbb{R}$ such that,

$$\rho'(x_\alpha) = \mathsf{Prob}_V(\alpha)$$

and $\rho'(x) = \rho(x)$ for every $x \in X_{\mathbb{N}}$. We now show that

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}}.$$

Recall that Prob_V is an Adams' probability assignment (Theorem 3.6). Hence, Prob_V satisfies Adams' postulates. Therefore:

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} \bigwedge_{U \subseteq B_\varphi} 0 \leq x_{\phi^U_{B_\varphi}} \leq 1$$

since $\rho'(x_{\phi^U_{B_\varphi}}) = \mathsf{Prob}_V(\phi^U_{B_\varphi})$ and using postulate P1. Moreover,

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} \sum_{U \subseteq B_\varphi} x_{\phi^U_{B_\varphi}} = 1$$

by postulates P2 and P4 since

$$\vDash_{\mathsf{c}} \bigvee_{U \subseteq B_\varphi} \phi^U_{B_\varphi}$$

by Proposition 3.2, and using the fact that $\rho'(x_{\phi^U_{B_\varphi}}) = \mathsf{Prob}_V(\phi^U_{B_\varphi})$ and $\vDash_{\mathsf{c}} \neg(\alpha_{j\ell} \wedge \alpha_{j\ell'})$ for every $j = 1, \ldots, k$ and $1 \leq \ell \neq \ell' \leq m_j$. Finally,

$$\bigwedge_{j=1}^{k'} \Big(x_{\alpha_j} = \sum_{\ell=1}^{m_j} x_{\alpha_{j\ell}} \Big).$$

since

$$\mathsf{Prob}_V(\alpha_j) = \mathsf{Prob}_V\big(\bigvee_{1 \leq \ell \leq m_j} \alpha_{j\ell} \big)$$

by postulate P3, and since, by postulate P4

$$\mathsf{Prob}_V\big(\bigvee_{1 \leq \ell \leq m_j} \alpha_{j\ell} \big) = \sum_{\ell=1}^{m_j} \mathsf{Prob}_V(\alpha_{j\ell}).$$

Hence, $\mathbb{R}\rho' \Vdash_{\mathsf{fo}} Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}}$. Moreover

$$\mathbb{R} \Vdash_{\mathsf{fo}} \forall(Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}} \supset \psi)$$

and so $\mathbb{R}\rho' \Vdash_{\mathsf{fo}} \psi$.

We now show, by induction on the structure of φ, that

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} \psi \quad \text{ iff } V\rho \Vdash \varphi.$$

Base: φ is $\int \alpha \mathbin{@} p$. Then

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} x_\alpha \mathbin{@} p \quad \text{ iff } \rho'(x_\alpha) \mathbin{@} p^{\mathbb{R}\rho'} \text{ iff } \mathsf{Prob}_V(\alpha) \mathbin{@} p^{\mathbb{R}\rho} \text{ iff } V\rho \Vdash \int \alpha \mathbin{@} p.$$

Step: φ is $\varphi_1 \supset \varphi_2$. Then

$$\begin{array}{cl} \mathbb{R}\rho' \Vdash_{\mathsf{fo}} \psi_1 \supset \psi_2 & \\ \text{iff} & \\ \mathbb{R}\rho' \nVdash_{\mathsf{fo}} \psi_1 \text{ or } \mathbb{R}\rho' \Vdash_{\mathsf{fo}} \psi_2 & \\ \text{iff} & \text{IH} \\ V\rho \nVdash \varphi_1 \text{ or } V\rho \Vdash \varphi_2 & \\ \text{iff} & \\ V\rho \Vdash \varphi & \end{array}$$

where ψ_1 and ψ_2 are formulas obtained from φ_1 and φ_2, respectively, by replacing each formula $\int \alpha \mathbin{@} p$ by $x_\alpha \mathbin{@} p$. Therefore, $\vDash \varphi$.

(ii) We now show that if $\mathscr{A}_{\mathsf{PPL}}(\varphi)$ is false then φ is not a valid formula of PPL. By contraposition, assume that $\vDash \varphi$. We prove that, for every assignment ρ over $\mathbb{R}$,

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} \forall(Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}} \supset \psi).$$

Assume that

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}}.$$

Let V be the stochastic valuation induced by ρ as defined in Proposition 6.2. Then, $V\rho \Vdash \varphi$ since $\vDash \varphi$ and, so, by the same proposition, $\mathbb{R}\rho \Vdash_{\mathsf{fo}} \psi$.

Therefore,

$$\mathscr{A}_{\mathsf{RCOF}}\left(\forall(Q^{\alpha_1,\ldots,\alpha_k}_{\alpha_{11},\ldots,\alpha_{km_k}} \supset \psi)\right)$$

returns true and so does $\mathscr{A}_{\mathsf{PPL}}(\varphi)$. □

8. Concluding remarks

Always within the setting of propositional logic we looked at ways of introducing probabilistic reasoning into logic. First, towards assigning probabilities to valuations we proposed to look at a random valuation as a stochastic process indexed by the set of propositional symbols. This novel notion (of stochastic valuation as we called it) allow us to be able to work with a countably infinite set of propositional symbols (we illustrate the relevance of this cardinality in the probabilistic halting problem as well as in the meeting problem). Moreover, it has the advantage of allowing the use of Kolmogorov's existence theorem for moving from the finite-dimensional probability distributions to the distribution in the underlying probability space. In particular, the existence theorem was quite useful in establishing the equivalence between Adams' probability assignments to formulas and stochastic valuations. Afterwards, we investigated a notion of probabilistic entailment in the scenario of leaving the propositional language unchanged. This notion turned out to be identical to classical entailment. Since it seems that not so much is gained by introducing probabilities without changing the language, we decided to set-up a small enrichment (PPL) of classical propositional logic by adding a language construct, inspired by [15], [23], [29], that allows the constraining (without nesting) of the probability of a formula. The resulting extension of classical propositional logic was shown to be rich enough for setting-up interesting theories and easy to axiomatize by relying on the decidable theory of real closed ordered fields (RCOF). In due course, we proved that the extension is conservative and still decidable.

Concerning future work, it seems worthwhile to investigate other meta-properties of PPL, starting with bounding the complexity of its decision problem. We expect this complexity to be much lower than the complexity of RCOF theoremhood, since we only need to recognize RCOF theorems of a very simple clausal form. Strong completeness of the PPL axiomatization was out of question because the semantics over $\mathbb{R}$ led to a non-compact entailment. Relaxing the semantics by allowing any model of RCOF may open the door to establishing strong completeness. Clearly, one should start by investigating whether Kolmogorov existence theorem can be carried over to every RCOF model. We would like to explore further definitions of probabilistic entailment namely involving the selection of a particular stochastic valuation using some criterion like, for example, the maximum Shannon entropy or others as discussed in [32]. The relevance of abduction in probabilistic reasoning was recognized in [17]. We would like to compute the required probability of the conjunction of the relevant hypotheses in order to ensure an envisaged probability for the conclusion. We expect to be able to find inspiration in the calculus presented in [38], given its abductive nature, towards developing an abduction calculus for PPL.

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A. Sernadas[†], Departamento de Matemática, Instituto Superior Técnico and Centro de Matemática, Aplicações Fundamentais e Investigação Operacional, Universidade de Lisboa, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal

J. Rasga, Departamento de Matemática, Instituto Superior Técnico and Centro de Matemática, Aplicações Fundamentais e Investigação Operacional, Universidade de Lisboa, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail: joao.rasga@tecnico.ulisboa.pt

C. Sernadas, Departamento de Matemática, Instituto Superior Técnico and Centro de Matemática, Aplicações Fundamentais e Investigação Operacional, Universidade de Lisboa, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail: cristina.sernadas@tecnico.ulisboa.pt